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Bayesian Kriging—Merging Observations and Qualified Guesses in Kriging[1]

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Frequently a user wants to merge general knowledge of the regionalized variable under study with available observations. Introduction of fake observations is the usual way of doing this. Bayesian kriging allows the user to specify a qualified guess, associated with uncertainty, for the expected surface. The method will provide predictions which are based on both observations and this qualified guess.

KEY WORDS: geostatistics, kriging, Bayesian statistics.

1. INTRODUCTION

In evaluation of spatial phenomena, both interpolation between, and extrapolation beyond, a set of observations, frequently take place. Many automatic techniques are developed for this purpose. Few of these techniques are able to provide acceptable predictions in areas which are sparsely sampled. Extrapolations, in particular, seem to cause problems for the procedures. Also in kriging theory, extrapolations create problems. Journel and Huijbregts (1978) and Omre (1983) are references for kriging theory. In kriging, predictions are based on a geostatistical model. The model includes assumptions about shape of the drift and variogram function. Influence of the former will increase with increasing distance between location of observations and location for prediction. In implementations of kriging theory presently available, predictions tend toward planes or second-order surfaces in areas far from observations. Users often find predictions in such areas unacceptable.

In many applications extrapolations are required. Traditionally the user introduces subjectivity through fake observations in areas which are sampled sparsely. To some extent, this will prevent wild predictions. Few techniques, however, are able to treat real observations and qualified guesses differently, as

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they should. Kriging theory allows for varying precision in observations and this provides a possibility for distinguishing the two sources of information. In Journel (1983), Kulkarni (1984), and Kostov and Journel (1985), handling of qualified guesses is treated in an indicator kriging setting. In this case, fake data can be specified through a probability distribution.

Introduction of fake observations to control the predicted surface seems improper for several reasons:

(1) Normally, the user has expectations about the general shape of the surface. To represent these expectations as a number of separate fake observations is not natural. To specify a surface would be more natural.

(2) By specifying fake observation in certain locations, these particular locations are given a different status than others. This may be observed in estimation variances of kriging. Expectations of the user are usually associated with all locations.

3 Usually, a qualitative difference exists in information carried by real observations and by the user. The former provides exact values of the regionalized variable whereas the latter normally reflects general expectations of the phenomenon under study. This difference is not distinguished when fake observations are introduced.

The fact that the user normally has expectations about the general behavior of the phenomenon suggests that user-specified information should be a part of the model. This will not be the case if fake observations are used. In Omre and Holden (1984), the model approach is chosen and a solution within universal kriging theory is found. The universal kriging theory, however, allows for no uncertainty in specifications of the model. Hence the user has to specify expectations without uncertainty. In this paper, kriging theory is extended to allow for uncertainty in specification of the model. The user may specify any expected surface with uncertainty. This qualified guess is included in the geostatistical model. Predictions are based on observations and this model.

The method also may be seen as an extension of a linear Bayesian theory to spatial problems (see Hartigan, 1969). A prior distribution of the expected surface is specified up to second order. This prior is combined with available observations, not to obtain a posterior estimate of the expected surface, but to make a posterior prediction in an arbitrary location. Familiar Bayesian relations are applied in the procedure. Berger (1980) is a useful reference to general Bayesian theory.

Because the method consists of a mixture of the kriging theory and the Bayesian approach, it is named Bayesian kriging. (The author is aware that Kulkarni, 1984, denoted his procedure Bayesian kriging. His procedure is qualitatively different from the one suggested here. However, the term soft kriging has recently been adopted for the technique used by Kulkarni; see Kostov and Journel, 1985.)

2. BASIC NOTATION AND RELATIONS

Let the regionalized variable under study be denoted $\{z(x); x \in A\}$, and let the underlying random function be

$$\{Z(x); x \in A\} \tag{1}$$

Characteristics of $\{Z(x); x \in A\}$ will not be known a priori.

Consider another regionalized variable $\{m(x); x \in A\}$, whose underlying random function is

$$\{M(x); x \in A\} \tag{2}$$

Assume its first two moments as known a priori

$$E[M(x')] = \mu_M(x') \qquad x' \in A$$

$$\text{Cov}\ [M(x'), M(x'')] = C_M(x', x'') \qquad x', x'' \in A \tag{3}$$

Note that the covariance may be dependent on both x' and x'', not only on their relative location.

This implies that the corresponding variogram function is defined as

$$\gamma_M(x', x'') = \tfrac{1}{2}\ \text{Var}\ [M(x') - M(x'')]$$

$$= [C_M(x', x') + C_M(x'', x'')]/2 - C_M(x', x'') \qquad x', x'' \in A \tag{4}$$

Let the following be true

$$E[Z(x') | M(x); x \in A] = a_0 + M(x') \qquad x' \in A$$

$$\text{Cov}\ [Z(x'), Z(x'') | M(x); x \in A] = C_{Z|M}(x' - x'') \qquad x', x'' \in A \tag{5}$$

with a_0 an unknown constant.

Then $\{M(x); x \in A\}$ may be interpreted as a guess, associated with uncertainty, for the shape of the expected function of the variable under study $\{Z(x); x \in A\}$. The constant a_0 is introduced to make the guess less sensitive to the actual level specified. The shape of the guess is primarily of interest.

The latter entails that $\{Z(x); x \in A\}$ is second-order stationary around its expected function $\{a_0 + m(x); x \in A\}$ and is an assumption familiar to geostatisticians. It also entails that

$$\text{Var}\ [Z(x') - Z(x'') | M(x); x \in A] = 2\gamma_{Z|M}(x' - x'') \qquad x', x'' \in A \tag{6}$$

Expressions for conditional expectation and conditional covariance are known from linear, Bayesian theory (see Hartigan, 1969)

$$E[Y_1] = E[E[Y_1 | Y_3]]$$

$$\text{Cov}\ [Y_1, Y_2] = E[\text{Cov}\ [Y_1, Y_2 | Y_3]] + \text{Cov}\ [E[Y_1 | Y_3], E[Y_2 | Y_3]]$$

with Y_1, Y_2, and Y_3 arbitrary random variables.

From this and expression (5)

$$\mu_Z(x') = E[Z(x')] = E[E[Z(x')|M(x); x \in A]] = a_0 + \mu_M(x'); x' \in A$$

$$\begin{aligned}
C_{ZM}(x', x'') &= \text{Cov } [Z(x'), M(x'')] \\
&= E[\text{Cov } [Z(x'), M(x'')|M(x); x \in A]] \\
&\quad + \text{Cov } [E[Z(x')|M(x); x \in A], E[M(x'')|M(x); x \in A]] \\
&= \text{Cov } [M(x'), M(x'')] \\
&= C_M(x', x'') \qquad x', x'' \in A \\
C_Z(x', x'') &= \text{Cov } [Z(x'), Z(x'')] \\
&= E[\text{Cov } [Z(x'), Z(x'')|M(x); x \in A]] \\
&\quad + \text{Cov } [E[Z(x')|M(x); x \in A], E[Z(x'')|M(x); x \in A]] \\
&= C_{Z|M}(x' - x'') + C_M(x', x'') \qquad x', x'' \in A \\
\gamma_Z(x', x'') &= \tfrac{1}{2} \text{ Var } [Z(x') - Z(x'')] \\
&= \tfrac{1}{2} (E[\text{Var } [Z(x') - Z(x'')|M(x); x \in A]] \\
&\quad + \text{Var } [E[Z(x') - Z(x'')|M(x); x \in A]]) \\
&= \gamma_{Z|M}(x' - x'') + \gamma_M(x', x'')
\end{aligned} \tag{7}$$

Expression (5) is a sufficient condition for expressions (7) to hold; a necessary and sufficient set of conditions would be

$$E[Z(x')|M(x'), M(x'')] = a_0 + M(x') \qquad x', x'' \in A$$

$$\text{Cov } [Z(x'), Z(x'')|M(x'), M(x'')] = C_{Z|M}(x' - x'') \qquad x', x'' \in A$$

The model above is specified up to second order and can be used in a linear Bayesian framework. The discussion will be exact if $\{Z(x); x \in A\}$ and $\{M(x); x \in A\}$ jointly are Gaussian random functions.

Recall that the two first moments of the random function $\{M(x); x \in A\}$ are considered known and that it can be interpreted as a qualified guess of the expected function of $\{Z(x); x \in A\}$. The random function of interest $\{Z(x); x \in A\}$ is unknown. Assume, however, that the latter can be observed in locations $x_i \in A$; $i = 1, \ldots, N$.

3. BAYESIAN KRIGING

Assume that the following set of observations from the random function of interest, $\{Z(x); x \in A\}$, is available

$$\{Z(x_i); i = 1, \ldots, N\}$$

Because the expected function of $\{Z(x); x \in A\}$ is known up to a constant, see expression (7), one may define the random function

$$\{Z^T(x) = Z(x) - \mu_M(x); x \in A\}$$

and the set of observations

$$\{Z^T(x_i) = Z(x_i) - \mu_M(x_i); i = 1, \ldots, N\}$$

Consider a linear estimator for $Z(x_0)$ with x_0 an arbitrary location within A

$$Z^*(x_0) = \sum_{i=1}^{N} \alpha_i Z^T(x_i) + \mu_M(x_0)$$

with $\{\alpha_i; i = 1, \ldots, N\}$ a set of constant weights to be determined.

Unbiasedness in the estimator requires

$$E[Z(x_0) - Z^*(x_0)] = 0$$

$$\Leftrightarrow$$

$$E[Z(x_0)] = E[Z^*(x_0)]$$

$$\Leftrightarrow$$

$$E[Z(x_0)] = \sum_{i=1}^{N} \alpha_i E[Z^T(x_i)] + \mu_M(x_0)$$

From (7) one obtains

$$E[Z(x_0) - Z^*(x_0)] = 0$$

$$\Leftrightarrow$$

$$a_0 = a_0 \sum_i \alpha_i$$

A necessary and sufficient condition on the set of weights for ensuring unbiasedness is

$$\sum_i \alpha_i = 1 \tag{8}$$

Estimation variance for the estimator is

$$\begin{aligned} \text{Var}\,[Z(x_0) - Z^*(x_0)] &= \text{Var}\,[Z(x_0)] - 2 \sum_i \alpha_i \,\text{Cov}\,[Z(x_0), Z^T(x_i)] \\ &\quad + \sum_i \sum_j \alpha_i \alpha_j \,\text{Cov}\,[Z^T(x_i), Z^T(x_j)] \\ &= \text{Var}\,[Z(x_0)] - 2 \sum_i \alpha_i \,\text{Cov}\,[Z(x_0), Z(x_i)] \\ &\quad + \sum_i \sum_j \alpha_i \alpha_j \,\text{Cov}\,[Z(x_i), Z(x_j)] \end{aligned} \tag{9}$$

because covariance is invariant to shifts in the variables.

By using expressions (4), (6), (7), and (9)

$$\text{Var}\,[Z(x_0) - Z^*(x_0)] = 2 \sum_i \alpha_i [\gamma_{Z|M}(x_0 - x_i) + \gamma_M(x_0, x_i)] - \sum_i \sum_j \alpha_i \alpha_j [\gamma_{Z|M}(x_i - x_j) + \gamma_M(x_i, x_j)] \quad (10)$$

In this expression, $\gamma_M(\cdot\,,\cdot)$ is known a priori. The $\gamma_{Z|M}(\cdot)$ is unknown, however, and has to be estimated. A procedure for estimating this variogram function will be discussed in Section 4. Let's proceed as if $\gamma_{Z|M}(\cdot)$ is known.

Note that estimation variance is ensured to be nonnegative if both $\gamma_{Z|M}(\cdot)$ and $\gamma_M(\cdot\,,\cdot)$ are conditionally positive-definite functions. This is caused by the additivity properties of this kind of function.

Consequently, $\gamma_{Z|M}(\cdot)$ may be chosen from the class of variogram functions usually applied in geostatistics (see Journel and Huijbregts, 1978).

The other variogram function $\gamma_M(x', x'')$ is dependent on locations x' and x'', not only their relative location. Hence it should be chosen from a larger family of functions. In Lemma AI (Appendix I), $\gamma_M(\cdot\,,\cdot)$ is shown to be conditional positive-definite if it is of the form

$$\gamma_M(x', x'') = \frac{[(\sigma_M(x') - \sigma_M(x'')]^2}{2} + \sigma_M(x')\,\sigma_M(x'')\,\gamma_S(x' - x'') \qquad x', x'' \in A$$

with $\sigma_M(x)$ a nonzero function defined over area A and $[1 - \gamma_S(x' - x'')]$ a positive-definite function.

From expression (4) for second-order stationary random functions, one gets

$$C_M(x', x'') = \sigma_M(x')\,\sigma_M(x'')\,C_S(x' - x'') \qquad x', x'' \in A$$

$$\text{with } \gamma_S(x' - x'') = 1 - C_S(x' - x'') \qquad x', x'' \in A$$

Hence, $C_S(\cdot)$ can be interpreted as the spatial correlation function, and $\sigma_M^2(x) = \text{Var}\,[M(x)]$, which may vary over the area A. Consequently, a fairly general class of conditional positive-definite functions is defined.

Recall that the second-order moments of $\{M(x); x \in A\}$ are assumed known a priori. Interpretation of $\sigma_M^2(x)$ as the variance of the qualified guess in location x simplifies the specification. In addition, a location-invariant spatial correlation $C_S(\cdot)$, valid everywhere within A, has to be given.

The set of weights $\{\alpha_i; i = 1, \ldots, N\}$ will be determined by minimizing variance under the unbiasedness constraint

$$\min_{\alpha_i;\, i=1,\ldots,N} \{\text{Var}\,[Z(x_0) - Z^*(x_0)]\}, \qquad \sum_i \alpha_i = 1 \quad (11)$$

By applying the Lagrange minimizing procedure, one finds the following Bayesian kriging system

$$\sum_i \alpha_i[\gamma_{Z|M}(x_i - x_j) + \gamma_M(x_i, x_j)] + \beta_1 = \gamma_{Z|M}(x_0 - x_j) + \gamma_M(x_0, x_j)$$

$$j = 1, \ldots, N$$

$$\sum_i \alpha_i = 1 \qquad (12)$$

with β_1 being a Lagrange multiplier.

Note that the varying precision in the qualified guess will have influence on the weights, and hence on the estimate obtained.

Special Cases

Some special cases of the Bayesian kriging system will be considered.

Case A:

Consider the case where location x_0 coincides with the location of one of the observations, let's say x_k; $1 \leq k \leq N$.

From expression (4) and (6), this entails

$$\gamma_{Z|M}(x_0 - x_k) = 0$$

$$\gamma_M(x_0, x_k) = 0$$

By choosing $\alpha_k = 1$ and $\alpha_i = 0$, $i = 1, \ldots, N$, $i \neq k$, one gets from expression (10)

$$\text{Var}\,[Z(x_0) - Z^*(x_0)] = 0$$

and the constraint in expression (8) will be fulfilled.

This implies

$$Z^*(x_k) = Z^T(x_k) + \mu_M(x_k) = Z(x_k)$$

Consequently, the Bayesian kriging technique is an exact interpolation technique. The predicted surface runs through observations regardless of the actual qualified guess and the surface has continuity according to the estimated variogram functions; hence observations have large influence in areas with dense sampling.

Case B:

Consider the case where location x_0 is farther from any observation than the range of both $\gamma_{Z|M}(\cdot)$ and $\gamma_S(\cdot)$. This implies

$$\gamma_{Z|M}(x_0 - x_i) = C \qquad i = 1, \ldots, N$$

$$\gamma_S(x_0 - x_i) = 1 \qquad i = 1, \ldots, N$$

$$\gamma_M(x_0, x_i) = [\sigma_M^2(x_0) + \sigma_M^2(x_i)]/2 \qquad i = 1, \ldots, N$$

with C a constant identical to Var $[Z(x) | M(x); x \in A]$.

The Bayesian kriging system, see expression (12), reduces to

$$\sum_i \alpha_i[\gamma_{Z|M}(x_i - x_j) + \gamma_M(x_i, x_j)] + \beta_1 = C + \frac{\sigma_M^2(x_0)}{2} + \frac{\sigma_M^2(x_i)}{2}$$

$$j = 1, \ldots, N$$

$$\sum_i \alpha_i = 1$$

By setting $\beta_1 = C + [\sigma_M^2(x_0)/2]$, the weights will be independent of location x_0.

Consequently the estimator

$$Z^*(x_0) = \sum_i \alpha_i Z^T(x_i) + \mu_M(x_0)$$

will be dependent on x_0 only through the expected function $\mu_M(\cdot)$. Hence, the difference in the estimates in these regions will be caused by the qualified guess only. Note that the estimation variance will vary with the precision in the guess through β_1. This shows that in areas far from observations, the qualified guess will dominate.

Case C:

Assume that the a priori qualified guess of the expected function of $\{Z(x); x \in A\}$ has second-order moments of the form

$$\text{Var}\,[M(x') - M(x'')] = 2\gamma_M(x', x'') = 2C\,\gamma_S(x' - x'') \quad (13)$$

This implies that the precision in the a priori guess is constant throughout the area A.

Then, from expression (10)

$$\text{Var}\,[Z(x_0) - Z^*(x_0)] = 2\sum_i \alpha_i \gamma_Z(x_0 - x_i) - \sum_i \sum_j \alpha_i \alpha_j \gamma_Z(x_i - x_j)$$

$$\text{with } \gamma_Z(x' - x'') = \gamma_{Z|M}(x' - x'') + \gamma_M(x' - x'') \quad (14)$$

In Section 4, the traditional variogram estimator based on $\{Z^T(x_i); i = 1, \ldots, N\}$ is shown to be an unbiased estimator for $\gamma_Z(x' - x'')$ in this case. Consequently, in practice this case will be reduced to ordinary kriging on the random residual function $\{Z^T(x); x \in A\}$.

In the particular case where the qualified guess can be specified without uncertainty, one gets

$$\text{Var}\,[M(x)] = 0 \qquad x \in A$$

This implies

$$\gamma_Z(x' - x'') = \gamma_{Z|M}(x' - x'')$$

and the reduction to the ordinary kriging system is obvious.

To summarize, it should be noted that two types of cases can be distinguished:

1. If precision of the qualified guess varies over area A, the Bayesian kriging system will provide estimates and estimation variances different from traditional kriging procedures.
2. If precision of the qualified guess is constant over area A, the Bayesian kriging system will, in practice, reduce to the traditional kriging procedure.

Bayesian kriging gives observations large influence in areas in which observations are dense. In areas with few observations, the qualified guess has larger impact. The variogram and the precision in the guess for the expected surface will dictate this trade-off.

4. VARIOGRAM ESTIMATION

From expression (7), one gets

$$\mathrm{Var}\,[Z(x') - Z(x'')] = 2[\gamma_{Z|M}(x' - x'') + \gamma_M(x', x'')] \qquad x', x'' \in A$$

$$\Leftrightarrow$$

$$\gamma_{Z|M}(x' - x'') = \tfrac{1}{2} E[[Z(x') - Z(x'')]^2] - \tfrac{1}{2}[\mu_M(x') - \mu_M(x'')]^2 - \gamma_M(x', x'') \qquad x', x'' \in A$$

Let h be a vector in the reference domain A, and define the set of pairs of indices D_h as

$$D_h : \{(i, j) | x_i - x_j = h \text{ or } x_j - x_i = h;\ i, j = 1, \ldots, N\}$$

$$N_h = \# D_h$$

An estimator for $\gamma_{Z|M}(h)$ is then

$$\hat{\gamma}_{Z|M}(h) = \frac{1}{2N_h} \sum_{(i,j) \in D_h} \{[Z(x_i) - Z(x_j)]^2 - [\mu_M(x_i) - \mu_M(x_j)]^2 - 2\gamma_M(x_i, x_j)\} \tag{15}$$

One may show that $\hat{\gamma}_{Z|M}(h)$ is an unbiased estimator for $\gamma_{Z|M}(h)$ for all h. The estimator $\hat{\gamma}_{Z|M}(h)$ should be plotted for a sequence of lags, and a positive-definite function should be fitted to these estimates. The resulting variogram function should replace the true function in expression (10) and (12).

In Case C in Section 3, one assumed that

$$\gamma_M(x', x'') = C\,\gamma_S(x' - x'') \qquad x', x'' \in A$$

i.e., the spatial covariance in the qualified guess is only dependent on $x' - x''$.

This entails that the variance is constant throughout the area A. In this case, $\gamma_Z(x', x'')$ is dependent on $x' - x''$ only, from expression (7). An unbiased estimator would be

$$\tilde{\gamma}_Z(x', x' + h) = \frac{1}{2N_h} \sum_{(i,j)\in D_h} [Z^T(x_i) - Z^T(x_j)]^2 \qquad x', x' + h \in A$$

Hence, for this case, the estimator for $\frac{1}{2}$ Var $[Z(x') - Z(x'')]$ will reduce to the traditional variogram estimator.

5. AN EXAMPLE

A brief example of the Bayesian kriging method in which data are constructed is presented. The objective is only to demonstrate the characteristics of the procedure.

Five observations are available (Fig. 1). The qualified guess for the expected function $\mu_M(\cdot)$ also is given in the figure. The variogram function $\gamma_{Z|M}(\cdot)$ is given to be spherical with sill value 3.0 and range value 3.0.

Three cases are evaluated:

I. When the expected function is assumed to be known exactly, i.e., Var $[M(x)] = 0$; all $x \in A$. This corresponds to subtracting a predefined drift from all observations and performing ordinary kriging on the residuals.

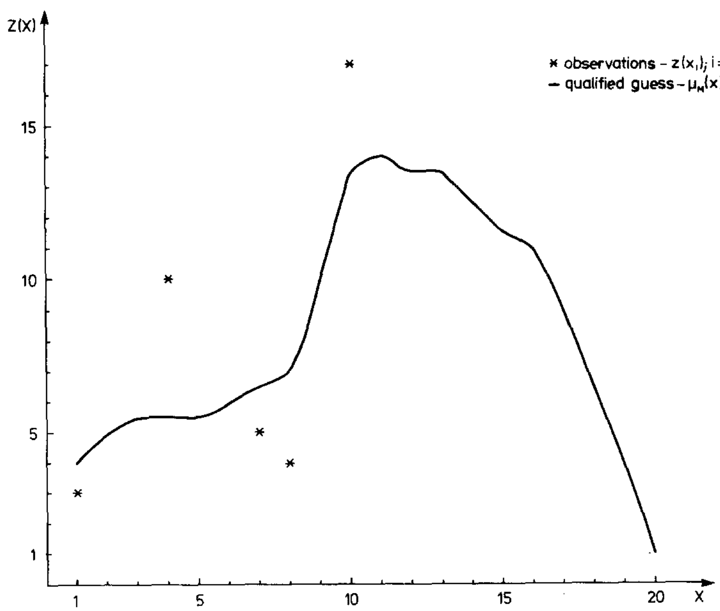

Fig. 1. Observations and the qualified guess.

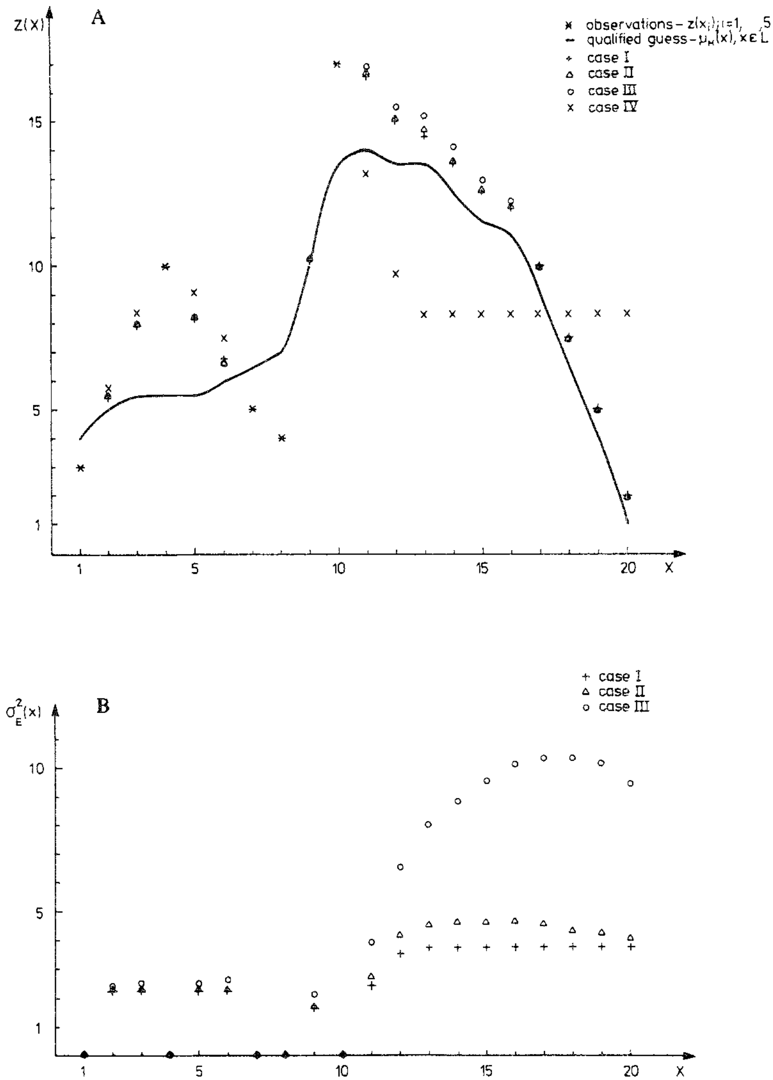

Fig. 2. (A) Results of estimates from the example. (B) Results of estimation variances from the example.

II. When the qualified guess is assumed to be associated with an uncertainty which has variance proportional to the value of the expected function, i.e., $\sigma_M^2(x) = \text{Var}\,[M(x)] = .25 * \mu_M(x)$. The spatial structure $\gamma_S(\cdot)$ is given by a spherical variogram function with unit sill and range value 8.0.

III. When the qualified guess is assumed similar to that in case II but variance also is increasing with increasing value of x.

Table 1. Results of Bayesian Kriging

Variable		Lag 1	2	3	4	5	6	7	8
Qualified guess $\mu_M(x)$		4.0	5.0	5.5	5.5	5.5	6.0	6.5	7.0
Observations $Z(x)$		3.0	—	—	10.0	—	—	5.0	4.0
Case I	$\sigma_M(x)$	0.	0.	0.	0.	0.	0.	0.	0.
Determined drift	Estimate/Estimated Variance	3.0/0.	5.5/2.2	8.0/2.2	10.0/0.	8.3/2.2	6.8/2.2	5.0/0.	4.0/0.
Case II	$\sigma_M(x)$	.50	.56	.59	.59	.59	.61	.64	.66
Drift uncertainty proportional to $\mu_M(x)$	Estimate/Estimate Variance	3.0/0.	5.5/2.3	8.0/2.3	10.0/0.	8.2/2.3	6.7/2.3	5.0/0.	4.0/0.
Case III	$\sigma_M(x)$	.60	.76	.89	.99	1.09	1.21	1.34	1.46
Drift uncertainty depends on $\mu_M(x)$ and x	Estimate/Estimated Variance	3.0/0.	5.5/2.4	8.0/2.5	10.0/0.	8.2/2.5	6.6/2.6	5.0/0.	4.0/0.
Case IV Ordinary kriging	Estimate/Estimated Variance	3.0/0.	5.8/4.4	8.4/4.4	10.0/0.	9.1/4.4	7.5/4.4	5.0/0.	4.0/0.

A fourth case is also included:

IV. When the ordinary kriging procedure is performed and the variogram function is spherical with sill value 6.0 and range value 3.0.

The Bayesian kriging procedure is performed in case I through III. Ordinary kriging is performed in case IV. The corresponding estimates and estimation variances are obtained in 20 locations corresponding to the integer values on the x axis (Fig. 1). Results are reported (Fig. 2 and Table 1). Estimation variances from case IV are not comparable to the others because the sill value of the variogram function is set arbitrarily. Hence, they are not reported graphically (Fig. 2B).

Note the following characteristics (Fig. 2 and Table 1):

1. All kriging estimates run through the observations and the corresponding estimation variances are all zero.
2. The Bayesian kriging procedure, case I through III, provides estimates which tend toward the qualified guess, corrected by a constant, in areas without observations. The ordinary kriging estimates level out on a constant value.
3. Uncertainties associated with the qualified guess have influence on the estimates, although limited influence in this particular example. Note that larger uncertainty entails relatively larger weight to nearby observations.
4. Estimation variances seem to be fairly sensitive to uncertainties in qualified guesses. Obviously, estimation variance increases with increasing uncertainty.

Lag											
9	10	11	12	13	14	15	16	17	18	19	20
10.0 — 0.	13.5 17.0 0.	14.0 — 0.	13.5 — 0.	13.5 — 0.	12.5 — 0.	11.5 — 0.	11.0 — 0.	9.0 — 0.	6.5 — 0.	4.0 — 0.	1.0 — 0.
10.3/1.6 .79	17.0/0. .92	16.6/2.4 .94	15.0/3.5 .92	14.5/3.7 .92	13.5/3.7 .88	12.5/3.7 .85	12.0/3.7 .83	10.0/3.7 .75	7.5/3.7 .64	5.0/3.7 .50	2.0/3.7 .25
10.3/1.7	17.0/0.	16.7/2.7	15.1/4.1	14.7/4.5	13.6/4.6	12.6/4.6	12.0/4.6	9.9/4.5	7.4/4.3	4.9/4.2	1.9/4.0
1.69	1.92	2.04	2.12	2.22	2.28	2.35	2.43	2.45	2.44	2.40	2.25
10.2/2.1	17.0/0.	16.9/3.9	15.5/6.5	15.2/8.0	14.1/8.8	12.9/9.5	12.2/10.1	9.9/10.3	7.4/10.3	4.9/10.1	1.9/9.4
10.3/3.2	17.0/0.	13.2/4.7	9.7/6.9	8.3/7.4	8.3/7.4	8.3/7.4	8.3/7.4	8.3/7.4	8.3/7.4	8.3/7.4	8.3/7.4

6. CLOSING REMARKS

In many applications in the earth sciences, observations of the phenomenon under study are few. The earth scientist often has thorough knowledge about the underlying process creating the phenomena, and alternative sources of indirect information may be available. The Bayesian kriging procedure provides two important features in these cases:

1. The user can include a qualified guess in the estimation procedure. In areas with observations, observations will dominate, whereas in areas without observations, the guess will be assigned increasing weight.

2. The user can assign uncertainties to the qualified guess. Consequently, more realistic estimates and estimation variances can be assessed.

In Kulkarni (1984) and Omre and Holden (1984), geotechnical engineering and description of petroleum reservoirs are mentioned as applications where the general knowledge of the user is of utmost importance. For other applications, as meteorology and air pollution analysis, underlying drifts may be determined by rough deterministic models. These estimates of drifts may be used as qualified guesses for expected surfaces. The Bayesian kriging method provides the possibility to evaluate uncertainties in the model specification in the kriging theory.

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APPENDIX I—AN EXTENDED CLASS OF CONDITIONAL, POSITIVE DEFINITE FUNCTIONS

Consider the following lemma:

Lemma A1

Let $\sigma(x)$ be an arbitrary, nonzero function with $x \in A$, and $[1 - \gamma_S(h)]$ a positive-definite function, and define

$$\gamma(x', x'') = \frac{[\sigma(x') - \sigma(x'')]^2}{2} + \sigma(x')\,\sigma(x'')\,\gamma_S(x'' - x')$$

Then, $-\gamma(x', x'')$ is a conditional, positive-definite function.

Definition of conditional, positive-definitness is given in Journel and Huijbregts (1978).

Proof:

Positive-definitness in $[1 - \gamma_S(h)]$ entails

$$\sum_i \sum_j \alpha_i \alpha_j [1 - \gamma_S(x_i - x_j)] \geq 0 \qquad \text{all } \alpha_i$$

$$\Leftrightarrow$$

$$\left(\sum_i \alpha_i\right)^2 - \sum_i \sum_j \alpha_i \alpha_j \left\{\frac{\gamma(x_i, x_j)}{\sigma(x_i)\,\sigma(x_j)} - \frac{[\sigma(x_i) - \sigma(x_j)]^2}{2\sigma(x_i)\,\sigma(x_j)}\right\} \geq 0 \qquad \text{all } \alpha_i$$

$$\Leftrightarrow$$

$$\left(\sum_i \alpha_i\right)^2 - \sum_i \sum_j \frac{\alpha_i}{\sigma(x_i)} \cdot \frac{\alpha_j}{\sigma(x_j)} \cdot \gamma(x_i, x_j)$$
$$+ \frac{1}{2} \sum_i \sum_j \frac{\alpha_i}{\sigma(x_i)} \cdot \frac{\alpha_j}{\sigma(x_j)} [\sigma(x_i) - \sigma(x_j)]^2 \geq 0 \qquad \text{all } \alpha_i$$

$$\Leftrightarrow$$

$$\sum_i \alpha_i' \sum_j \alpha_j' \sigma^2(x_j) - \sum_i \sum_j \alpha_i' \alpha_j' \gamma(x_i, x_j) \geq 0 \qquad \text{all } \alpha_i' \text{ with } \alpha_i' = \alpha_i/\sigma(x_i)$$

$$\Leftrightarrow$$

$$-\sum_i \sum_j \alpha_i' \alpha_j' \gamma(x_i, x_j) \geq 0 \qquad \text{all } \alpha_i' \text{ such that } \sum_i \alpha_i' = 0$$

Consequently, $-\gamma(\cdot\,,\cdot)$ is a conditional, positive-definite, function. QED.

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