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RESEARCH INTO THE CONSTRUCTION OF MATHEMATICAL TEXTS

ABSTRACT. Investigations of the actual reading process and of the structure of a mathematical text are important for elaborating didactical methods for teaching reading a mathematical text. The article reports on a broader project concerning the structure of a mathematical text. The structure of several hundred texts has been analyzed. Academic textbooks and mathematical monographs (especially in Polish) written by mathematicians have been used as a source. Characteristic features of proof construction, that may influence the course of the reading process, have been isolated. For instance, transmission means have been investigated such as text segmentation which uses delimiters and so called procedure schemes.

Mathematical texts are one of the main means of transmission of mathematics. Active studying of such texts is a basic form of acquiring knowledge and mastering mathematical methods at different levels of mathematical education. Practice of teaching and common observations point out, however, that there are specific difficulties in the reception of such types of texts. Studying mathematical text is hard, even for university students. Reading such a text demands special technique. That technique is not given to the students together with the alphabet. It should be taught and this is one of the important tasks of the modern school.

Didactical problems referring to the functioning of the mathematical text in acquiring and transmitting knowledge were presented at ICME-1 Congress by A.Z. Krygowska (Krygowska, 1969). Getting to know the complex processes and mechanisms of reading mathematical text has essential significance for didactics of mathematics. Systematic, although still small research on that subject matter was started relatively not long ago. The school practice still follows customary rules and a subjective experience in that domain. The current statements that the mathematical text is difficult, abstract and needs a careful reading etc. have little value both theoretically and practically. They do not include any constructive starting point for elaborating a rational proposal how to teach reading mathematical text

Research in two directions is necessary to work out motivated projects of teaching reading mathematical text in school. On the one hand, one should study the actual process of reading mathematical text (both with novices and advanced readers); on the other hand, one should analyse the construction of the mathematical text itself. This article is a sketchy report of the second type of research which has been more widely presented in (Konior, 1983). Considering the differences among particular types of mathematical texts, the research has

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been restricted to the texts of proofs. The construction of over 700 such texts has been analysed thoroughly. Academic textbooks and mathematical monographs (especially in Polish) written by mathematicians-specialists have been used as a source.In a similar way one should study the text of school textbooks (the research presented here did not include such texts); for didactics of mathematics it is necessery to analyse texts directed to readers at different levels of mathematical knowledge and experience.

The starting point of the research is the general statement that the classical mathematical text has its specific construction. Such a text uses means and techniques that are less or not at all used in non-mathematical texts. The aim of the research is insight into the specific character of that text. The question is to reveal specific features of the proof text construction, to express and characterize ways of written presentation of mental construction, to reveal methods of guiding reader's work and of wording endeavours, worked out in mathematical tradition and practice, to be used in pedagogical transmission. Thus, the research efforts tend towards revealing the components and mechanisms in the construction of a classical proof text that can determine the process of its reading, and determine the process of its reception one way or the other.

The analysis which has been done shows that besides common, verbal and symbolic, transmission other sources of information important for the reader appear in the proof texts. So various paralinguistic means are used in the construction of the mathematical text (for example: type face, section, different indentions, different ways of numbering cases and distribution of the text on a page; the contrast which is formed by two adjoining parts of the text, one of which is verbal the other symbolic, plays sometimes the role of such a means).They play the role of systems that support a verbal-symbolic language in acts of communication. The components of the texts, being the carriers of information, create a multidimensional language in which a verbal communication act is a basic but not the only kind and way of the transmission. The process of communication in the proof text is thus a multiple process. Some channels serve to intensify verbal information, others transmit additional information concerning the essence of a matter, yet others serve to initiate suitable behaviour and activity of a reader.

One of the chief methods with the help of which the author of the proof guides his reader's work is segmentation of the text. Segmentation means separation of the whole text into a few parts which is done deliberately in order to reflect the logical structure of the proof and facilitate the reconstruction of the whole proof construction.

The next question is then about the ways and means used to achieve the segmentation of the text. They can be divided into verbal and non-verbal. Very often such segmentation signs, i.e. indicators of the limits of the segments of the text, are not expressed explicitly. The carriers of such signs can be expressions related to the subject matter, but at the same time containing the additional information about a limit of some text segment. Signs of the limit of some text segment, very often inserted in a subtle way into the mathematical "shorthand" of the text of the proof, are called *delimitators*. They play an essential role in the process of reading the proof, provided that the reader recognizes them and interprets them correctly. One of the characteristic editorial steps in the process of the text segmentation of a proof is constructing the so-called *delimiting frame*. It consists in isolating some fragment of the text (which according to the author's editorial conception should be stressed, for some formal or heuristic reasons) by means of two signs: a delimitator of the beginning and a delimitator of the end.

The role of delimitator is often played by *plan of procedure* with the help of which the author opens the whole text (general plan) or some of its fragments (stage plan). Becoming aware of the plan and then remembering it in the process of reading the text is helpful in reconstructing the proof. Apart from explicit programmes of further operations, there are often occasional hints, indications, indirect suggestions, etc. This often occurs in case of stage plans.

The interesting thing is not only the fact that in the text of a proof there are plans included but that the author very often separately indicates the boundaries of validity of such a plan. Having realized the plan, the author closes the area of operations indicated in it by means of a special delimitator of the end. As a result there appears in the text an isolated fragment, indicated by a delimiting frame. But it is not only a question of simply isolating a fragment of the text from the whole by means of a frame. The essential thing is that the reader's work within that fragment of the text is stimulated by the initial plan and is over when its realization is indicated. The fragments of the text isolated in the manner presented above, because of their heuristic function in the process of reading, will be called the *areas of directing the reader by the plan.*

We will illustrate the components and phenomena described above by means of an example. An individual example, however, can make only a partial illustration of very complex and refined phenomena which can be observed against the background of the whole group of analyzed texts. Here is the text of a proof of the Cantor-Bernstein theorem (Kuratowski, 1977): If $m \leq n$ and $n \leq m$, then $m = n$ for any cardinal numbers m, n. (In the quoted text of the proof some formal arrangements will be made and commented further on; the picture will also be excluded from the original text and inserted on the next page).

Proof. Let $\overline{\overline{X}} = m$. Since $n \leq m$, the set X contains a subset Y of power n. But since $m \leq n$, the set X is of power equal to that of some subset of the set Y; i.e. there exists a one-to-one mapping f defined on X such that

(29) $f(X) \subset Y \subset X$.

We have to define a one – to – one mapping g of X into Y.
Let us set

(30) $Z = Y - f(X), \qquad S = Z \cup f(Z) \cup ff(Z) \cup \ldots$

(see Figure 1). We define g as follows:

(31) $$g(x) = \begin{cases} x & \text{for } x \in S, \\ f(x) & \text{for } x \in X - S. \end{cases}$$

We shall first prove that

(32) $g(X) = Y$.

Since $S \subset X$,

(33) $X = S \cup (X - S)$.

And therefore

(34) $g(X) = g(S) \cup g(X - S) = S \cup f(X - S)$

by virtue of (31). At the same time (because of (30) and Chapter IV, (14)):

$$f(S) = f(Z) \cup ff(Z) \cup fff(Z) \cup \ldots,$$

and hence applying (30):

$$S = Z \cup f(S).$$

From this and (34) and (33), we obtain

$$g(X) = S \cup f(X - S) = Z \cup f(S) \cup f(X - S) = Z \cup f(X),$$

but by (30) we have

$$Z \cup f(X) = [Y - f(X)] \cup f(X) = Y.$$

We have thus proved formula (32).

It remains to show that g is one – to – one.
Since (according to (31)) g is one-to-one on each of the sets S and $X - S$ separately, we ought to prove that

(36) $g(S) \cap g(X - S) = \phi$.

Now by (31) we have

(37) $g(S) = S$ and $g(X - S) = f(X - S) = f(X) - f(S)$;

at the same time, $f(X) = f(X) - Z$ because $f(X) \cap Z = \phi$,
and hence

$$f(X) - f(S) = f(X)\,[Z \cup f(S)] = f(X) - S$$

because of (35).
Hence, we have $S \cap [f(X) - f(S)] = \phi$, whence formula (36) follows by virtue of (37).

This completes theproof of the Cantor – Bernstein theorem.

(see Figure 1 in which X is the largest rectangle, Y is the second in size, $f(X)$ is the third, and so on, $X - S$ is the shaded part).

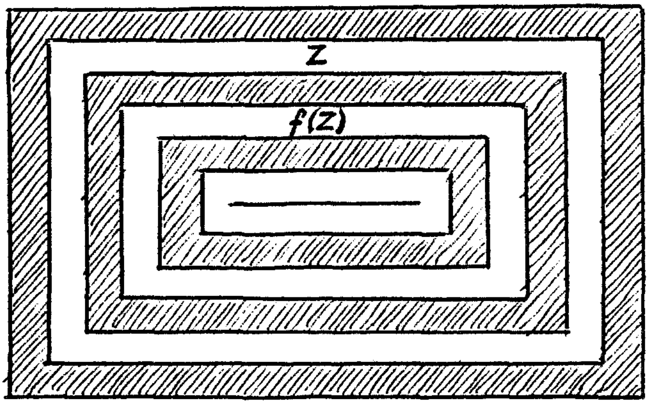

Fig. 1.

Analysing the quoted text we notice that, after a short introduction, a plan appears: "We have to define a one-to-one mapping g of X onto Y". It is a general plan, which together with the final formula at the end: "This completes the proof of the Cantor-Bernstein theorem", creates a delimiting frame and separates the main area of directing the reading of the text. That area is represented in the reproduced text graphically by the greatest rectangle frame. The initial and final formulae functioning as delimitators have been additionally underlined for the analysis of the text.

The author of the proof includes into the text also stage plans. The role of such a plan, directing locally the reader's work is played by the indication: "We shall first prove that $g(X) = Y$". That indication, together with the sentence at the end: "We have thus proved formula (32)", which reverts to the plan, isolates the smaller area, being a subarea of the main area of directing the reading. The second such subarea is opened by a stage plan, beginning with the words: "It remains to show that...". It is closed by an additional final statement: "Whence formula (36) follows by virtue of (37)".

Further analysis of the proof texts reveals a tendency towards a structure such that the areas of directing cover the whole text. Often subareas separated in a given area of directing the reading create together some kind of a system. Sentence formulae which open them refer to each other indicating subordinate (or primary) character of the parts of the text, and some hierarchy of the fragments of the proof construction. A system of areas constructed in such a way and each particular area of directing become an important means for directing the reader's work.

In routine reading of the text of a proof we do not pay attention to how delimitators work. Anyway, not all of them are expressed directly, as in the presented example, not always they are "on the surface". Some of them are very refined signals. An educated reader takes advantage of each such signal. The situation of a beginner is quite different. One may think that awareness of reading in a given area of directing is needed and even necessary for determining the place and role of each step of a proof within the whole of its construction. It is also necessary to foresee the next steps of a proof and integrate these steps in order to form a whole and achieve sufficient synthesis.

The components of that kind determine, among other features, the specific character of the mathematical text. They do not usually appear in the texts which the reader has encountered in his education from the beginning and due to which he has already worked out his own reading strategies. He transfers these habitual ways of reading to the mathematical texts, but often unsuccessfully.

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