Integr. equ. oper. theory 42 (2002) 229-242
0378-620X/02/020229-14 $ 1.50+0.20/0


Integral Equations and Operator Theory

# POSITIVE CARATHÉODORY INTERPOLATION ON THE POLYDISC

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The positive Carathéodory interpolation problem in the Agler-Herglotz class on the polydisc is solved, along with a several variable version of the Naimark dilation theorem. In addition, the positive Carathéodory interpolation problem for general holomorphic functions is discussed and numerical results are presented.

## 1 Introduction

Given a set of indices $\Lambda = \{0, \dots, n_1\} \times \dots \times \{0, \dots, n_d\} \subset \mathbb{Z}^d$, and complex numbers $c_k$, $k \in \Lambda$, with $c_0 \in \mathbb{R}$, one may consider the following problems:

1. *Carathéodory interpolation problem*: find an analytic function $\phi : \mathbb{D}^d \to \mathbb{C}$ so that $\text{Re}\, \phi(z) \geq 0, z \in \mathbb{D}^d$ (that is, $\phi$ belongs to the *Carathéodory class*), $\phi(0) = c_0$ and $\frac{\phi^{(k)}(0)}{k!} = 2c_k$, $k \in \Lambda \setminus \{0\}$.

2. *Moment problem*: Find a positive measure $\sigma$ on $\mathbb{T}^d$ with moments $\widehat{\sigma}(k) := \int_{\mathbb{T}^d} z^k d\sigma = c_k$, $k \in \Lambda$.

3. *Autoregressive filter problem*: find a polynomial $p(z) = \sum_{k \in \Lambda} p_k z^k$ with $p(z) \neq 0$, $z \in \overline{\mathbb{D}}^d$ ($p(z)$ is *stable*), so that the Fourier coefficients $\widehat{f}(k)$ of $f(z) := \frac{p_0}{|p(z)|^2}$ satisfy $\widehat{f}(k) = c_k$, $k \in \Lambda$.

Here we used the notations $z^k = (z_1, \dots, z_d)^{(k_1, \dots, k_d)} := z_1^{k_1} \cdots z_d^{k_d}$, $\phi^{(k)} = \frac{\partial^{k_1}}{\partial z_1^{k_1}} \cdots \frac{\partial^{k_d}}{\partial z_d^{k_d}} \phi$, and $k! = (k_1, \dots, k_d)! := k_1! \cdots k_d!$.

In the classical one-variable case ($d = 1$) there is a tight connection between these three problems. The Riesz-Herglotz representation of functions in the Carathéodory class connects the problems 1 and 2, so that they are only solvable simultaneously. Next, if $p(z)$ is a solution to 3, then $f(z)d\mu(z)$, where $\mu$ is the normalized Lebesgue measure, is a solution to 2. Finally, if 2 has an strictly positive absolutely continuous solution $f(z)d\mu$ (with $f(z) > 0$ on $\mathbb{T}$), then a combination of the maximum entropy principle and the Riesz-Fejer factorization lemma shows that it has in fact a solution of the form $\frac{p_0}{|p(z)|^2} d\mu$, with $p$ a stable polynomial, thus providing a solution for 3. In fact, as is well-known, 1 and 2 are solvable if and only if $T \geq 0$ and 3 is solvable if and only if $T > 0$, where $T$ is the finite Toeplitz matrix $T = (c_{i-j})_{i,j=0}^{n}$. There is a vast literature on this subject, and as a guide to it we refer to the books [17], [3], [14], and [12].

When $d > 1$ one may again observe that if $p$ is a solution to 3 then we have that $\sigma = \frac{p_0}{|p|^2} d\mu$ is a solution to 2. In addition, it follows from Theorem 1 in [19] that a solution to 1 gives a solution to 2. In contrast to the single variable case none of the other directions hold (even in the strict positive case). Consider the following examples.

**Example 1.** Let $\Lambda = \{0,1\}^2$ and $c_{00} = 1$, $c_{10} = c_{01} = c_{11} = 0.9$. Then 2 has a solution namely $0.1 d\mu + 0.9\delta_{(1,1)}$, where $d\mu$ is the normalized Lebesgue measure on $\mathbb{T}^2$ and $\delta_p$ is the Dirac mass at the point $p$. Furthermore, with these data problem 1 does not have a solution as

$$\begin{pmatrix} 1 & 0.9 & 0.9 & 0.9 \\ 0.9 & 1 & \bar{x} & 0.9 \\ 0.9 & x & 1 & 0.9 \\ 0.9 & 0.9 & 0.9 & 1 \end{pmatrix}$$

is not positive semidefinite when $x = 0$. The autoregressive filter problem does have a solution as with the choice $x = 0.81$ the conditions of Theorem 1.1 in [16] are satisfied.

**Example 2.** Let $\Lambda = \{0,1\}^2$ and $c_{00} = 1$, $c_{10} = 0.3$, $c_{01} = 0.7$ and $c_{11} = 0.8$. One may check that the matrix

$$\begin{pmatrix} 1 & 0.7 & 0.3 & 0.8 \\ 0.7 & 1 & \bar{x} & 0.3 \\ 0.3 & x & 1 & 0.7 \\ 0.8 & 0.3 & 0.7 & 1 \end{pmatrix}$$

is positive definite e.g. when $x = 0$. By Theorem 5.10 in [15] therefore a (strictly positive) solution to problem 2 exists. When $x = \frac{c_{10}\overline{c_{01}}}{c_{00}} = 0.21$ the matrix has determinant equal to $-\frac{589}{250000}$ and is therefore not positive definite. Hence by Theorem 1.1 in [16] it follows that no solution to 3 exists for these data. Finally, a solution to problem 1 does exist as we shall see in Section 4.

In this paper we shall address the $d$-variable Carathéodory interpolation problem. The two-variable autoregressive filter problem was solved in [16] where a necessary and sufficient condition for the existence of a solution was given in terms of the existence of a solution of a matrix completion problem. Numerical solutions to the autoregressive filter problem were presented in [11]. For the moment problem the connection with its dual problem of representing trigonometric polynomials as sums of squares, is very useful. This connection allowed [10] and [23] to construct examples of so-called "non-extendable patterns". Subsequent results may be found in [24], [15], [5] and [20].

The paper is organized as follows. In Section 2 we use the results of [1] and [8] to obtain a solution to the Carathéodory problem in the class of functions introduced by Agler. We thus completely solve the Carathéodory problem in case $d = 2$ and give sufficient conditions in case $d \geq 3$. In Section 3 we record some necessary conditions for the Carathéodory problem. Finally in Section 4 we briefly discuss some numerical results.

# 2 Carathéodory interpolation in the Agler-Herglotz class

Based on ideas in [1] and [8] we address in this section the Carathéodory interpolation problem in the Agler-Herglotz class. Other related papers in the area are [2], [13] and [22].

We first remind the reader of a class of functions introduced by J. Agler [1]. Let $\mathcal{E}$ be a Hilbert space, and $\mathcal{L}(\mathcal{E})$ denotes the space of bounded linear operators on $\mathcal{E}$. We denote $\mathbb{N}_0 = \{0,1,2,\dots\}$. Let $\phi$ be a holomorphic $\mathcal{L}(\mathcal{E})$-valued function defined on the polydisc $\mathbb{D}^d = \{(z_1,\dots,z_d) : |z_i| < 1, i = 1,\dots,d\}$, and let $\phi(z) = \sum_{k\in\mathbb{N}_0^d}\phi_k z^k$, $(z_1,\dots,z_d)^{(k_1,\dots,k_d)} := z_1^{k_1}\cdots z_d^{k_d}$, $\phi_k \in \mathcal{L}(\mathcal{E})$, be its series expansion. For a commuting collection of strict contractions $R = (R_1,\dots,R_d) \in \mathcal{L}(\mathcal{K})^d$ we may define $\phi(R) = \phi(R_1,\dots,R_d)$ to be an operator in $\mathcal{L}(\mathcal{E})\otimes\mathcal{L}(\mathcal{K}) \equiv \mathcal{L}(\mathcal{E}\otimes\mathcal{K})$ by $\phi(R) = \sum_{k\in\mathbb{N}_0^d}\phi_k\otimes R^k$, where $(R_1,\dots,R_d)^{(k_1,\dots,k_d)} := R_1^{k_1}\cdots R_d^{k_d}$. We say that $\phi$ is *Agler-Herglotz* if Re $\phi(R)\geq 0$ for all commuting collections of strict contractions $R = (R_1,\dots,R_d)$. Here Re $A = \frac{1}{2}(A+A^*)$ denotes the real part of the operator $A$. We shall denote the class of Agler-Herglotz $\mathcal{L}(\mathcal{E})$-valued functions on $\mathbb{D}^d$ by $\mathcal{A}_d(\mathcal{E})$. By taking $R_j = z_j I$ one sees immediately that $\phi\in\mathcal{A}_d(\mathcal{E})$ implies that Re $\phi(z)\geq 0$, $z\in\mathbb{D}^d$. The converse holds when $d = 1,2$ (see [1], which is based on a result by Ando [4]), but not when $d\geq 3$ (follows from the results of either [21] or [25], and performing a Cayley transform). It was shown in [1] that a holomorphic $\phi$ with $\phi(0) = \frac{1}{2}I$ belongs to $\mathcal{A}_d(\mathcal{E})$ if and only if there exists a Hilbert space $\mathcal{H} = \mathcal{H}_1\oplus\cdots\oplus\mathcal{H}_d$, an isometry $V:\mathcal{E}\to\mathcal{H}$ and a unitary $U:\mathcal{H}\to\mathcal{H}$ so that

$$\phi(z) = \frac{1}{2}I + V^*U(I - Z(z)U)^{-1}Z(z)V, z\in\mathbb{D}^d, \tag{2.1}$$

where $(Z(z))(\oplus_{i=1}^d h_i) := \oplus_{i=1}^d(z_i h_i)$.

We next state our interpolation problem. Let $\leq$ be the partial order on $\mathbb{N}_0^d$ defined by $(k_1,\cdots,k_d)\leq(l_1,\cdots,l_d)$ if and only if $k_i\leq l_i$, $i = 1,\dots,d$. A subset $K$ of $\mathbb{N}_0^d$ will be called *lower-inclusive* if $k\in K$ and $l\leq k$ imply $l\in K$. Given a finite lower-inclusive subset $K$ of $\mathbb{N}_0^d$ and operators $C_k\in\mathcal{L}(\mathcal{E})$, $k\in K$, *the Carathéodory interpolation* problem asks for a $\phi\in\mathcal{A}_d(\mathcal{E})$ so that $\phi_k = C_k$, $k\in K$, where $\phi(z) = \sum_{k\in\mathbb{N}_0^d}\phi_k z^k$.

Before we can state our main result, we first have to recall the notion of words. Let $A_1,\dots,A_d$ be operators on $\mathcal{H}$. An expression of the form

$$A_1^{n_{11}}\cdots A_d^{n_{1d}}\cdots A_1^{n_{p1}}\cdots A_d^{n_{pd}}$$

is called a *word* in $(A_1,\dots,A_d)$ of *multilength* $(\sum_{j=1}^p n_{j1},\dots,\sum_{j=1}^p n_{jd})$. Typically, factors with an exponent 0 are left out, and subexpressions $A_i^pA_i^q$ are contracted to $A_i^{p+q}$. Words are equal when after leaving out factors with a zero exponent and contracting as above, the expressions are equal. So, for example, there are exactly three different words in $(A,B)$ of multilength $(2,1)$, namely $A^2B$, $ABA$, and $B^2A$. Notice that even though it may happen that these three operators are equal (e.g., when $A$ and $B$ commute), the words are considered to be different. When $w$ is a word in $(A_1,\dots,A_d)$, its multilength is denoted by $ml(w)(\in\mathbb{N}_0^d)$. For $k\in\mathbb{N}_0^d$ we denote the set of all words of multilength $k$ in $(A_1,\dots,A_d)$ by $W_k(A_1,\dots,A_d)$.

Let $K$ be a finite subset of $\mathbb{N}_0^d$ with cardinality $|K|$. We let $\mathcal{E}^{|K|}$ denote the Hilbert space of $|K|$-tuples $\xi = (\xi_k)_{k\in K}$ with $\|\xi\|^2 := \sum_{k\in K} \|\xi_k\|^2 (< \infty$, since $K$ is finite). Notice that instead of indexing the coordinates of the tuples by $1, \dots, |K|$, we prefer to index them with the elements of $K$. This will be convenient when we define operators on $\mathcal{E}^{|K|}$. If $F_k \in \mathcal{L}(\mathcal{E})$, $k \in K$, the notation $F = \text{col}\,(F_k)_{k\in K}$ stands for the operator $F : \mathcal{E} \to \mathcal{E}^{|K|}$ defined by $F\xi = (F_k\xi)_{k\in K}$. Next, for $T \in \mathcal{L}(\mathcal{K})$ we denote by $M_T$ the conjugacy operator $M_T(X) = TXT^*$ on $\mathcal{L}(\mathcal{K})$. Note that $X \geq 0$ implies that $M_T(X) \geq 0$. Finally, denote by $e_i$, $i = 1, \dots, d$, the $i$th standard vector in $\mathbb{N}_0^d$, and let $\delta_{kl}$ denote the Kronecker delta function on $\mathbb{N}_0^d$.

**Theorem 2.1.** *Given a nonempty finite lower-inclusive set $K \subset \mathbb{N}_0^d$ and operators $C_k \in \mathcal{L}(\mathcal{E})$, $k \in K$, with $C_0 = \frac{1}{2}I$, the following are equivalent:*

*(i) there exists a $\phi(z) = \sum_{k\in\mathbb{N}_0^d} \phi_k z^k \in \mathcal{A}_d(\mathcal{E})$ so that $\phi_k = C_k$, $k \in K$;*

*(ii) there exist positive semidefinite operators $G_1, \dots, G_d$ on $\mathcal{E}^{|K|}$ so that $\prod_{j\neq i}(I - M_{T_j})(G_i) \geq 0$, $i = 1, \dots, d$, and*

$$X + X^* = G_1 + \ldots + G_d. \tag{2.2}$$

*Here $X = (C_{k-j})_{k,j\in K}$, $C_k = 0$ for $k \notin K$, and $T_j = (t_{k,l}^{(j)})_{k,l\in K}$, where $t_{k,l}^{(j)} = I$ if $k = l + e_j$ and $t_{k,l}^{(j)} = 0$ otherwise;*

*(iii) there exist positive definite operators $\Gamma_1, \dots, \Gamma_d \in \mathcal{L}(\mathcal{E}^{|K|})$ so that*

$$EC^* + CE^* = \Gamma_1 - T_1\Gamma_1T_1^* + \ldots + \Gamma_d - T_d\Gamma_dT_d^*, \tag{2.3}$$

*where*

$$C = \text{col}\,(C_k)_{k\in K}, E = \text{col}\,(\delta_{0k})_{k\in K},$$

*and $T_j$ is as in (ii).*

*(iv) there exists a Hilbert space $\mathcal{H} = \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_d$, an isometry $V : \mathcal{E} \to \mathcal{H}$ and a unitary $U \in \mathcal{L}(\mathcal{H})$ so that $C_k = V^*(\sum_{w\in W_k(UP_1,\dots,UP_d)} w)V$, $k \in K \setminus \{0\}$, where $P_i$ is the orthogonal projection in $\mathcal{L}(\mathcal{H})$ with image $\mathcal{H}_i$, $i = 1, \dots, d$.*

**Proof.** For the equivalence of (ii) and (iii) observe that $T_j^{|K|} = 0$, and thus for all $j = 1, \dots, d$ we have that $I - M_{T_j}$ is invertible on $\mathcal{L}(\mathcal{E}^{|K|})$. But then the equalities $\prod_{j=1}^d (I - M_{T_j})(X + X^*) = EC^* + CE^*$ and $\Gamma_i = \prod_{j\neq i}(I - M_{T_j})(G_i)$, $i = 1, \dots, d$, yield the equivalence.

Next, the equivalence of (i) and (iv) follows directly from writing equation (2.1) in its series expansion. A tedious but straightforward computation will show that for $k \in \mathbb{N}_0^d \setminus \{0\}$ we have that $\phi_k = V^*(\sum_{w\in W_k(UP_1,\dots,UP_d)} w)V$.

It remains to show the equivalence of (i) and (iii). First suppose that (i) holds, and write $\phi$ as in (2.1). Identify $\mathcal{E}^{|K|}$ with $H^2(\mathbb{D}^d, \mathcal{E}, K) := \{f(z) = \sum_{k\in K} f_k z^k, z \in \mathbb{D}^d : f_k \in \mathcal{E}\}$.

We may view $T_j : H^2(\mathbb{D}^d, \mathcal{E}, K) \to H^2(\mathbb{D}^d, \mathcal{E}, K)$ as the restriction of the multiplication operator with symbol $z_j$, namely

$$(T_j f)(z) = P_K(z_j f(z)),$$

where $P_K$ is the projection $P_K(\sum g_k z^k) = \sum_{k\in K} g_k z^k$. Likewise, $X$ (as defined in (ii)) may be viewed as the restriction of the multiplication operator with symbol $\phi$, namely

$$(Xf)(z) = P_K(\phi(z)f(z)).$$

Define

$$\Omega = [\Omega_1 \ \cdots \ \Omega_d] : \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_d \to H^2(\mathbb{D}^d, \mathcal{E}, K)$$

by $\Omega(\xi)(z) = P_K(V^*U(I - Z(z)U)^{-1}\xi)$. Since $K$ is finite, this indeed defines a bounded operator. Then (analogous as in Lemma 3.2 in [8]) one easily checks that

$$[T_1\Omega_1 \ \cdots \ T_d\Omega_d]\,(\xi)(z) = P_K(V^*U(I - Z(z)U)^{-1}Z(z)\xi). \tag{2.4}$$

Letting $\Gamma_i = \Omega_i\Omega_i^*$ we get that

$$CE^* + EC^* = \Gamma_1 - T_1\Gamma_1T_1^* + \ldots + \Gamma_d - T_d\Gamma_dT_d^*.$$

For the last equality, let $h \in H^2(\mathbb{D}^d, \mathcal{E}, K)$ and $x \in \mathcal{E}$ (which we may also view as the constant function in $H^2(\mathbb{D}^d, \mathcal{E})$ with value $x$). Then, using (2.1) and (2.4), we get that

$$\langle (X^*h)(0), x\rangle_{\mathcal{E}} = \langle X^*h, x\rangle_{H^2(\mathbb{D}^d,\mathcal{E},K)} = \langle h, Xx\rangle_{H^2(\mathbb{D}^d,\mathcal{E},K)} =$$
$$\langle h, \frac{1}{2}x + P_K(V^*U(I - Z(z)U)^{-1}Z(z)Vx)\rangle_{H^2(\mathbb{D}^d,\mathcal{E},K)}$$
$$\langle \frac{1}{2}h(0), x\rangle_{\mathcal{E}} + \langle V^*(\Omega_1^*T_1^*h \oplus \cdots \oplus \Omega_d^*T_d^*h), x\rangle_{\mathcal{E}},$$

and thus we have that

$$(X^*h)(0) = \frac{1}{2}h(0) + V^*(\Omega_1^*T_1^*h \oplus \cdots \oplus \Omega_d^*T_d^*h). \tag{2.5}$$

Moreover, again using (2.4), we get that

$$\langle h, \Omega\xi\rangle_{H^2(\mathbb{D}^d,\mathcal{E},K)} = \langle h, P_K(V^*U(I - ZU)^{-1}(I - ZU + ZU)\xi)\rangle_{H^2(\mathbb{D}^d,\mathcal{E},K)} =$$
$$\langle h, V^*U\xi\rangle_{H^2(\mathbb{D}^d,\mathcal{E},K)} + \langle h, [T_1\Omega_1 \ \cdots \ T_d\Omega_d]\,U\xi\rangle_{H^2(\mathbb{D}^d,\mathcal{E},K)} =$$
$$\langle U^*Vh(0), \xi\rangle_{\mathcal{E}} + \langle U^*(\Omega_1^*T_1^*h \oplus \cdots \oplus \Omega_d^*T_d^*h), \xi\rangle_{\mathcal{H}_1\oplus\cdots\oplus\mathcal{H}_d}.$$

Thus

$$\Omega_1^*h \oplus \cdots \oplus \Omega_d^*h = U^*Vh(0) + U^*(\Omega_1^*T_1^*h \oplus \cdots \oplus \Omega_d^*T_d^*h). \tag{2.6}$$

Equations (2.5) and (2.6) yield that

$$\begin{pmatrix} U^* & U^*V \\ V^* & \frac{1}{2}I \end{pmatrix} \begin{pmatrix} \Omega_1^*T_1^*h \oplus \cdots \oplus \Omega_d^*T_d^*h \\ h(0) \end{pmatrix} = \begin{pmatrix} \Omega_1^*h \oplus \cdots \oplus \Omega_d^*h \\ (X^*h)(0) \end{pmatrix}. \tag{2.7}$$

Notice that when

$$\begin{pmatrix} U^* & U^*V \\ V^* & \frac{1}{2}I \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

we get that

$$\|a\|^2 - \|x\|^2 = \langle U^*x + U^*Vy, U^*x + U^*Vy\rangle - \|x\|^2 =$$
$$2\text{Re}\ \langle U^*x, U^*Vy\rangle + \langle y, y\rangle = 2\text{Re}\ \langle V^*x + \frac{1}{2}y, y\rangle = 2\text{Re}\ \langle b, y\rangle,$$

where we used that $U$ is unitary and that $V$ is isometric. Thus from (2.7) we obtain that

$$2\text{Re}\ \langle (X^*h)(0), h(0)\rangle = \|\Omega_1^*h\|^2 + \ldots + \|\Omega_d^*h\|^2 - (\|\Omega_1^*S_1^*h\|^2 + \ldots + \|\Omega_d^*S_d^*h\|^2).$$

It is straightforward to check that this is equivalent to the statement that

$$CE^* + EC^* = (I - M_{T_1})(\Gamma_1) + \cdots + (I - M_{T_d})(\Gamma_d).$$

Next, suppose that $\Gamma_i$ exist as (iii). Equation (2.3) implies that

$$2\text{Re}\ \langle (X^*h)(0), h(0)\rangle = \|\Gamma_1^{1/2}h\|^2 - \|\Gamma_1^{1/2}T_1^*h\|^2 + \ldots + \|\Gamma_d^{1/2}h\|^2 - \|\Gamma_d^{1/2}T_d^*h\|^2,$$

for all $h \in H^2(\mathbb{D}^d, \mathcal{E}, K)$. But then

$$\|\frac{1}{2}h(0) + (X^*h)(0)\|^2 + \sum_{i=1}^{d} \|\Gamma_i^{1/2}T_i^*h\|^2 = \|\frac{1}{2}h(0) - (X^*h)(0)\|^2 + \sum_{i=1}^{d} \|\Gamma_i^{1/2}h\|^2.$$

Thus the map

$$\begin{pmatrix} \Gamma_1^{1/2}T_1^*h \oplus \cdots \oplus \Gamma_d^{1/2}T_d^*h \\ \frac{1}{2}h(0) + (X^*h)(0) \end{pmatrix} \to \begin{pmatrix} \Gamma_1^{1/2}h \oplus \cdots \oplus \Gamma_d^{1/2}h \\ \frac{1}{2}h(0) - (X^*h)(0) \end{pmatrix}$$

defines an isometry from $\{\Gamma_1^{1/2}T_1^*h \oplus \cdots \oplus \Gamma_d^{1/2}T_d^*h \oplus (\frac{1}{2}h(0) + (X^*h)(0)) : h \in \mathcal{M}\}$ into $\mathcal{M} \oplus \cdots \mathcal{M} \oplus \mathcal{E}$, where $\mathcal{M} = H^2(\mathbb{D}^d, \mathcal{E}, K)$. Extend the isometry to a unitary operator $\mathcal{U}$ on $\mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_d \oplus \mathcal{E}$. By composing $\Gamma_i^{1/2}$ with the embeddings of $\mathcal{M}$ into $\mathcal{H}_i$, we obtain mappings $\Phi_i : \mathcal{M} \to \mathcal{H}_i$ $(i = 1, \ldots, d)$ such that

$$\mathcal{U}^* : \begin{pmatrix} \Phi_1 T_1^*h \oplus \cdots \oplus \Phi_d T_d^*h \\ \frac{1}{2}h(0) + (X^*h)(0) \end{pmatrix} \to \begin{pmatrix} \Phi_1 h \oplus \cdots \oplus \Phi_d h \\ \frac{1}{2}h(0) - (X^*h)(0) \end{pmatrix}.$$

Decompose

$$\mathcal{U}^* = \begin{pmatrix} \mathcal{U}_{11}^* & \mathcal{U}_{21}^* \\ \mathcal{U}_{12}^* & \mathcal{U}_{22}^* \end{pmatrix} : (\mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_d) \oplus \mathcal{E} \to (\mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_d) \oplus \mathcal{E}.$$

Let $\xi \in \mathcal{E}$, and consider it as a constant function. Then, since $T_i^*\xi = 0$ and $(X^*\xi)(0) = \frac{1}{2}\xi$, we get that $\mathcal{U}_{22}^*(\xi) = 0$. But, since this holds for all $\xi \in \mathcal{E}$ we get that $\mathcal{U}_{22} = 0$. Let now $V = -\mathcal{U}_{12}$ and $U = \mathcal{U}_{11} - \mathcal{U}_{12}\mathcal{U}_{21}$. Then it is not hard to check that $V$ is an isometry, $U$ is unitary, and $V^*U = \mathcal{U}_{21}$. Furthermore,

$$\begin{pmatrix} U^* & U^*V \\ V^* & \frac{1}{2}I \end{pmatrix} \begin{pmatrix} \Phi_1 T_1^*h \oplus \cdots \oplus \Phi_d T_d^*h \\ h(0) \end{pmatrix} = \begin{pmatrix} \Phi_1 h \oplus \cdots \oplus \Phi_d h \\ (X^*h)(0) \end{pmatrix}. \tag{2.8}$$

Define, as before,

$$\Omega = [\Omega_1 \ \cdots \ \Omega_d] : \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_d \to H^2(\mathbb{D}^d, \mathcal{E}, K)$$

by $\Omega(\xi)(z) = P_K(V^*U(I - Z(z)U)^{-1}\xi)$. Then (2.6) holds. Combining (2.6) and (2.8) we get that

$$U^*((\Omega_1^* - \Phi_1)T_1^*h \oplus \cdots \oplus (\Omega_d^* - \Phi_d)T_d^*h) = (\Omega_1^* - \Phi_1)h \oplus \cdots \oplus (\Omega_d^* - \Phi_d)h, h \in \mathcal{M}.$$

Thus

$$\sum_{i=1}^{d}(\Omega_i^* - \Phi_i)^*(\Omega_i^* - \Phi_i) - \sum_{i=1}^{d} T_i(\Omega_i^* - \Phi_i)^*(\Omega_i^* - \Phi_i)T_i^* = 0.$$

Apply now Lemma 2.2 in [8] (note that $T_i^N = 0$ for $N \geq |K|$) to get that $\Omega_i^* = \Phi_i$, $i = 1, \ldots, d$. Consequently, if we let

$$\phi(z) = \frac{1}{2}I + V^*U(I - Z(z)U)^{-1}Z(z)V,$$

then for $\xi \in \mathcal{E}$ and $h \in \mathcal{M}$ we have that

$$\langle h, T_\phi \xi\rangle = \langle h, (\frac{1}{2}I + V^*U(I - Z(z)U)^{-1}Z(z)V)\xi\rangle =$$

$$\langle h, \frac{1}{2}\xi\rangle + \langle V^*(\Phi_1 T_1^* h \oplus \cdots \oplus \Phi_d T_d^* h), \xi\rangle =$$

$$\langle \frac{1}{2}h(0), \xi\rangle + \langle -\frac{1}{2}h(0) + (X^*h)(0), \xi\rangle = \langle (X^*h)(0), \xi\rangle = \langle h, X\xi\rangle.$$

And thus

$$\langle T_\phi^* h, S_1^{n_1} \cdots S_d^{n_d}\xi\rangle = \langle T_\phi^* S_1^{n_1 *} \cdots S_d^{n_d *} h, \xi\rangle =$$
$$\langle X^* T_1^{n_1 *} \cdots T_d^{n_d *} h, \xi\rangle = \langle X^* h, S_1^{n_1} \cdots S_d^{n_d}\xi\rangle.$$

But now it follows that $\phi$ has the required properties. □

The process of extracting an isometry out of the given data as was done in the proof of (iii) → (i) is sometimes referred to as a "lurking isometry" technique (see, e.g., [7]). A possible alternative approach to the above problem is to use the results in [8] directly in combination with a Cayley transform.

Notice that the equivalence of (iii) and (iv) in Theorem 2.1 may be interpreted as a multivariable version of the Naimark dilation theorem on a finite index set. Recall that the Naimark dilation theorem states that a sequence of operators $C_k \in \mathcal{L}(\mathcal{E})$, $k = 1, 2, \ldots$, may be represented as $C_k = V^*U^kV$ with $V : \mathcal{E} \to \mathcal{H}$ an isometry and $U : \mathcal{H} \to \mathcal{H}$ a unitary, if and only if for all $k \in \mathbb{N}_0$ the lower triangular Toeplitz operator matrix $(C_{i-j})_{i,j=0}^k$, where $C_0 = \frac{1}{2}I$ and $C_{-1} = C_{-2} = \ldots = 0$, has a positive semidefinite real part. The same is true for a finite collection of operators $C_k \in \mathcal{L}(\mathcal{E})$, $k = 1, 2, \ldots, n$. Theorem 2.1 now extends the classical result to the case of a finite lower inclusive subset of $\mathbb{N}_0^d$.

By combining the results in [1] and [19] we can, in addition, state the following two-variable generalization of the Naimark dilation theorem.

**Corollary 2.2.** *Consider the doubly indexed sequence of operators $\{C_k\}_{k\in\mathbb{N}_0^2}$ on a Hilbert space $\mathcal{E}$ with $C_0 = I$. Define $C_{-k} = C_k^*$, $k \in \mathbb{N}_0^2$, and $C_k = 0$, $k \in \mathbb{Z}^2 \setminus (\mathbb{N}_0^2 \cup -\mathbb{N}_0^2)$. Then the sequence $\{C_k\}_{k\in\mathbb{N}_0^2}$ is positive definite in the sense that for every finite $K \subset \mathbb{Z}^2$ the operator matrix $(C_{k-l})_{k,l\in K}$ is positive semidefinite, if and only if there exists a Hilbert space $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, an isometry $V : \mathcal{E} \to \mathcal{H}$ and a unitary $U \in \mathcal{L}(\mathcal{H})$ so that*

$$C_k = V^*(\sum_{w\in W_k(UP_1,UP_2)} w)V, \; k \in K \setminus \{0\}, \tag{2.9}$$

*where $P_i$ is the orthogonal projection in $\mathcal{L}(\mathcal{H})$ with image $\mathcal{H}_i$, $i = 1,2$.*

**Proof.** By Corollary 1 in [19] the sequence $\frac{1}{2}I$, $\{C_k\}_{k\in\mathbb{N}_0^2\setminus\{0\}}$ are the Taylor coefficients of a Carathéodory function $\phi$ if and only if $\{C_k\}_{k\in\mathbb{N}_0^2}$ is a positive definite sequence. By [1] the function $\phi$ belongs to the Carathéodory class if and only if it may be represented as in (2.1), yielding the description (2.9). □

# 3 Carathéodory Interpolation for holomorphic functions

In this section we explore the Carathéodory interpolation problem in the class $\mathcal{M}_d(\mathcal{E})$ of holomorphic $\mathcal{L}(\mathcal{E})$-valued functions $\phi$ defined on $\mathbb{D}^d$ with positive real part, i.e., $\operatorname{Re}\phi(z) \geq 0$ for all $z \in \mathbb{D}^d$. We shall confine ourselves to the situation where $K^{(d)}(n)$ is the subset of $\mathbb{N}_0^d$ consisting of points $x = (x_1, \ldots, x_d)$ satisfying $|x| := \sum_{i=1}^d x_i \leq n$.

We have the following necessary condition. Denote

$$K_j^{(d)} = \{(x_1, \ldots, x_d) \in \mathbb{N}_0^d \;:\; x_1 + \ldots + x_d = j\}.$$

Further, for $j = 1, \ldots, d$ let $S_j : H^2(\mathbb{D}^d, \mathbb{C}) \to H^2(\mathbb{D}^d, \mathbb{C})$ be the multiplication operator with symbol $z_j$, namely

$$(S_j f)(z) = z_j f(z).$$

For $k = (k_1, \ldots, k_d)$ we let $S^k$ denote the operator $S_1^{k_1} \cdots S_d^{k_d}$ on $H^2(\mathbb{D}^d, \mathbb{C})$.

**Theorem 3.1.** *Let $n \in \mathbb{N}$ and $K^{(d)}(n) = \cup_{j=0}^n K_j^{(d)}$. Given are operators $C_k \in \mathcal{L}(\mathcal{E})$, $k \in K^{(d)}(n)$, with $C_0 = \frac{1}{2}I$. If there exists $f(z) = \sum_{k\in\mathbb{N}_0^d} f_k z^k \in \mathcal{M}_d(\mathcal{E})$ with $f_k = C_k$, $k \in K$, then the operator*

$$\Gamma := (\Gamma_{p-q})_{p,q=0}^n \tag{3.1}$$

*has positive semidefinite real part. Here $\Gamma_p = \sum_{k\in K_p^{(d)}} C_k \otimes S^k$. When $n = 1$ the converse is also valid.*

**Proof of Theorem 3.1.** First suppose that $f \in \mathcal{M}_d(\mathcal{E})$ exists with $f_k = C_k$, $k \in K^{(d)}(n)$. We can find unitary liftings $U_i : \mathcal{K} \to \mathcal{K}$ of $S_i$ so that $U_1, \ldots, U_d$ commute. Let now $g(w) = \sum_{j=0}^\infty w^j (\sum_{k\in K_j^{(d)}} f_k \otimes U^k)$, $w \in \mathbb{D}$, where $U^k = U_1^{k_1} \cdots U_d^{k_d}$. Then $g \in \mathcal{M}_1(\mathcal{E} \otimes \mathcal{K})$, and thus by the classical one-variable result we get that $\Gamma + \Gamma^* \geq 0$.

Let now $n = 1$ and suppose that $\Gamma + \Gamma^* \geq 0$. We let $e_1, \dots, e_d$ denote the standard basis in $\mathbb{N}_0^d$. Since $\Gamma + \Gamma^* \geq 0$ it follows that $\| \sum_{i=1}^d z_i C_{e_i} \| < 1$ for all $z = (z_1, \dots, z_d) \in \mathbb{D}^d$. But then

$$f(z) := \frac{1}{2} I + (I - \sum_{i=1}^d z_i C_{e_i})^{-1} \sum_{i=1}^d z_i C_{e_i}, z \in \mathbb{D}^d,$$

has the required properties. □

We observe that even in the case $n = 1$ the condition on $C_k$, $k \in K^{(d)}(n)$, in Theorem 3.1 is much weaker than the condition in Theorem 2.1. In fact, the following data provides an example for which the Carathéodory interpolation problem is solvable in $\mathcal{M}_d(\mathcal{E})$ but not in $\mathcal{A}_d(\mathcal{E})$.

**Example.** Let $\mathcal{H}$ be a Hilbert space of dimension more than one, and $U_1$ and $U_2$ be non-commuting unitary operators on $\mathcal{H}$. Put $\mathcal{E} = \mathcal{H} \oplus \mathcal{H}$ and let $T_i$, $i = 1, 2, 3$, be the pairwise commuting contractions that were introduced in [21], i.e.,

$$T_1 = \begin{pmatrix} 0 & 0 \\ U_1 & 0 \end{pmatrix}, T_2 = \begin{pmatrix} 0 & 0 \\ U_2 & 0 \end{pmatrix}, T_1 = \begin{pmatrix} 0 & 0 \\ I_{\mathcal{H}} & 0 \end{pmatrix},$$

where $I_{\mathcal{H}}$ is the identity operator on $\mathcal{H}$. By the main result in [18], there exist an $n \in \mathbb{N}$ and $n \times n$ matrices $A_1, A_2$ and $A_3$ so that

$$\|A_1 \otimes T_1 + A_2 \otimes T_2 + A_3 \otimes T_3\| > 1 \geq \max_{z_1, z_2, z_3 \in \mathbb{T}} \|z_1 A_1 + z_2 A_2 + z_3 A_3\|. \tag{3.2}$$

Let now $C_{(0,0,0)} = \frac{1}{2} I$, $C_{(1,0,0)} = A_1$, $C_{(0,1,0)} = A_2$, and $C_{(0,0,1)} = A_3$. The right hand inequality in (3.2) implies that the condition in Theorem 3.1 is satisfied. Thus the Carathéodory interpolation problem with data $C_k$, $k \in K^{(3)}(1)$, has a solution in $\mathcal{M}_3(\mathcal{E})$. The left hand inequality in (3.2) implies, however, that the Carathéodory interpolation problem with data $C_k$, $k \in K^{(3)}(1)$, does not have a solution in $\mathcal{A}_3(\mathcal{E})$. Indeed, if a solution $\phi \in \mathcal{A}_3(\mathcal{E})$ exists, then the single variable function

$$f(z) := \phi(zT_1, zT_2, zT_3)$$

should have the property that Re $f(z) \geq 0$ for $z \in \mathbb{D}$. In particular,

$$\frac{df}{dz}(0) = A_1 \otimes T_1 + A_2 \otimes T_2 + A_3 \otimes T_3$$

should be a contraction, which contradicts (3.2).

Though the Carathéodory interpolation problems in $\mathcal{M}_d(\mathcal{E})$ and $\mathcal{A}_d(\mathcal{E})$ are in general different, there are also cases when the Carathéodory interpolation problem in both classes are only solvable simultaneously. This happens for instance when $\mathcal{E} = \mathbb{C}$ and $n = 1$.

**Proposition 3.2.** *Given are complex numbers $c_k$, $k \in K^{(d)}(1)$, with $c_0 = \frac{1}{2}$. The following are equivalent.*

*(i) There exist* $f(z) = \sum_{k\in\mathbb{N}_0^d} f_k z^k \in \mathcal{A}_d(\mathbb{C})$ *with* $f_k = c_k$, $k \in K^{(d)}(1)$.

*(ii) There exist* $f(z) = \sum_{k\in\mathbb{N}_0^d} f_k z^k \in \mathcal{M}_d(\mathbb{C})$ *with* $f_k = c_k$, $k \in K^{(d)}(1)$.

*(iii)* $\sum_{i=1}^d |c_{e_i}| \leq 1$, *where* $e_1, \ldots, e_d$ *is the standard basis in* $\mathbb{N}_0^d$.

**Proof.** We will prove (i) $\to$ (ii) $\to$ (iii) $\to$ (i).

Since $\mathcal{A}_d(\mathbb{C}) \subset \mathcal{M}_d(\mathbb{C})$, the implication (i) $\to$ (ii) follows.

Suppose that (ii) holds. Choose complex numbers $\alpha_i$, $i = 1, \ldots, d$, of modulus 1 so that $\alpha_i c_{e_i} \geq 0$, and let $g(w) = f(\alpha_1 w, \ldots, \alpha_d w)$, $w \in \mathbb{D}$. Then $\mathrm{Re}\, g(w) \geq 0$ for $w \in \mathbb{D}$, and by the one-variable results we get that $|g'(0)| \leq 1$. But this yields $\sum_{i=1}^d |c_{e_i}| \leq 1$.

Suppose that (iii) holds. Consider the functions $g_1(z_1) = \frac{1}{2}|c_{e_1}| + c_{e_1} z_1, \ldots,$ $g_{d-1}(z_{d-1}) = \frac{1}{2}|c_{e_{d-1}}| + c_{e_{d_1}} z_{d-1}$, $g_d(z_d) = \frac{1}{2}(1 - |c_{e_1}| - \ldots - |c_{e_{d-1}}|) + c_{e_d} z_d$. By the one-variable result, there exist for $i = 1, \ldots, d$ functions $f_i \in \mathcal{A}_1(\mathbb{C})$ so that $f_i(0) = g_i(0)$ and $f_i'(0) = g_i'(0)$. But then $f(z) = f_1(z_1) + \ldots + f_d(z_d)$ is a function satisfying (i). □

Notice that the proof of (iii) $\to$ (i) in Proposition 3.2 goes through for operators $C_{e_i}$. That is, if $\sum_{i=1}^d \|C_{e_i}\| \leq 1$, then there exists a solution to the Carathéodory interpolation problem in $\mathcal{A}_d(\mathcal{E})$ with given data $C_0 = \frac{1}{2}I, C_{e_1}, \ldots, C_{e_d}$. Clearly, the condition $\sum_{i=1}^d \|C_{e_i}\| \leq 1$ is not necessary in general as one can start with a Hilbert space $\mathcal{H}$ of dimension greater than or equal to $d \geq 2$, a unitary $U : \mathcal{H} \to \mathcal{H}$, and nontrivial orthogonal projections $P_1, \ldots, P_d$ on $\mathcal{H}$ with $\oplus_{i=1}^d \mathrm{Ran}\, P_i = \mathcal{H}$, and put $C_{e_i} = UP_i$. Then $\frac{1}{2}I + (I - \sum_{i=1}^d z_i C_{e_i})^{-1} \sum_{i=1}^d z_i C_{e_i} \in \mathcal{A}_d(\mathcal{H})$ but $\sum_{i=1}^d \|C_{e_i}\| = d > 1$.

There are many questions that remain. For instance, are in the scalar case the finite data Carathéodory interpolation problems in $\mathcal{M}_d(\mathbb{C})$ and $\mathcal{A}_d(\mathbb{C})$ different? In other words, does there exist a finite lower-inclusive set $K$ and data $c_k \in \mathbb{C}$, $k \in K$, for which the Carathéodory interpolation problem is solvable in $\mathcal{M}_d(\mathbb{C})$ but not in $\mathcal{A}_d(\mathbb{C})$? Another open problem is the question whether the condition $\Gamma + \Gamma^* \geq 0$ in Theorem 3.1 is also sufficient for the existence of a solution in $\mathcal{M}_d(\mathcal{E})$? By Theorem 10 in [6] this condition $\Gamma + \Gamma^* \geq 0$ is necessary and sufficient for a positive measure $\sigma$ to exist with moments $\widehat{\sigma}(0) = I$, $\widehat{\sigma}(k) = C_k$, $k \in K^{(d)}(n) \setminus \{0\}$, $\widehat{\sigma}(k) = C_k^*$, $k \in -K^{(d)}(n) \setminus \{0\}$, and $\widehat{\sigma}(k) = 0$, $k \in \{k \in \mathbb{Z}^d \setminus \mathbb{N}_0^d \,:\, -n \leq k_1 + \ldots + k_d \leq n\}$. In order to get a solution to the Carathéodory interpolation problem we need by Theorem 1 in [19] that $\widehat{\sigma}(0) = I$, $\widehat{\sigma}(k) = C_k$, $k \in K^{(d)}(n) \setminus \{0\}$, $\widehat{\sigma}(k) = C_k^*$, $k \in -K^{(d)}(n) \setminus \{0\}$, and $\widehat{\sigma}(k) = 0$, $k \in \mathbb{Z}^d \setminus \mathbb{N}_0^d$. It should be noted that Theorem 10 in [6] may also be used for index sets other than $K^{(d)}(n)$, namely any set $0 \in K \subset \mathbb{N}_0^d$ that lies on one side of a hyperplane.

# 4 Numerical Results

The condition in Theorem 2.1(iii) may be checked numerically by semidefinite programming. Using Matlab's LMIlab we performed a few experiments. We briefly describe three of them.

Let $K = \{0,1,2\}^2$ and $c_{00} = 1/2, c_{01} = 0, c_{02} = 0, c_{10} = 1/2\sqrt{2}, c_{11} = 1/2, c_{12} = -1/4\sqrt{2}, c_{20} = 1/2, c_{21} = 1/2\sqrt{2}$ and $c_{22} = -1/4$. In order to build the matrices, we order

$K$ using the lexicographical order. Using LMIlab we find the matrices

$$\begin{pmatrix} .7437 & .1813 & -.1282 & .5259 & .5000 & -.2629 & .3718 & .6165 & -.1218 \\ .1813 & .1282 & -.0906 & .1282 & .1813 & -.0641 & .0906 & .1922 & .0000 \\ -.1282 & -.0906 & .0641 & -.0906 & -.1282 & .0453 & -.0641 & -.1359 & -.0000 \\ .5259 & .1282 & -.0906 & .3718 & .3536 & -.1859 & .2629 & .4359 & -.0861 \\ .5000 & .1813 & -.1282 & .3536 & .3782 & -.1768 & .2500 & .4442 & -.0609 \\ -.2629 & -.0641 & .0453 & -.1859 & -.1768 & .0930 & -.1315 & -.2180 & .0431 \\ .3718 & .0906 & -.0641 & .2629 & .2500 & -.1315 & .1859 & .3082 & -.0609 \\ .6165 & .1922 & -.1359 & .4359 & .4442 & -.2180 & .3082 & .5320 & -.0861 \\ -.1218 & .0000 & -.0000 & -.0861 & -.0609 & .0431 & -.0609 & -.0861 & .0305 \end{pmatrix}$$

and

$$\begin{pmatrix} .2563 & -.1813 & .1282 & .1813 & -.0000 & -.0906 & .1282 & .0906 & -.1282 \\ -.1813 & .1282 & -.0906 & -.1282 & .0000 & .0641 & -.0906 & -.0641 & .0906 \\ .1282 & -.0906 & .0641 & .0906 & -.0000 & -.0453 & .0641 & .0453 & -.0641 \\ .1813 & -.1282 & .0906 & .3718 & -.1723 & .0578 & .2629 & .0641 & -.1768 \\ -.0000 & .0000 & -.0000 & -.1723 & .1218 & -.0861 & -.1218 & -.0000 & .0609 \\ -.0906 & .0641 & -.0453 & .0578 & -.0861 & .0930 & .0408 & -.0320 & .0022 \\ .1282 & -.0906 & .0641 & .2629 & -.1218 & .0408 & .1859 & .0453 & -.1250 \\ .0906 & -.0641 & .0453 & .0641 & -.0000 & -.0320 & .0453 & .0320 & -.0453 \\ -.1282 & .0906 & -.0641 & -.1768 & .0609 & .0022 & -.1250 & -.0453 & .0945 \end{pmatrix}$$

for $\Gamma_1$ and $\Gamma_2$, respectively. The corresponding $U$ and $V$ are

$$V = \begin{pmatrix} 0.8543 \\ -0.1178 \\ 0.4141 \\ -0.2913 \end{pmatrix}, U = \begin{pmatrix} 0.7061 & 0.0036 & 0.6952 & 0.1341 \\ -0.0012 & 0.7054 & 0.1318 & -0.6964 \\ 0.6947 & 0.1346 & -0.7066 & 0.0014 \\ 0.1368 & -0.6959 & 0.0006 & -0.7050 \end{pmatrix}.$$

The projections $P_1$ and $P_2$ are the projections onto $\text{span}\{e_1, e_2\}$ and $\text{span}\{e_3, e_4\}$. respectively, where $e_1, \dots, e_4$ is the standard basis in $\mathbb{C}^4$. The example was constructed using

$$V = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, U = \begin{pmatrix} 1/2\sqrt{2} & 1/2\sqrt{2} \\ 1/2\sqrt{2} & -1/2\sqrt{2} \end{pmatrix}, P_1 = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}, P_2 = \begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix}.$$

In an attempt to create a scalar valued example for which the Carathéodory interpolation has a solution in $\mathcal{M}_3(\mathbb{C})$ but not in $\mathcal{A}_3(\mathbb{C})$ we tried the following based on Varapoloulos' example [25]. Let $f(z) = \sum_{k \in \mathbb{N}_0^3} f_k z^k := \frac{1-\psi}{2+2\psi}$, where

$$\psi(z) = \frac{1}{5}(z_1^2 + z_2^2 + z_3^2) - \frac{2}{5}(z_1 z_2 + z_2 z_3 + z_1 z_3),$$

and take the given data $\{f_k, k \in K_3^{(3)}\}$. Based on our algorithm, though, we found that the Carathéodory interpolation problem is solvable in $\mathcal{A}_3(\mathbb{C})$. Clearly $f$ was not the solution as $f \notin \mathcal{A}_3(\mathbb{C})$.

Lastly, when we perform the algorithm on the data in Example 2 of the introduction, we obtain after rounding the following positive semidefinite matrices:

$$\Gamma_1 = \begin{pmatrix} 0.3000 & 0.4000 & -0.1000 & 0.0952 \\ 0.4000 & 0.5523 & -0.1523 & 0.1125 \\ -0.1000 & -0.1523 & 0.0523 & -0.0173 \\ 0.0952 & 0.1125 & -0.0173 & 0.0603 \end{pmatrix},$$

and

$$\Gamma_2 = \begin{pmatrix} 0.7000 & 0.3000 & 0.4000 & 0.7048 \\ 0.3000 & 0.1477 & 0.1523 & 0.2875 \\ 0.4000 & 0.1523 & 0.2477 & 0.4173 \\ 0.7048 & 0.2875 & 0.4173 & 0.7397 \end{pmatrix}.$$

In order to see that the matrices are positive semidefinite one may note that $(-1,1,1,0)^T$ belongs to the kernel of both. Furthermore, after omitting the third column and row in both matrices the determinants of the leading principal submatrices in exact arithmetic are

$$\frac{3}{10}, \frac{569}{100000}, \frac{106167}{976562500}, \frac{7}{10}, \frac{1339}{100000}, \frac{993207}{3906250000},$$

yielding the positive semidefiniteness of both $\Gamma_1$ and $\Gamma_2$. Next, one may easily check that

$$EC^* + CE^* - \Gamma_1 + T_1\Gamma_1 T_1^* - \Gamma_2 + T_2\Gamma_2 T_2^* = 0$$

where $C, E, T_1$ and $T_2$ are as in Theorem 2.1. Thus the Carathéodory interpolation problem is solvable for this data set. By adding a small enough $\epsilon > 0$ to $c_{00}$ one may even construct an example for which the autoregressive filter problem is not solvable but a solution $\phi$ to the Carathéodory interpolation problem exists with $\inf_{z \in \mathbb{D}^2} \operatorname{Re} \phi(z) > 0$.

**Acknowledgements**

*The author is partially supported by NSF Grants DMS 9800704 and DMS 9988579. In addition, the author was supported by a Research Summer Grant from the College of William and Mary. The author gratefully acknowledges the hospitality of the Department of Mathematics and Statistics at the University of New Mexico, where this research was carried out.*

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2000 Mathematics Subject Classification. Primary 47A57, 30E05; Secondary 42B05

Submitted: October 12, 2000
Revised: November 12, 2000