Probab. Theory Relat. Fields 92, 511–527 (1992)

Probability Theory and Related Fields
© Springer-Verlag 1992

Uniqueness of the infinite component in a random graph with applications to percolation and spin glasses

A. Gandolfi,[1,★] M.S. Keane[2], and C.M. Newman[3]

[1] Department of Statistics, University of California, Berkeley, CA 94720, USA
[2] Delft University of Technology, Mekelweg 4, 2628 CD Delft, The Netherlands
[3] Courant Institute of Mathematical Sciences, 251 Mercer Street, New York, NY 10012, USA

Received July 6, 1989; in revised form November 26, 1991

Summary. We extend the theorem of Burton and Keane on uniqueness of the infinite component in dependent percolation to cover random graphs on $\mathbb{Z}^d$ or $\mathbb{Z}^d \times \mathbb{N}$ with long-range edges. We also study a short-range percolation model related to nearest-neighbor spin glasses on $\mathbb{Z}^d$ or on a slab $\mathbb{Z}^d \times \{0, \ldots, K\}$ and prove both that percolation occurs and that the infinite component is unique for $V = \mathbb{Z}^2 \times \{0, 1\}$ or larger.

1 Introduction

Consider a countable set V and a subset E of the set of unordered pairs of elements in V: we call elements in V vertices and elements in E edges. Consider a subset $\tilde{E} \subset E$ and suppose vertices are connected through the elements of $\tilde{E}$; then V falls apart into connected components. To have a good definition we will only refer to maximal connected components, having the property that they are not properly contained in any other connected component. The global connectivity picture can now be given by one of the following possibilities:

(1) all components are finite;
(2) there are some infinite components, but also finite ones;
(3) all vertices of V are connected.

Different terminologies have been developed to describe the various cases, so we define case (1) as absence of percolation, case (2) as occurrence of percolation and case (3) as *connectedness of the graph* $(V, \tilde{E})$. A further study can be carried out in case (2) counting the number of distinct infinite components, called infinite clusters in percolation theory, which do occur: if there is only one infinite cluster we say *uniqueness of the infinite component* holds. Note that we adopt terminologies coming both from random graph theory and from percolation theory.

★ A.G. was partially supported from AFOSR through grant no. 90–0090

We now suppose that $\tilde{E}$ is described by a probability measure P on $\{0,1\}^E$. Having an assignment η of the values 0 and 1 to the edges, we declare an edge $e \in E$ open, and thus belonging to $\tilde{E}$, if the value assumed by η in e is 1 and closed, and not belonging to $\tilde{E}$, if this value is 0.

In this paper we want to show that for a broad class of choices of V and E and of probability measures on $\{0,1\}^E$ two dichotomic laws hold:

(a) *Connectedness of the random graph*: when with probability one the graph described by P is either totally connected or falls apart into infinitely many components.
(b) *Uniqueness of the random infinite component*: if with P probability one there is either no infinite component or a unique one.

Actually we will prove results concerning (b) and those concerning (a) will be an immediate consequence.

These kinds of results, together with the search for conditions to ensure that one of the two possibilities described in (a) or (b) occurs, have already a long history, and we now review some of the more relevant steps. But let us first remark that we are considering infinite random graphs and the results are thus different from those obtained in the graphs studied originally by Erdös and Rényi [9] and surveyed for example by Bollobás [3], in which the number of vertices is finite and in which asymptotic properties are considered when the number of vertices approaches infinity. Nevertheless there are some features which can be considered common: in particular, also in the finite case, for a large class of probability distributions, either the graph has a probability tending to one to be totally connected or the number of components tends to infinity (see [3]).

Another difference, this time not so significant, is between bond percolation models, which are those considered here, and site ones, in which vertices are randomly distributed and all edges of a set E are open. Indeed many arguments can be easily translated from one model to the other. We prefer the bond (edge) one because the relation between graph theory and percolation theory is somewhat clearer, because of the relevance to statistical mechanical models of bond percolation (see Sect. 4 below), and finally because long-range models, which we treat in this paper, are intrinsic to bond percolation.

We recall that n.n. (nearest-neighbor) models are those in which only edges between vertices at distance one are considered (with the distance always taken to be the Euclidean distance for $V \subseteq \mathbb{Z}^d$) and long-range models will be for us those in which all edges are considered. We use the term connectedness when (a) holds and uniqueness when (b) holds. The problems of connectedness and uniqueness were successively solved in the following settings.

First we have n.n. stationary models in $\mathbb{Z}^2$ for which uniqueness was proved for distributions of edges which are independent (Harris [18] and Fisher [10]), Markov (Coniglio et al. [6]) or simply positively correlated models (the so-called FKG condition) with some additional geometrical requirements (Gandolfi et al. [15]). We mention that some of the results of these papers are not included in the present one.

As we move to $\mathbb{Z}^d$ we first have results for n.n. and long-range distributions for the independent case yielding connectedness of the infinite graph (Grimmett et al. [16], in which also necessary and sufficient conditions are given to decide which of the possibilities occurs) and uniqueness (Aizenman et al. [1], with a simplified proof in Gandolfi et al. [14]). Then uniqueness was obtained for Gibbs measures (with some additional requirements, Gandolfi [13]).

In the meantime n.n. percolation in $\mathbb{Z}^d$ was analyzed with the only requirements of stationarity and finite energy of the probability measure (finite energy essentially means that it is possible to change locally an event preserving its positive probability). Partial results in this direction were obtained by Newman and Schulman [27] and a simple and elegant proof of uniqueness under these conditions was given by Burton and Keane [4]. The argument in this last paper shows that if two (and hence many) distinct infinite clusters occur, then the surface of a cube cannot accommodate all the disjoint open paths which are nevertheless forced by the regularities of the measure to intersect that surface.

We want to push forward this argument, first of all to all long-range models. This will be achieved by realizing that the volume of the cube itself is not sufficient to accommodate the disjoint open paths still forced by the geometrical properties of the measure to intersect it (see proof of Theorem 1).

We are also able to reduce the requirements about the set of vertices; Theorem 1, for example, covers $\mathbb{Z}^d \times \mathbb{N}$ as well as $\mathbb{Z}^d$ and Theorem 1′ covers other vertex sets. Nonetheless we cannot treat in generality other classes of graphs: the first example to which our results do not apply is long-range percolation in the quadrant $\mathbb{N} \times \mathbb{N}$. Previous results for the n.n. independent problem in $\mathbb{Z}^d \times \mathbb{N}$ are given by Kesten in [24], where he proves uniqueness for these models; Kesten's results were extended to $\mathbb{N}^k$ and $\mathbb{Z}^d \times \mathbb{N}^k$ by Barsky et al. [2]. Kesten [24] also shows connectedness for long-range independent percolation in $\mathbb{Z}^d \times \mathbb{N}^k$, $d \geqq 0$, finding the conditions under which the graph is totally connected.

Theorem 1 and our other results also do not entirely require finite energy, but only a one-sided version of it. We will say that P has positive finite energy for $e \in E$ if the conditional probability that e is open, given the configuration of all other edges in E, is almost surely positive; we will say that P obeys the positive finite energy condition if the set of such e's is large enough to connect any pair of vertices in V. Theorem 1 is stated in the next section after a lemma, which will be needed in its proof. Meanwhile, we state a special case of Theorem 1:

Theorem 0. *Let $V = \mathbb{Z}^d$ and $E =$ the set of all pairs of vertices from V. The random graph determined by a probability measure P on $\{0, 1\}^E$ satisfies uniqueness if P is* (a) *stationary and* (b) *obeys the positive finite energy condition.*

We remark that if one further restricts Theorem 0 to the case where P is assumed ergodic, then most of the technical issues which arise in the proof of Theorem 1 are eliminated. The reader is encouraged to consider this special case while looking at the proof of Theorem 1 given in the next section.

It may be appropriate here to mention some examples in which (a) or (b) is violated. Infinitely many infinite clusters can occur even with stationarity and the positive finite energy condition for percolation on a graph where the number of vertices at distance R from a fixed vertex grows exponentially, such as a homogeneous tree T or $T \times \mathbb{Z}^d$ (Grimmett and Newman [17]). Other examples can be found in certain exactly solved percolation models in $\mathbb{Z}^d$ called ergodic percolation (Meester [25]) where the positive finite energy condition does not hold. A nice class of examples in which one obtains a finite number of distinct infinite clusters in $\mathbb{Z}^d$ may be constructed by considering i.i.d. variables $\{X_v \colon v \in \mathbb{Z}^d\}$ taking values in

$\{1, \ldots, q\}$ with probabilities $p_i \equiv P(X_v = i)$ for $i = 1, \ldots, q$. Take $E = \{$n.n. edges$\}$ and then define $\eta_e = 1$ if and only if $e = \{v, v'\}$ has $X_v = X_{v'}$. There will be one infinite cluster for each i such that there is independent n.n. site percolation in $\mathbb{Z}^d$ at density p_v. For example, with $d = 3$, $q = 2$ and $p_1 = p_2 = \frac{1}{2}$, there will be exactly two distinct infinite clusters since the critical value for n.n. site percolation in $\mathbb{Z}^3$ is strictly below 1/2 (Campanino and Russo [5]).

There is a second line of development for the connection–uniqueness problem which is concerned with models in $\mathbb{N}$. Connectedness for long-range distributions on $\mathbb{N}$ has been shown in great generality by Kalikow and Weiss [19]. They also find for independent distributions the conditions under which the graph is completely connected or there are infinitely many finite components with probability one, work generalized by Kesten to $\mathbb{Z}^d \times \mathbb{N}^k$ as already mentioned [24]. Conditions for this transition, including the explicit computation of the critical value of the parameter in one-parameter families, has been found for non-homogeneous distributions by Shepp [29] and Durrett and Kesten [7], thus "solving" a large class of these models. We limit ourselves to the homogeneous case by which we mean that the distribution is invariant under the (induced map given by the one-sided) shift $n \to n + 1$. In Sect. 3 we give a proof of uniqueness for all these models, provided they have finite energy: results on connectedness for homogeneous models are an immediate consequence.

As an application of Theorem 1 (or the original Burton–Keane theorem) we show in Sect. 4 that the cluster of edges is unique in a model related to spin glasses. In the process of verifying the conditions of Theorem 1, we show that indeed percolation occurs when $V = \mathbb{Z}^d$, $d \geqq 3$ or even in a slab with $V = \mathbb{Z}^2 \times \{1, \ldots, K\}$ with $K \geqq 2$ for n.n. models (so that it has a meaning to worry about the number of infinite components): this result is in accordance with the belief that phase transitions should occur for dimensions higher than 2 in these models. We have not determined whether or not percolation occurs when $V = \mathbb{Z}^2$.

In the next section we begin with some definitions and an introductory lemma, before stating and proving Theorem 1 and the closely related Theorem 1′.

2 The main result

We now proceed by fixing the notation. Let V be a countable set; the elements of V will be called *vertices*. The set of *edges* between vertices in V will be a subset E of $V_2 = \{\{v_1, v_2\}, v_i \in V, i = 1, 2\}$. Each $e \in E$ will be identified by the two vertices which define it and these will be called the *end-points* of e: $e = \{v_1, v_2\}$ for $v_1, v_2 \in V$. The edges are not directed. A (vertex self-avoiding) *path* γ in E between distinct vertices v and v' in V is a finite sequence of distinct vertices $(v_0 = v, v_1, \ldots, v_n = v')$ such that the edge $\{v_{i-1}, v_i\} \in E$ for $i = 1, \ldots, n$. v and v' will be said to be connected by γ. We will identify γ with the set of these edges and write for example that $\gamma \subset E$.

To represent edges which are open or closed we consider $H = \{0, 1\}^E$ in which a topology is given by the *cylinders* of the form $C = \{\eta \in H: \eta_{e_{(1)}} = \alpha_1, \ldots, \eta_{e_{(n)}} = \alpha_n\}$ with *base* $\{e_{(1)}, \ldots, e_{(n)}\} \subset E$ for $n \in \mathbb{N}$, $\alpha_i \in \{0, 1\}$. For a subset $E' \subset E$ a topology is similarly obtained by the cylinders with base contained in E', to which we shortly refer as cylinders in E' for short. Given $\eta \in H$ and a vertex $v \in V$ we define the *component* I_v of v (in η) on (V, E) as the maximal subset of V which has the property that all its vertices are connected to v by a path γ whose edges are open, i.e. $\eta_e = +1$

when $e \in \gamma$. We will also consider *components on a subset* $V' \subset V$, by which we mean maximal connected components on (V', E') in η' where $E' \subset E$ is the set of edges whose end-points are both in V' and $\eta' \in (0,1)^{E'}$.

For $v \in V$ we consider maps (to be specified) $T_v: V \to V$. Note that given such a map T_v we have an induced action on E defined by $T_v(e) = T_v(\{v_1, v_2\}) = (T_v v_1, T_v v_2)$, $e \in E$, where we use the same notation for the induced maps because it will be always clear at which level we are considering the map. We will say that E is T_v-invariant if $\{v_1, v_2\} \in E$ if and only if $(T_v v_1, T_v v_2\}$ is in E. There is then an induced map on H defined by $T_v(\eta)_e = \eta_{T_v(e)}$, $\eta \in H$ and on the subsets A of H by $T_v(A) = \{T_v(\eta)\colon \eta \in A\}$. If a σ-algebra $\mathscr{A}$ of subsets of H is given, then T_v is $\mathscr{A}$ measurable if $T_v^{-1}(A) \in \mathscr{A}$ for all $A \in \mathscr{A}$ and a probability measure P defined on $\mathscr{A}$ is T_v-invariant if $P(T_v^{-1}(A)) = P(A)$ for all $A \in \mathscr{A}$.

We start now with a technical lemma. It relates the conditional probabilities of a probability measure P given a sub σ-algebra with the same conditional probabilities of the ergodic components of P, when P is decomposed into probability measures which are ergodic with respect to a map T. This result essentially shows that if finite energy (or positive finite energy) holds for P then it holds also for its ergodic components.

Lemma 1. *Let E be a countable set and T an invertible map of E onto itself. Let $H = \{0,1\}^E$, $e \in E$ and let $\mathscr{A}$ and $\mathscr{A}_{E\setminus\{e\}}$ be the σ-algebras generated by the cylinders in E and $E\setminus\{e\}$ respectively. As usual let us denote by the same letter the extension of the map T to H and $\mathscr{A}$.*

Suppose that $T^n(e) \neq e$ for all $n \geqq 1$. Then there exists a function φ on H, measurable with respect to $\mathscr{A}_{E\setminus\{e\}}$, such that for any T-invariant probability measure P defined on $\mathscr{A}$,

$$\varphi(\eta_{E\setminus\{e\}}) = P(\eta_e = 1 \mid \mathscr{A}_{E\setminus\{e\}})(\eta_{E\setminus\{e\}}) \quad P\text{-almost everywhere}$$

and therefore for almost all ergodic components $\tilde{P}$ in the ergodic decomposition of P related to T

$$\tilde{P}(\eta_e = 1 \mid \mathscr{A}_{E\setminus\{e\}}) = P(\eta_e = 1 \mid \mathscr{A}_{E\setminus\{e\}}) \quad \tilde{P}\text{-almost everywhere.}$$

Proof. The main idea in the proof is a proper use of the ergodic theorem. Indeed if $C \subset H$ is a cylinder set and I_c denotes its indicator function, then $\lim_{n\to+\infty} \frac{1}{n}\sum I_C(T^n\eta) = \mu_\eta(C)$ exists for P-almost all $\eta \in H$ whenever P is a T-invariant probability measure, and if P is ergodic this limit no longer depends on η. From now on we consider only $\eta \in H$ for which $\mu_\eta(C)$ exists for all cylinders C and since there are only countably many cylinders, the set of these η's has probability one for any T-invariant measure. Choose a sequence of finite subsets $B_m \subset E$ such that $e \in B_m \subset B_{m+1}$ and $\bigcup_m B_m = E$. We now define cylinders depending on a given η. For $m \in \mathbb{N}$ let $C_m(\eta)$ be the cylinder set of the configurations coinciding with η in $B_m\setminus\{e\}$; i.e., $C_m(\eta) = \{\eta'\colon \eta'_{e'} = \eta_{e'} \text{ for all } e' \in B_m\setminus\{e\}\}$. Furthermore, let $C_m^1(\eta)$ be the cylinder set with the same properties as $C_m(\eta)$ but with e forced to be open: $C_m^1(\eta) = \{\eta' \in C_m(\eta)\colon \eta'_e = 1\}$.

Let $\varphi_m(\eta) = \mu_\eta(C_m^1(\eta))/\mu_\eta(C_m(\eta))$ when this is defined (i.e. for the η's we are considering for which the denominator is nonzero). Because of the assumption on $T^n(e)$, $\mu_\eta(C)$ does not depend on η_e for any C. If $\tilde{P}$ is ergodic, then with $\tilde{P}$ probability one, $\mu_n(C) = \tilde{P}(C)$ and hence $\varphi_m(\eta)$ depends only on the values of η in $B_m\setminus\{e\}$ and

$$\varphi_m(\eta) = \tilde{P}(\eta_e = 1 \mid \mathscr{A}_{B_m\setminus\{e\}})(\eta)$$

for $\tilde{P}$-almost all $\eta \in H$. Now apply the martingale convergence theorem to the sequence $\varphi_m(\eta)$ of functions measurable with respect to $\mathscr{A}_{B_m\setminus\{e\}}$, where $\mathscr{A}_{B_m\setminus\{e\}}$ converges as a sequence of σ-algebras to $\mathscr{A}_{E\setminus\{e\}}$, to see that

$$\varphi(\eta) = \lim_{m\to+\infty} \varphi_m(\eta) = \tilde{P}(\eta_e = 1 \mid \mathscr{A}_{E\setminus\{e\}})(\eta)$$

holds $\tilde{P}$-almost everywhere. To extend the equality to all invariant measures we can use uniqueness (a.e.) of the conditional probability and remark that using the ergodic decomposition and denoting by ρ_P the probability measure on the space of probability measures that realizes this decomposition, the following holds for any $A \in \mathscr{A}_{E\setminus\{e\}}$:

$$\begin{aligned}\int_A \varphi \, \mathrm{d}P &= \int \left(\int_A \varphi \, \mathrm{d}\tilde{P} \right) \rho_P(\mathrm{d}\tilde{P}) \\ &= \int \tilde{P}(A \cap \{\eta_e = 1\}) \rho_P(\mathrm{d}\tilde{P}) = P(A \cap \{\eta_e = 1\}) .\end{aligned}$$

This shows that φ satisfies the equation claimed in the lemma. Next consider P and note that for ρ_P-almost all $\tilde{P}$ in the ergodic decomposition of P the set in which $P(\eta_e = 1 \mid \mathscr{A}_{E\setminus\{e\}}) = \varphi$ has probability one. If $\tilde{P}$ is one of these components then the second equality of the lemma holds for $\tilde{P}$-almost all $\eta \in H$. □

We are now ready to state our main result for percolation on $\mathbb{Z}^d$ or $\mathbb{Z}^d \times \mathbb{N}$.

Theorem 1. *Let V be $\mathbb{Z}^d$ or $\mathbb{Z}^d \times \mathbb{N}$ and let $T_v: V \to V$ be defined by $T_v(w) = v + w$. Let E be a subset of $V_2 = \{\{v_1, v_2\}: v_i \in V,\ i = 1, 2\}$ such that E is T_v-invariant for each $v \in V$ and let P be a probability measure on $H = \{0,1\}^E$ invariant under T_v for all $v \in V$. Assume P satisfies the positive finite energy condition; i.e., assume that the set E' of edges e such that*

$$P(\eta_e = 1 \mid \mathscr{A}_{E\setminus\{e\}}) > 0 \quad P\text{-almost everywhere},$$

is large enough so that for every $v_1, v_2 \in V$, there is some path $\gamma \subset E'$ connecting v_1 and v_2. Then, in the random graph defined by P, there is at most one infinite component with P probability one.

Proof. The proof is divided into several steps.

1. First we reduce the range of values the number of infinite components can assume, using Lemma 1, to 0, 1 or ∞.

Denote the origin $(0, \ldots, 0)$ by v_0. First note that for some $v \in V$, $v \neq v_0$ also $-v \in V$ and thus T_v becomes an invertible map of V onto itself, whose extension to E has also such a property. This was the requirement of Lemma 1, together with the fact, here obviously satisfied, that $T_v^n(e) \neq e$ for all $e \in E'$, for $n \geqq 1$.

The second conclusion of Lemma 1 shows then that almost all ergodic components $\tilde{P}$ of P in the decomposition related to T_v satisfy positive finite energy on the same subset E'.

For an ergodic measure $\tilde{P}$ the number of infinite components is constant almost everywhere and positive finite energy on E' easily implies this number must be 0, 1 or ∞ (as in [1] or [27], think of another number, find this many components intersecting a finite region and join them through E', which gives a prohibited positive probability to a smaller number).

Considering P again, it remains only to exclude the possibility of a positive probability for having infinitely many components. We turn immediately to P to stress that almost the whole proof can be done without using ergodic properties (in some cases they do not appear at all as in Theorem 2 below).

2. To achieve a contradiction suppose there are infinitely many infinite components with P positive probability. The strategy is now to find at least three of these distinct infinite components and join them to a fixed vertex, the origin, in such a way that they remain disjoint apart from vertices near the origin; this has to occur with positive probability.

By the assumptions on E', each pair of vertices can be connected by a path using edges in E'. Therefore, we can find $\gamma_{v0} \subset E'$, a union of at most three paths, such that v_1, v_2, v_3 and v_0 are all connected by γ_{v_0}. Next, we repeatedly use the positive finite energy condition for the edges of γ_{v_0} in the following form. Let A be any event such that $P(A) > 0$ (in our case this will be the event that v_1, v_2, v_3 are in three distinct infinite components) and let $e \in E$; then, $P(\{\eta_e = 1\} \cap A) > 0$. The equivalence of this formulation to the one given in the hypothesis is easily seen and the result is that the event τ^1_{v0}, as defined in the next equation has nonzero probability:

$$\begin{aligned} P(\tau^1_{v_0}) = P(&\gamma_{v_0} \text{ is open, each of the three } v_i\text{'s is in the same infinite component, but} \\ &\text{there is an altered configuration of the } \eta_e\text{'s for } e \in \gamma_{v0} \text{ such that } v_1, v_2 \text{ and } v_3 \\ &\text{would be in three disjoint infinite components}) \\ = \sigma &> 0. \end{aligned}$$

Consider the set of edges $E\backslash\gamma_{v0}$. When τ^1_{v0} occurs there are at least three distinct infinite components $C^{(1)}_{v0}, \ldots, C^{(i_{v_0}(\eta))}_{v0}$ on $(V, E\backslash\gamma_{v0})$ which are then connected through edges in γ_{v_0}. These components of $(V, E\backslash\gamma_{v0})$ will be called *branches* of v_0.

3. The event just defined for the origin can occur also around other vertices and this will be our definition of τ^1_v in $\mathbb{Z}^d$; but since we want to preserve the probability of the events, in the case of $\mathbb{Z}^d \times \mathbb{N}$, a better definition is necessary.

To define similar events for $v \in V$ let $\tau^1_v = T_v^{-1}(\tau^1_{v_0})$. Note that τ^1_v only refers to edges having both end-points in $T_v(V)$ and nothing is assumed about edges having at least one end-point in $V\backslash T_v(V)$ (if this set is not empty). The branches $C^{(i)}_v$ of v are related to those of v_0 by $C^{(i)}_v(\eta) = T_v(C^{(i)}_{v0}(T_v(\eta)))$ for $\eta \in \tau^1_v$. Furthermore define $C_v = \bigcup_i C^{(i)}_v$ for $v \in V$.

4. We now want to show that if $\tau^1_v \cap \tau^1_w$ occurs for two sites v and w the branches $C^{(i)}_v$ and $C^{(j)}_w$ will satisfy the hypotheses of Lemma 2 (given below) when w and v are "far enough apart".

We define a certain subset $V' \subseteq V$; pairs of sites from V' will be far enough apart. First choose a box B_1 containing $v_i, i = 0, \ldots, 3$ and all end-points of edges in γ_{v0}, where box in the context of $\mathbb{Z}^d$ means a set of vertices of the form $[-k, k]^d \cap V$ and in the context of $\mathbb{Z}^d \times \mathbb{N}$ means $([-k, k]^d \times [0, k]) \cap V$, Denote next by $v \sim w$ the relation $v \in T_w(v)$ and $w \in T_v(V)$. If $v \sim w$ we simply ask that $T_v(B_1) \cap T_w(B_1) = \emptyset$. On the other hand, if for instance, $v \notin T_w(V)$, which implies $w \in T_v(V)$, we ask that $T_v(B_1) \cap T_w(V) = \emptyset$. Choose a subset $V' \subseteq V$ such that for any $v, w \in V'$ the previous requirements are fulfilled. V' can and will be chosen to have positive density. Now we want to see what the simultaneous occurrence of τ^1_v and τ^1_w, for $v, w \in V'$, implies.

Let us first take the case where $v \sim w$ (this is the only case in $\mathbb{Z}^d$). Either $w \notin I_v$ (the component of v), and thus $v \notin I_w$, or there is an open path γ' connecting w and v. If $w \notin I_v$ or if every such γ' contains a vertex in $V\backslash T_v(V) = V\backslash T_w(V)$ then we have

$(\{v\} \cup C_v) \cap (\{w\} \cup C_w) = \emptyset$, because branches are defined only using vertices in $T_v(V)$. Suppose therefore that there is such an open γ' whose end-points are all vertices in $T_v(V)$. First let us remark that w is in a branch of v. Indeed the open path γ' connects v and w and there are at least three distinct infinite components of $(T_w(V), T_w E \setminus \gamma_w)$ which are connected only through γ_w ($\gamma_w = T_w(\gamma_{v0})$); therefore at most one can contain vertices which are end-points of edges of γ_v since γ_v and γ_w are disjoint (since $T_v(B_1) \cap T_w(B_1) = \emptyset$), and γ_v would connect two branches of w outside γ_w. This implies that at least two infinite paths emanate from w without using edges of γ_v and thus end-points of edges in these paths have to be part of a branch, say $C_v^{(1)}$, of v, together with end-points of γ' and w itself. For the same reason v is in a branch, say $C_w^{(1)}$, of w. No other branch of v can have a common vertex with the remaining branches $C_w^{(i)}$, $i \neq 1$, of w, which are those branches not containing any end-point of edges in γ_v, since otherwise there would be a path not containing edges in γ_v and connecting two branches of v. Thus $C_v^{(1)} \supseteq \{w\} \cup C_w \setminus C_w^{(1)}$ and, for the same reason, $C_w^{(1)} \supseteq \{v\} \cup C_v \setminus C_v^{(1)}$.

Now we consider the case where $w \in T_v(V)$ but $v \notin T_w(V)$; if $w \notin I_v$ or if every open path connecting v and w contains edges whose end-points are vertices of $V \setminus T_v(V)$, then again $(C_v \cup \{v\}) \cap (C_w \cup \{w\}) = \emptyset$. Suppose instead that there is an occupied path γ' from v to w whose end-points are all contained in $T_v(V)$. This time we conclude immediately that w is in a branch of v, which we denote again by $C_v^{(1)}$, and that $C_v^{(1)} \supseteq \{w\} \cup C_w$; indeed no vertex of C_w can be an end-point of an edge in γ_v since we assumed $T_w(V) \cap T_v(B_1) = \emptyset$, so they are all contained in $C_v^{(1)}$, together with w.

5. The next step in the argument will be to show, based on probabilistic reasons, that for a positive density of vertices $v \in V'$, τ_v^1 occurs. This would already be conclusive for n.n. models; to overcome the problem of having a possibly long-range model we first provide a box around v in which each of the branches of v contains many vertices. Together with the density of occurrence of τ_v^1's and the disjointness of the various branches (from Lemma 2 below) this will require the existence of more vertices than there actually are.

Take $V' \subset V$ as before. Now V' was chosen to have a positive density, i.e. $\lim_{n \to +\infty} |B_n \cap V'|/B_n \cap V| = \rho > 0$, where $\{B_n\}_{n \in \mathbb{N}}$ is a sequence of boxes such that $B_n \supset B_{n-1}$ and $\bigcup_{n \geqq 0} B_n = V$. Let $K \in \mathbb{N}$ be such that $K(\sigma/2)\rho/2 > 1$. (Recall σ from the definition of τ_{v0}^1.) Then we can choose a box $B_2 \supset B_1$ such that τ_{v0}^2, defined in the next equation, satisfies

$$P(\tau_{v0}^2) = P(\tau_{v0}^1 \text{ occurs and all the branches of } v_0 \text{ contain at least } K \text{ vertices in } B_2 \setminus B_1) \geqq \sigma/2 > 0\ .$$

Now define $\tau_v^2 = T_v^{-1}(\tau_{v0}^2)$, for $v \in V$. Next consider a box $B_3 \supset B_2$ such that $\mathring{B}_3 = \{v \in B_3 \colon T_v(B_2) \in B_3\}$ satisfies

$$\frac{|\mathring{B}_3 \cap V'|}{|B_3|} \geqq \rho/2\ .$$

Such a box exists because of the positive density of V'. (*Remark.* We use here the fact that our graph (V, E) is subexponential: this is the reason why this proof cannot work for trees; see also Theorem 1′ below). The mean number of sites v in $\mathring{B}_3 \cap V'$ for which τ_v^2 occurs is $|\mathring{B}_3 \cap V'|\sigma/2$ and thus there must be at least this number with positive probability. Thus, τ^3, the event that this number exceeds $|B_3|(\sigma/2)\rho/2$, occurs with nonzero probability.

To apply Lemma 2 given below take $\eta \in \tau^3$ and let R be the set of $v \in \mathring{B}_3 \cap V'$ for which τ_v^2 occurs. To define S take $v \in R$ and let us remove v from C_v if it was contained in it; denote with the same symbol $C_v^{(i)}$ the branches of v intersected with B_3 and from one of which v has been removed (if it was contained in C_v). Now let $S = \bigcup_{v \in R}(C_v \cup \{v\})$. From the previous discussion, we have seen that condition (a) of Lemma 2 is satisfied when as subsets of S we take for each $v \in R$ the branches of v. When $v \sim w$ conditions (I) or (IV) of (b) are satisfied; otherwise, conditions (I), (III) or (III) of (b) are satisfied. Lemma 2 then implies that $|S| \geqq K(|R| + 2)$, since the definition of τ_v^2 implies that every branch contains at least K vertices in B_3. Therefore we can conclude that with P positive probability

$$|B_3 \cap S| \geqq K(|B_3|(\sigma/2)\rho/2 + 2) > |B_3| \,.$$

This, by contradiction, proves the theorem. □

The next lemma, which was used in the proof of Theorem 1, is a combinatorial one which extends Lemma 2 of Burton and Keane [4].

Lemma 2. *Given a set S and a finite subset $R \subset S$, suppose that*
(a) *for all $v \in R$ there exists a finite family $(C_v^{(1)}, C_v^{(2)}, \ldots, C_v^{(n_v)})$ of $n_v \geqq 3$ disjoint non-empty subsets of S not containing v;*
(b) *for all $v, w \in R$ one of the following cases occurs (where we define $C_w = \bigcup_i C_w^{(i)}$ for any $w \in R$):*
(I) $(\{v\} \cup C_v) \cap (\{w\} \cup C_w) = \emptyset$;
(II) *$\exists i$ s.t. $C_v^{(i)} \supset \{w\} \cup C_w$;*
(III) *$\exists j$ s.t. $C_w^{(j)} \supset \{v\} \cup C_v$;*
(IV) *$\exists i,j$ s.t. $C_v^{(i)} \supset \{w\} \cup C_w \setminus C_w^{(j)}$ and $C_w^{(j)} \supset \{v\} \cup C_v \setminus C_v^{(i)}$.*
Then $|S| \geqq (\min_{v \in R}(\min_i |C_v^{(i)}|))(|R| + 2)$.

Proof. First we observe that the assumptions imply that there exist $v \in R$ and $i \in \mathbb{N}$, $i \leqq n_v$, such that $C_v^{(i)} \cap R = \emptyset$. In fact, choose any $w_1 \in R$ and $j_1 \in \{1, 2, \ldots, n_{w_1}\}$. If $C_{w_1}^{(j_1)} \cap R = \emptyset$ take $v = w_1$ and $i = j_1$; otherwise consider $w_2 \in C_{w_1}^{(j_1)} \cap R$ and j_2 such that $C_{w_2}^{(j_2)} \subset C_{w_1}^{(j_1)}$. The existence of such a j_2 can be easily derived by (b); in fact for $w_2 \in C_{w_1}^{(j)}$ case (I) cannot occur and neither can case (II) with $w_2 = v$ and $w_1 = w$ because by the assumptions $w_2 \in C_{w_1}^{(j_1)} \subset C_{w_1}$ but $w_2 \notin C_{w_2}$ and thus $C_{w_1} \nsubseteq C_{w_2}^{(i)}$ for any i. Then clearly $|C_{w_2}^{(j_2)} \cap R| < |C_{w_1}^{(j_1)} \cap R|$ so that repeating this procedure we obtain v and i with the required properties.

Next take $R \setminus \{v\}$ and $S \setminus C_v^{(i)}$ and note that properties (a) and (b) still hold. In fact $R \setminus \{v\} \subset S \setminus C_v^{(i)}$ since $C_v^{(i)} \cap R = \emptyset$. Property (a) still holds since for $w \in R \setminus \{v\}$, of the previous family $\{C_w^{(1)}, C_w^{(2)}, \ldots, \}$ only one set, say $C_w^{(1)}$, may have changed into $C_w^{(1)} \setminus C_v^{(i)}$, but in this case $\exists j \neq i$ such that $C_w^{(1)} \supset C_v^{(j)}$ and thus $C_w^{(1)} \setminus C_v^{(i)} \neq \emptyset$. This follows easily from (b). Property (b) still holds since of the previous families related to two points w_1 and w_2 not equal to v no element has become empty and inclusion relations are unchanged under the transformation $C_w^{(j)} \to C_w^{(j)} \setminus C_v^{(i)}$.

Furthermore the proof that property (a) still holds shows that $\min_{v \in R}(\min_i |C_v^{(i)}|)$ can only have been increased. This results in an application of induction, noting that if $|R| = 1$, then $|S| \geqq 3\min_{v \in R}(\min_i C_v^{(i)})$. □

The next theorem is proved by essentially the same arguments used for Theorem 1; details are left to the reader. Neither Theorem 1 nor Theorem 1′ contains the other. We include Theorem 1′ because it covers some examples, such as percolation on the graphs of finitely generated groups, which may be of some interest.

Theorem 1′. *Let V be a countable set and let $E = V_2$, the set of all edges between pairs of elements of V. Let P be a probability measure on $\{0,1\}^E$ and let G be a set (or without loss of generality, a group) of bijections, $T: V \to V$, whose natural extensions to $\{0,1\}^E$ leave P invariant. Suppose there are elements T_* of G and 0 of V such that the following properties hold:*

(i) (*Finite energy*) *The set E', of edges $e = \{v_1, v_2\}$ such that P has finite energy for e, i.e.,*

$$0 < P(\eta_e = 1 \mid A_{E\setminus\{e\}}) < 1 \quad P\text{-almost everywhere},$$

and such that $\{T_*^n v_1, T_*^n v_2\} \neq \{v_1, v_2\}$ for all $n \geqq 1$, is large enough so that all points in V are connected by paths in V'.

(ii) (*Subexponential growth of volumes*) *There is an increasing sequence of finite subsets C_n of V containing D and converging to V such that for any K,*

$$\lim_{L\to\infty} \frac{|\{x \in V\colon \exists T \in G \text{ s.t. } T(0) = x \text{ and } T(C_K) \subseteq C_L\}|}{|C_L|} = 1,$$

where $|A|$ denotes the cardinality of A.

Then in the random graph defined by P, there is at most one infinite component with P probability one.

Remarks. 1. It is probably possible to write down a theorem sufficiently general to include both Theorems 1 and 1′ as special cases (and prove it). We shall spare the reader the agony of reading such a theorem by not writing one down.

2. We mention a simple consequence for connectedness of random graphs. Under the condition of Theorem 1 or Theorem 1′, if the random graph has the property that every vertex $v \in V$ is connected to infinitely many other vertices with probability one, then connectedness holds. This gives a different proof of the result in [16], as well as an extension of it to half spaces (see also [24]) and dependent measures.

3 Long-range models on $\mathbb{N}$

There has recently been some interest in long-range models on $\mathbb{N}$, where $V = \mathbb{N}$ and E is the set of all edges $\mathbb{N}_2$ [7, 19, 29]. In this specific case the proof of Theorem 1 and Theorem 1′ cannot be directly applied because the space is not invariant under an invertible map. Nevertheless it is possible to modify the proof to show that also for long-range models on $\mathbb{N}$ there is at most one infinite component. We only require that the probability measure has finite energy, which is a very natural assumption, and is invariant under translations. This second assumption rules out interesting cases [7, 29] in which a transition occurs from connection to non-connection of the graph, but it includes, for instance, the case when all edges $e \in E$

have independent probabilities $p_e = p_{\{v_1, v_2\}} = p_{|v_1 - v_2|}$ to be open and $\sum_{i \in \mathbb{N}} p_i = \infty$, which is studied in [19] with different techniques; the result proven there that the graph is totally connected with probability one is achieved here as an easy consequence of the next theorem, which more generally shows that the infinite component is unique.

Theorem 2. *Let $E = \mathbb{N}_2$ be the set of all edges between vertices in $\mathbb{N}$ and let P be a probability measure on $\{0, 1\}^E$. Suppose that:*

(a) *P is invariant under T_n, in the sense that $P(T_n^{-1}(A)) = P(A)$ for all events A, where T_n is the map on the σ-algebra generated by the cylinder sets in E induced by the map T_n: $T_n(m) = m + n$, for $n, m \in \mathbb{N}$.*

(b) *P has finite energy, i.e. for all $e \in E$*

$$1 > P(\eta_e = 1 | \mathscr{A}_{E \setminus \{e\}})(\eta_{E \setminus \{e\}}) > 0 \quad P\text{-almost everywhere },$$

where $\eta \in \{0, 1\}^E$, $\eta_{E \setminus \{e\}}$ is its restriction to $E \setminus \{e\}$ and $\mathscr{A}_{E \setminus \{e\}}$ is the σ-algebra generated by the cylinder sets in $E \setminus \{e\}$.
Then there is at most one infinite component with P probability one.

Proof. Since the result is achieved by adapting Theorem 1 we will only indicate the principal modifications. The main difference is that we now directly prove that there cannot be *two* or more disjoint infinite components with positive probability. So we start by supposing that there exist two vertices $v_1, v_2 \in \mathbb{N}$ such that

$$P(v_1 \text{ and } v_2 \text{ are in two distinct infinite components}) > 0 .$$

Finite energy allows us to assume now that with positive probability $\sigma > 0$ the origin v_0 is in two distinct branches $C_{v_0}^{(1)}$ and $C_{v_0}^{(2)}$; denote this event by τ_{v_0}.

For $n \in \mathbb{N}$, let $\tau_n = T_n^{-1}(\tau_{v_0})$ and suppose τ_n and τ_m occur for $n, m \in \mathbb{N}$, with $n < m$.

Note that by our definition of τ_n the two branches $C_n^{(1)}$ and $C_n^{(2)}$ (which are the τ_n analogues of $C_{v_0}^{(1)}$ and $C_{v_0}^{(2)}$) are infinite components of the graph restricted to $[n, \infty)$ and we think of them as not containing the vertex $n \in \mathbb{N}$ and being therefore completely disjoint. Then two possible cases can occur:
(1) $m \notin C_n = C_n^{(1)} \cup C_n^{(2)}$, in which ase $(C_m \cup \{m\}) \cap (C_n \cup \{n\}) = \emptyset$;
(2) $m \in C_n$, in which case there exists a branch of n, say $C_n^{(1)}$ such that $C_n^{(1)} \supset \{m\} \cup C_m$. Recall that $P(\tau_{v_0}) = \sigma > 0$ and let $K \in \mathbb{N}$ be such that $K(\sigma/2) > 1$. As in the proof of Theorem 1 we find sets B_2 and $B_3 \subset \mathbb{N}$ such that $\tau_{v_0}^2$, the event defined as

$$\tau_{v_0}^2 = \{\tau_{v_0} \text{ occurs and both branches } C_{v_0}^{(1)} \text{ and } C_{v_0}^{(2)} \text{ contain}$$
$$\text{at least } K \text{ vertices in } B_2\} ,$$

has $P(\tau_{v_0}^2) > \sigma/2$ and $\mathring{B}_3 = \{n \in B_3 \colon T_n(B_2) \subset B_3\}$ satisfies $|\mathring{B}_3|/|B_3| \geqq \frac{1}{2}$.

Let $\tau_n^2 = T_n^{-1}(\tau_{v_0}^2)$. The mean number of integers in $\mathring{B}_3$ for which τ_n^2 occurs is $|\mathring{B}|\sigma/2$ and thus there must be at least this number with positive probability. We want now to imitate Lemma 2 to show that in this case there are at least $|\mathring{B}_3|K\sigma/2$ vertices in B_3 and achieve a contradiction.

Consider the set $R \subset \mathring{B}_3$ of vertices for which τ_n^2 occurs and the set $S = \bigcup_{n \in R} (\{n\} \cup C_n \cap B_3\}$. Take $m, n \in R$ with, say, $n < m$. Note that (1) or (2) must occur. Then it is not difficult to see that

$$|S| \geqq \left(\min_{n \in R} \min_{i = 1, 2} |C_n^{(i)} \cap B_3| \right) \cdot (|R| + 1) \geqq K \cdot (|R| + 1) .$$

This is true if $|R| = 1$ and it can be shown by induction for all values of $|R|$. Indeed it is enough to take $m \in R$ such that there is a branch, say $C_m^{(1)}$, with the property $C_m^{(1)} \cap R = \emptyset$ and to note that for each element $n \in R \setminus \{m\}$ there are two branches in $S \setminus C_m^{(1)}$ satisfying (1) and (2). Application of induction follows again by observing that the procedure we just described does not decrease the minimum size of the branches left.

Since $|S| \geqq K|R| > |B_3|K\sigma/2 > |B_3|$ and $S \subset B_3$ we have achieved the required contradiction and proved the theorem. □

4 Spin glass models – random cluster models

We consider in this section dependent percolation models which are related to spin glasses. The interest in spin glass models is in the behavior of configurations of spins under a suitable Gibbs distribution, but as Fortuin and Kasteleyn have discovered [11, 12, 21] for Ising ferromagnets it is possible to obtain the Gibbs distribution for the spins (which are located at the vertices of a graph) by a construction which starts from a distribution of variables defined on the edges between spins. Following Kasai and Okiji [20] and Swendsen and Wang [30] (see also Edwards and Sokal [8] and Newman [26]), we start from the definition of such a distribution and we study some of its percolation properties as they are related, even if less directly than in the Ising ferromagnet, to the magnetic properties of the spin distribution.

Consider again a set of vertices V, which we assume to be $\mathbb{Z}^d$, and the set of edges $E = V_2$. We start from a finite box $B \subset V$, which we take of the form $B = \{x \in \mathbb{Z}^d \colon \|x\| \leqq k\}$, $k \in \mathbb{N}$, $\|\cdot\|$ being the sup-norm in $\mathbb{Z}^d$. Let E_B be the set of all edges between vertices of B. Let $J_B = \{a, f\}^{E_B}$, so that $j \in J_B$ is a prescription of the edges being ferromagnetic (f) or antiferromagnetic (a) and denote J_B by J when $B = \mathbb{Z}^d$; the choice of the values a's of f's is made independently for each edge with a and f equally likely. Once $j \in J$ is fixed we construct the random cluster measure on the edge variables $\eta \in H = \{0, 1\}^E$. Let therefore $\eta = \eta_B \in H_B = \{0, 1\}^{E_B}$ be a prescription of occupation variables for the edges of B. Once η is given the set of vertices of B can be split into maximal connected components (where the connection is once again through open edges) called *clusters* and we denote by $cl(\eta)$ the *number of these components*. The next step is the "coloring" of the vertices of B, which will produce a configuration $\omega \in \{-1, 1\}^B$. This step is carried out by assigning to each vertex a spin value $+1$ or -1 with the following prescription. If an edge e is closed, i.e. such that $\eta_e = 0$, then the spin variables ω can assume in the two vertices v_1 and v_2 any of the two values ± 1, but if the edge is open then $\omega_{v_1} = \omega_{v_2}$ if $j_e = f$ and $\omega_{v_1} \neq \omega_{v_2}$ if $j_e = a$. Note that this can lead to a contradiction for some choices of j and η, in which case we say that η is *frustrated*. Note also that if η is *unfrustrated*, i.e. it is possible to assign the spin variables without contradiction, then exactly two such assignments are possible. Finally, let us denote by $U(\eta) = U_j(\eta)$ a function being 1 if η is unfrustrated for the given j and 0 if η is frustrated. As usual we think here of two values of the spin variables, but results are valid when a different number is considered.

For a given j, the probability distribution on the edge variables will be

$$P^j_{\mathscr{P},B}(\eta) = \left(\prod_{e:\,\eta_e=1} p_e\right)\left(\prod_{e:\,\eta_e=0} (1-p_e)\right) 2^{cl(\eta)} U_j(\eta) Z^{-1}_{\mathscr{P},B}(j)\,,$$

where $\mathscr{P} = \{p_e\}_{e \in E}$, $0 \leqq p_e \leqq 1$, $j \in J_B$, $\eta \in H_B$ and $Z_{\mathscr{P},B}^{-1}(j)$ is a normalizing factor.

If $j = j_F \equiv f$ (i.e., in the totally ferromagnetic case) we have the Fortuin–Kasteleyn representation for the Ising model (see [26]) which is then obtained by independently and symmetrically coloring each cluster of B and letting $B \uparrow \mathbb{Z}^d$.

There are many possible choices for $\mathscr{P}$ and two of interest are:

(1) nearest-neighbor (n.n.) models:

$$p_e = \begin{cases} p & \text{if } \|e\| = 1 \\ 0 & \text{otherwise} \end{cases},$$

where $0 \leqq p \leqq 1$.

(2) long-range models: $p_e = p_{\|e\|}$, for example with $\lim_{x \to +\infty} x^2 p_x = \beta \in \mathbb{R}$.

To simplify the exposition we will limit outselves to n.n. models from now on, but it will be clear that similar computations, and in particular the uniqueness described in Corollary 2, can be obtained for long-range models as well. In the notations we will therefore replace $\mathscr{P}$ by the *single parameter p*.

We have not mentioned so far the boundary conditions we are taking. This amounts to fixing edge variables and spin values off B and affects the definition of $cl(\eta)$ as well as the ensuing coloring process. It is possible that for fixed values of the parameter p different boundary conditions can produce different weak limits when $B = B_n \uparrow \mathbb{Z}^d$ (i.e. when $B_n \subseteq B_{n+1}$ and $\mathbb{Z}^d = \bigcup_n B_n$). This is related to the existence of more than one (infinite volume) Gibbs distribution for the spin variables. We will generally take *periodic boundary conditions*; i.e. B will be a rectangle in $\mathbb{Z}^d$ treated as a torus (i.e. with "opposite" sites in B identified).

An important remark is that when we consider the joint distribution of j and η, any weak limit obtained with periodic boundary conditions will be invariant under all $\mathbb{Z}^d$-translations.

For a given j, a sufficient condition for the occurrence of more than one Gibbs distribution for the spin variables is that in some Gibbs distribution, $\mathrm{Cov}(\omega_0, \omega_x) \not\to 0$ as $\|x\| \to \infty$. We will consider this possibility for the Gibbs distributions that arise, from a translation invariant distribution of j and η obtained as described above, by conditioning on j and using the independent coloring process to construct ω from η. We will use a subscript p to keep track of the parameter p in the model (in physical terms, $p = 1 - \exp(-K/T)$, where T is the temperature and K is a positive constant, so that $p \to 1$ as $T \to 0$); thus conditioned on a fixed j, Cov_p will denote the covariance in the Gibbs distribution of ω and P_p^j will denote the corresponding distribution for η.

One of the results of Fortuin and Kasteleyn [11, 12, 21] is that in the totally ferromagnetic case

$$\mathrm{Cov}_p(\omega_0, \omega_x) = P_p^{j_F}\,(0 \text{ is connected to } x \text{ by a path of open edges})$$

$$=: P_p^{j_F}(A_{0,x}),$$

where $P_p^{j_F}$ is the weak limit of $P_{p,B}^{j_F}$ with periodic boundary conditions. In the general case we have only a one-sided result (see e.g. [26]):

$$|\mathrm{Cov}_p(\omega_0, \omega_x)| \leqq P_p^j(A_{0,x}).$$

Nevertheless, it is clear that percolation is necessary for $\mathrm{Cov}_p(\omega_0, \omega_x) \not\to 0$ as $\|x\| \to \infty$, and as a first partial result we study $P_p^j(A_{0,x})$, $j \in J$ and the behavior of the infinite components of open edges.

Since we are interested in properties which are true for almost all configurations $j \in J$ we consider the joint distribution $P_{p,B}$ of j and η variables for edges in E_B, which we call the (finite volume) *random interaction random cluster model.* As explained above, consider periodic boundary conditions and let P_p be the weak limit of P_{p,B_n} along a suitable subsequence $B_n \uparrow \mathbb{Z}^d$. Let P_p^{edge} be the marginal of P_p on the edge variables. The next theorem provides a lower bound to the conditional probability, under P_p^{edge}, that $\eta_e = +1$ given the η values for $e' \neq e$. As a consequence we obtain that for p large enough, in dimension $d \geqq 3$ (and in fact for the slab $\mathbb{Z}^2 \times \{0, 1\}$), there will be percolation of open edges under P_p^j for almost all $j \in J$ (Corollary 1). The lower bound implies also that the positive finite energy condition of Theorem 1 holds for P_p; we will therefore be able to conclude that the infinite cluster of open edges is unique P_p^j-almost always for almost all $j \in J$ (Corollary 2).

Theorem 3. *Let P_p^{edge} be the marginal on the edge variables of a weak limit P_p of finite volume random interaction random cluster models. Then for $e \in E$*

$$P_p^{\text{edge}}(\eta_e = +1 \,|\, \mathscr{A}_{E\setminus\{e\}})(\eta_{E\setminus\{e\}}) \geqq p/2 \,,$$

for P_p^{edge}-almost all $\eta_{E\setminus\{e\}} \in \{0, 1\}^{E\setminus\{e\}}$, where $A_{E\setminus\{e\}}$ is the σ-algebra generated by the variables in $E\setminus\{e\}$.

Proof. Let us call e' the edges in $E\setminus\{e\}$ and let $P_{p,B}$ be the finite volume measure with edges marginal $P_{p,B}^{\text{edge}}$ converging to P_p^{edge}. To prove the theorem it is enough to estimate, uniformly in B, the following:

$$(*) \qquad \min_{\bar{\eta}_e} \frac{P_{p,B}^{\text{edge}}(\eta_e = +1, \bar{\eta}_{e'})}{P_{p,B}^{\text{edge}}(\eta_e = 0, \bar{\eta}_{e'})} \,,$$

where $\bar{\eta}_e = \bar{\eta}_{e'}^B$ means a choice of $\eta_{e'}$ for $e' \in E\setminus\{e\} \cap B = E_B\setminus\{e\}$.

The explicit form of the measure $P_{p,B}^{\text{edge}}$ is

$$\begin{aligned} P_{p,B}^{\text{edge}}(\eta_e = +1, \bar{\eta}_{e'}) &= \sum_{j \in \{a, f\}^{E_B}} P_{p,B}(\eta_e = +1, \bar{\eta}_{e'}, j) \\ &= \sum_{j \in \{a, f\}^{E_B}} p^{N_1}(1-p)^{N_0} 2^{cl(\eta)} U_j(\eta) Z_p^{-1}(j) \,, \end{aligned}$$

where η is the configuration assuming the values $+1$ and $\bar{\eta}_{e'}$ in e and e' respectively, N_0 and N_1 are the numbers of edges $e'' \in E_B$ such that $\eta_{e''} = 0$ or 1 respectively.

To study $(*)$ we can thus estimate

$$(**) \qquad \min_{\bar{\eta}_{e'}, \bar{j}_{e'}} \frac{\sum_{j_e = a, f} P_{p,B}(\eta_e = +1, j_e, \bar{\eta}_{e'}, \bar{j}_{e'})}{\sum_{j_e = a, f} P_{p,B}(\eta_e = 0, j_e, \bar{\eta}_{e'}, \bar{j}_{e'})} \,,$$

where $\bar{j}_{e'}$ is a configuration of the j variables for $e' \in E_B\setminus\{e\}$.

Let us consider separately the four cases $\eta_e = 1, 0$ and $j_e = a, f$. There are still three different situations which can occur once $\bar{\eta}_{e'}$ and $\bar{j}_{e'}$ are given. Either v_1 and v_2, the two end-points of e_1, are not connected (I) by any path of edges in $\bar{\eta}_{e'}$ or they are; in this second case either $U_{j_e = f, \bar{j}_{e'}}(\eta_e = +1, \bar{\eta}_{e'}) = 0$ or $U_{j_e = a, \bar{j}_{e'}}(\eta_e = +1, \bar{\eta}_{e'}) = 0$, in other words if v_1 and v_2 are connected in $\bar{\eta}_{e'}$ then the edge variable η_e being $+1$ forces either $j_e = a$ (II) or $j_e = f$ (III).

This explains Table 1, which gives the value of $P_{p,B}(\eta_e, j_e, \bar{\eta}_{e'}, \bar{j}_{e'})$ for the indicated values of η_e and j_e and the three possible cases (I), (II) and (III). In each entry the factor C_i, $i = \mathrm{I, II, III}$ depends only on $\bar{j}_{e'}$, $\bar{\eta}_{e'}$ and is constant on each line and thus irrelevant in (**).

The normalizing factor $Z_p(j)$ depends on the entire j, but we denote it by $Z(a)$ or $Z(f)$ to mean that $j_e = a$ or f and $\bar{j}_{e'}$ is fixed.

We can evaluate (**) as

(***)
$$\min\left[\left(\frac{p}{2Z(f)} + \frac{p}{2Z(a)}\right)\frac{1}{(1-p)\left(\frac{1}{Z(f)} + \frac{1}{Z(z)}\right)},\right.$$
$$\left.\frac{p}{(1-p)}\frac{\frac{1}{Z(a)}}{\frac{1}{Z(a)} + \frac{1}{Z(f)}}, \frac{p}{(1-p)}\frac{1}{\frac{Z(f)}{Z(a)} + 1}\right]$$
$$= \min\left[\frac{p}{2(1-p)}, \frac{p}{(1-p)}\frac{1}{1 + \frac{Z(a)}{Z(f)}}, \frac{p}{(1-p)}\frac{1}{1 + \frac{Z(f)}{Z(a)}}\right].$$

Next it remains to evaluate the ratio $Z(a)/Z(f)$ and we follow the same scheme which led from (**) to Table 1.

Indeed writing $Z(a) = Z(j_e = a, \bar{j}_{e'})$ explicitly, separating the cases $\eta_e = +1$ and $\eta_e = -1$, and assuming $\bar{\eta}_{e'}$ as fixed we achieve the same computation as in Table 1 apart from the normalizing factors: this is summarized in Table 2.

We conclude that

$$\min_{\bar{j}_{e'}} \frac{Z(a)}{Z(f)} = (1-p), \qquad \max_{\bar{j}_{e'}} \frac{Z(a)}{Z(f)} = \frac{1}{1-p}.$$

Table 1

Case	$j_e = f$		$j_e = a$	
	$\eta_e = 0$	$\eta_e = 1$	$\eta_e = 0$	$\eta_e = 1$
I	$\frac{(1-p)}{Z(f)} C_{\mathrm{I}}$	$\frac{p}{2Z(f)} C_{\mathrm{I}}$	$\frac{(1-p)}{Z(a)} C_{\mathrm{I}}$	$\frac{p}{2Z(a)} C_{\mathrm{I}}$
II	$\frac{(1-p)}{Z(f)} C_{\mathrm{II}}$	0	$\frac{(1-p)}{Z(a)} C_{\mathrm{II}}$	$\frac{p}{Z(a)} C_{\mathrm{II}}$
III	$\frac{(1-p)}{Z(f)} C_{\mathrm{III}}$	$\frac{p}{Z(f)} C_{\mathrm{III}}$	$\frac{(1-p)}{Z(a)} C_{\mathrm{III}}$	0

Table 2

Case	$Z(a)$	$Z(f)$
I	$\frac{2(1-p)+p}{2}D_{\mathrm{I}}$	$\frac{2(1-p)+p}{2}D_{\mathrm{I}}$
II	$(1-p)D_{\mathrm{II}}$	D_{II}
III	D_{III}	$(1-p)D_{\mathrm{III}}$

We evaluate $(***)$ as $\min(p/2(1-p), p/(2-p)) = p/(2-p)$, which is also the minimum in $(*)$ and yields

$$P_{p,B}^{\mathrm{edge}}(\eta_e = +1 \mid \mathscr{A}_{E\setminus\{e\}})(\eta_{E\setminus\{e\}}) \geqq \frac{1}{1+\dfrac{2-p}{p}} = \frac{p}{2}$$

uniformly in B. □

The first consequence concerns the occurrence of percolation. Consider a finite box B and its edges E_B. It is not difficult to see that for two probability measures μ_1 and μ_2 on $\{0,1\}^{E_B}$ the inequality

$$\min_{\eta_{E\setminus\{e\}}} \mu_1(\eta_e = +1 \mid \mathscr{A}_{E_B\setminus\{e\}})(\eta_{E\setminus\{e\}}) \geqq \max_{\eta_{E\setminus\{e\}}} \mu_2(\eta_e = +1 \mid \mathscr{A}_{E_B\setminus\{e\}})(\eta_{E\setminus\{e\}})$$

implies that μ_1 stochastically (or FKG) dominates μ_2 in the sense that for any increasing function f on $\{0,1\}^{E_B}$ we have $\int f d\mu_1 \geqq \int f d\mu_2$ (see for instance Russo [28]).

Theorem 3 implies that $P_{p,B}^{\mathrm{edge}}$ stochastically dominates uniformly in B the Bernoulli measure $\mu_{p/2}$, where $p/2$ is the density of edge variable assuming the value $+1$; since for $d \geqq 3$ for bond percolation in $\mathbb{Z}^d$ (or for percolation in $\mathbb{Z}^2 \times \{0,1\}$) the density at which edge percolation occurs is strictly smaller than 1/2 (see [22]) we can now prove the following:

Corollary 1. *If $d \geqq 3$ and p is large enough then P_p^j (the origin is in an infinite cluster of open edges) > 0 for almost all $j \in J$, where P_p^j is the weak limit of $P_{p,B}^j$ along a sequence of boxes converging to $\mathbb{Z}^d$ for which $P_{p,B}$ weakly converges.*

Proof. From Theorem 3 and the previous remarks we know that P_p^{edge} (there exists an infinite cluster of open edges) $= 1$. The same holds for P_p and applying Fubini's theorem P_p^j (there exists an infinite cluster of open edges) $= 1$ for almost all $j \in J$. Additivity of the measure and finite energy yield the result. □

A second consequence of Theorem 3 is about uniqueness of the infinite cluster of open edges.

Corollary 2. *Let P_p^j be as in Corollary 1 in any dimension. Then P_p^j (there exists an unique infinite cluster of open edges) $= 1$ for almost all $j \in J$.*

Proof. We apply Theorem 1 to P_p^{edge} which is preserved by the full group of translations of $\mathbb{Z}^d$ and has (positive) finite energy on the full lattice E by Theorem 3. A subsequent application of Fubini's theorem yields the result. □

Acknowledgements. The authors would like to thank G. Grimmett and H. Kesten for useful discussion and suggestions

References

1. Aizenman, M., Newman, C.M., Kesten, H.: Uniqueness of the infinite cluster and continuity of connectivity functions for short and long range percolation. Commun. Math. Phys. **111**, 505–531 (1987)
2. Barsky, D.J., Grimmett, G.R., Newman, C.M.: Percolation in half spaces: equality of critical densities and continuity of the percolation probability. Probab. Theory Relat. Fields **90**, 111–148 (1991)
3. Bollobás, B.: Random graphs. London: Academic Press 1985
4. Burton, R.M., Keane, M.: Density and uniqueness in percolation. Commun. Math. Phys. **121**, 501–505 (1989)
5. Campanino, M., Russo, L.: An upper bound for the critical probability for the three-dimensional cubic lattice. Ann. Probab. **13**, 478–491 (1985)
6. Coniglio, A., Nappi, C.R., Peruggi, F., Russo, L.: Percolation and phase transition in the Ising model. Commun. Math. Phys. **51**, 315–323 (1976)
7. Durrett, R., Kesten, H.: The critical parameter for connectedness of some random graphs. In: Baker, A., Bollobas, B., Hajnal, A. (eds.) A tribute to Paul Erdös, pp. 161–176. Cambridge: Cambridge University Press 1990
8. Edwards, R.G., Sokal, A.D.: Generalization of the Fortuin-Kasteleyn-Swendsen-Wang representation and Monte Carlo algorithm. Phys. Rev. D **38**, 2009–2012 (1988)
9. Erdös, P., Rényi, A.: On random graphs I. Publ. Math. **6**, 290–297 (1959)
10. Fisher, M.E.: Critical probabilities for cluster size and percolation problems. J. Math. Phys. **2**, 620–627 (1961)
11. Fortuin, C.M.: On the random-cluster model III. The simple random-cluster model. Physica **59**, 545–570 (1972)
12. Fortuin, C.M., Kasteleyn, P.W.: On the random-cluster model. I. Introduction and relation to other models. Physica **57**, 536–564 (1972)
13. Gandolfi, A.: Uniqueness of the infinite cluster for stationary Gibbs states. Ann. Probab. **17**, 1403–1415 (1989)
14. Gandolfi, A., Grimmett, G., Russo, L.: On the uniqueness of the infinite cluster in the percolation model. Commun. Math. Phys. **114**, 549–552 (1988a)
15. Gandolfi, A., Keane, M.S., Russo, L.: On the uniqueness of the infinite occupied cluster in dependent two-dimensional site percolation. Ann. Probab. **16** (3), 1147–1157 (1988b)
16. Grimmett, G.R., Keane, M.S., Marstand, J.M.: On the connectedness of a random graph. Math. Proc. Camb. Philos. Soc. **96**, 151–166 (1984)
17. Grimmett, G.R., Newman, C.M.: Percolation in $\infty + 1$ dimensions. In: Grimmett, G., Welsh, D. (eds.). Hammersley Festschrift. Oxford University Press 1988
18. Harris, T.E.: A lower bound for the critical probability in a certain percolation process. Proc. Camb. Philos. Soc. **56**, 13–20 (1960)
19. Kalikow, S., Weiss, B.: When are random graphs connected. (1988) preprint.
20. Kasai, Y., Okiji, A.: Percolation problem describing $\pm J$ Ising spin glass system. Prog. Theor. Phys. **79**, 1080–1094 (1988)
21. Kasteleyn, P.W., Fortuin, C.M.: Phase transitions in lattice systems with random local properties. J. Phys. Soc. Japan **26**, 11–14 (1969)
22. Kesten, H.: Percolation theory for mathematicians. Boston Basel Stuttgart: Birkhäuser 1982
23. Kesten, H.: Correlation length and critical probabilities of slabs for percolation. (Preprint, 1988)
24. Kesten, H.: Connectivity of certain graphs on halfspaces, quarter spaces. Proceedings Probability Conference Singapore 1989 (to appear)
25. Meester, R.W.J.: An algorithm for calculating critical probabilities and percolation functions in percolation models defined by rotations. Ergodic Theory Dyn. Syst. **9**, 495–509 (1989)
26. Newman, C.M.: Ising models and dependent percolation. In: Block, H.W., Sampson, A.R., Savits, T.H. (eds.) Topics in statistical dependence (IMS Lect. Notes – Monograph Series, vol. 16, pp. 395–401
27. Newman, C.M., Schulman, L.S.: Infinite clusters in percolation models. J. Stat. Phys. **26**, (3) 613–628 (1981)
28. Russo, L.: An approximate zero-one law. Z. Wahrscheinlichkeitstheor. Verw. Geb. **61**, 129–139 (1982)
29. Shepp, L.A.: Connectedness of certain random graphs. Jsr. J. Math. **67**, 23–33 (1989)
30. Swendsen, R.H., Wang, J.S.: Nonuniversal critical dynamics in Monte Carlo simulations. Phys. Rev. Lett. **58**, 86–88 (1987)