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State space formulation to viscoelastic fluid flow of magnetohydrodynamic free convection through a porous medium

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Summary. In this work we formulate the state space approach for one-dimensional problems of viscoelastic magnetohydrodynamic unsteady free convection flow through a porous medium past an infinite vertical plate. Laplace transform techniques are used. The resulting formulation is applied to a thermal shock problem and to a problem for the flow between two parallel fixed plates both without heat sources. Also a problem with a distribution of heat sources is considered. A numerical method is employed for the inversion of the Laplace transforms. Numerical results are given and illustrated graphically for the problem considered.

Notation

C_ϱ	specific heat at constant pressure
g	acceleration due to gravity
ϱ	density
t'	time
u'	velocity component parallel to the plate
H_x'	induced magnetic field
x', y'	coordinates system
T'	temperature distribution
T_0'	temperature of the plate
T'_∞	temperature of the fluid away from the plate
μ_0	limiting viscosity at small rates to shear
$v_0{}^*$	μ/ϱ
v_m	magnetic diffusivity
α	Alfven velocity
β^*	coefficient of volume expansion
λ	thermal conductivity
λ^*	thermal diffusivity
G	Grashof number
Pr	Prandtl number
L	some characteristic length
k_0	the elastic constant
K'	permeability of the porous medium

1 Introduction

The study of viscoelastic fluids has become of increasing importance in the last few years. This is mainly due to their many applications in petroleum drilling, manufacturing of foods and paper, and many other similar activities. The boundary-layer concept for such fluids is of special importance owing to its application to many engineering problems, among which we cite the possibility of reducing frictional drag on the hulls of ships and submarines.

Flow through a porous medium in the presence of a magnetic field is of special importance due to its application to many scientific and engineering problems [1].

Yamamoto and Iwamura [2] investigated the flow streaming into a porous and permeable medium with an arbitrary smooth surface. Rudraiah and Prabhamani [3] studied the effect of thermal diffusion on convective viscous fluid flow in a porous medium. Straus [4] and Schubert and Straus [5] studied convection in porous media.

Walters [6] and Beard and Walters [7] deduced the governing equations for the boundary layer flow for a prototype viscoelastic fluid which they have designated as liquid B' when this liquid has a very short memory. Many other authors have contributed to the subject. Soundalgekar et al. [8] have studied the behaviour of an oscillating flow past an infinite porous plate with mass transfer. Raptis et al. [9]–[11] have investigated the free convection and mass transfer flow of a viscous and viscoelastic fluid past a vertical wall. Singh and Singh [12] have studied the magnetohydrodynamic flow of a viscoelastic fluid past an accelerated plate. The response of laminar skin friction, temperature and heat transfer to the fluctuations in the stream velocity in the presence of a transverse magnetic field has been discussed by Sherief and Ezzat [13].

In most of the above applications, the method of solution developed by Lighthill [14] and Stuart [15] is utilized. This method is applicable only to problems of simple harmonic vibrations. This prompted many authors to use other methods of solution when dealing with the problems of a nonvibrating fluid. Gupta [16] and Riley [17] have used an approximate Pohlhausen method, Wilks and Hunt [18] have used the method of similarity solution. Saponkoff [19] and Vajravelu and Sastri [20] have used perturbation methods to solve problems of free convection in hydromagnetic flows.

In the above-mentioned works the effect of the induced magnetic field was neglected.

The main objective of this work is to investigate the free convection flow of an electrically conducting viscoelastic fluid (liquid B') past an infinite flat plate subject to a transverse magnetic field when we take into account the effect of the induced magnetic field. The solution is obtained using a method proposed by Ezzat [21], [22] in hydromagnetic free convection flows.

In this approach, the governing equations are written in matrix form using a state vector that consists of the Laplace transforms in time of the velocity, the induced magnetic field, temperature, and their gradients. Their integration, subjected to zero initial conditions, is carried out by means of the matrix exponential method. Influence functions in the Laplace transform domain are explicitly developed.

The inversion of the Laplace transform is carried out using a numerical technique [23].

2 Formulation of the problem

We investigate the free convective heat transfer in an incompressible viscoelastic hydromagnetic flow past an infinite vertical porous plate. The x'-axis is taken along the plate in the direction of the flow and the y'-axis normal to it. Let u' be the component of the velocity in the x' direction and H_0 be the strength of a constant magnetic field in the y' direction. All the fluid properties are assumed constant except that the influence of the density variation with temperature is considered only in the body force term. In the energy equation, terms representing viscous and Joule's dissipation are neglected as they are assumed to be very small in free convection flows [24]. Also in the energy equation, the term representing the volumetric heat source is taken as a function of the space variables. With these assumption, the equations that govern unsteady one-dimensional free convection flow in an incompressible viscoelastic conducting fluid through a porous medium bounded by an infinite non-magnetic vertical plate in the presence of a constant magnetic field are [7]:

$$\frac{\partial u'}{\partial t'} = g\beta^*(T' - T'_\infty) + \nu_0{}^* \frac{\partial^2 u'}{\partial y^2} - k_0{}' \frac{\partial^3 u'}{\partial y^{2\prime}\, \partial t'} - \frac{\nu_0{}^*}{K'}\, u + \frac{\alpha'^2}{H_0}\left(\frac{\partial H_x{}'}{\partial y'}\right), \tag{1}$$

$$\frac{\partial H_x{}'}{\partial t'} = \nu_m \frac{\partial^2 H_x{}'}{\partial y'^2} + H_0 \frac{\partial u'}{\partial y'}, \tag{2}$$

$$\frac{\partial T'}{\partial t'} = \lambda^* \frac{\partial^2 T'}{\partial y'^2} + \frac{Q'}{\lambda}. \tag{3}$$

Let us introduce the following non-dimensional variables

$$y = \frac{y'}{L}, \quad t = \frac{\nu_0{}^* t'}{L^2}, \quad u = \frac{Lu'}{\nu_0{}^*}, \quad \Pr = \frac{\nu_0{}^*}{\lambda^*},$$

$$H_x = \frac{H_x{}'}{H_0}, \quad K = \frac{K'}{L^2}, \quad \theta = \frac{T' - T'_\infty}{T_0{}' - T'_\infty}, \quad Q = \frac{L^2 Q'}{\lambda(T_0{}' - T'_\infty)\, \varrho C_p}, \tag{4}$$

$$\mathrm{G} = \frac{g\beta^* L^3 (T_0{}' - T'_\infty)}{\nu_0{}^{*2}}, \quad \alpha = \frac{\alpha' L}{\nu_0{}^*}, \quad k_0 = \frac{k_0{}'}{L^2}.$$

In view of the transformation equations (4), (1), (2), and (3) yield

$$\left(k_0 \frac{\partial^3}{\partial y^2\, \partial t} - \frac{\partial^2}{\partial y^2} + \frac{\partial}{\partial t} + \frac{1}{K}\right) u = \mathrm{G}\, \theta + \alpha^2 \frac{\partial H_x}{\partial y}, \tag{5}$$

$$\left(\frac{\partial^2}{\partial y^2} - b \frac{\partial}{\partial t}\right) H_x = -b \frac{\partial u}{\partial y}, \tag{6}$$

$$\left(\frac{\partial^2}{\partial y^2} - \Pr \frac{\partial}{\partial t}\right) \theta = -Q, \tag{7}$$

where $b = \nu_0{}^*/\nu_m$.

We shall also assume that the initial state of the medium is quiescent. Taking the Laplace transform, defined by the relation

$$\bar{f}(s) = \int_0^\infty e^{-st} f(t)\, dt,$$

of both sides of Eqs. (5), (6) and (7), we obtain

$$\left(a \frac{\partial^2}{\partial y^2} - s - \frac{1}{K}\right) \bar{u} = -\mathrm{G}\, \bar{\theta} - \alpha^2 \frac{\partial \bar{H}_x}{\partial y}, \tag{8}$$

$$\left(\frac{\partial^2}{\partial y^2} - bs\right) \bar{H}_x = -b \frac{\partial \bar{u}}{\partial y}, \tag{9}$$

$$\left(\frac{\partial^2}{\partial y^2} - \Pr s\right) \bar{\theta} = -\bar{Q}, \tag{10}$$

where $a = 1 - k_0 s$.

We shall choose as state variables the temperature increment θ, the velocity component u, the induced magnetic field H_x and their gradients. Equations (8), (9) and (10) can be written as

$$\frac{\partial\bar{\theta}}{\partial y} = \bar{\theta}', \tag{11}$$

$$\frac{\partial\bar{u}}{\partial y} = \bar{u}', \tag{12}$$

$$\frac{\partial\bar{H}_x}{\partial y} = \bar{H}_x', \tag{13}$$

$$\frac{\partial\bar{\theta}'}{\partial y} = \mathrm{Pr}\, s\bar{\theta} - \bar{Q}, \tag{14}$$

$$\frac{\partial\bar{u}'}{\partial y} = \frac{1}{a}\left(\left(s + \frac{1}{K}\right)\bar{u} - \mathrm{G}\,\bar{\theta} - \alpha^2\bar{H}_x'\right), \tag{15}$$

$$\frac{\partial H_x'}{\partial y} = bsH_x - b\bar{u}'. \tag{16}$$

The above equations can be written in matrix form as

$$\frac{d\bar{V}(y,s)}{dy} = A(s)\,\bar{V}(y,s) + B(y,s), \tag{17}$$

where

$$\bar{V}(y,s) = \begin{bmatrix} \bar{\theta}(y,s) \\ \bar{u}(y,s) \\ \bar{H}_x(y,s) \\ \bar{\theta}'(y,s) \\ \bar{u}'(y,s) \\ \bar{H}_x'(y,s) \end{bmatrix},$$

$$A(s) = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \mathrm{Pr}\, s & 0 & 0 & 0 & 0 & 0 \\ \dfrac{-\mathrm{G}}{a} & \dfrac{\left(s + \dfrac{1}{K}\right)}{a} & 0 & 0 & 0 & \dfrac{-\alpha^2}{a} \\ 0 & 0 & bs & 0 & -b & 0 \end{bmatrix},$$

$$B(y,s) = -\bar{Q}(y,s)\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

The formal solution of Eq. (17) can be expressed as

$$\bar{V}(y, s) = \exp\,[A(s)\,y]\left(\bar{V}(0, s) + \int_0^y \exp\,[-A(s)\,z]\,B(z, s)\,dz\right). \tag{18}$$

In the special case when there is no heat source acting inside the medium, Eq. (18) simplifies to

$$\bar{V}(y, s) = \exp\,[A(s)\,y]\,\bar{V}(0, s). \tag{19}$$

The characteristic equation of the matrix **A(s)** is

$$ak^6 - a_{11}k^4 + a_{21}k^2 - a_{31} = 0, \tag{20}$$

where

$$a_{11} = \left(s + \frac{1}{K}\right) + abs + b\alpha^2 + a\,\mathrm{Pr}\,s,$$

$$a_{21} = s\left[ab\,\mathrm{Pr}\,s + \alpha^2 b\,\mathrm{Pr} + (b + \mathrm{Pr})\left(s + \frac{1}{K}\right)\right],$$

$$a_{31} = \mathrm{Pr}\,bs^2\left(s + \frac{1}{K}\right).$$

The roots $\pm k_1$, $\pm k_2$, and $\pm k_3$, of Eq. (20) satisfy the relations

$$k_1{}^2 + k_2{}^2 + k_3{}^2 = \frac{a_{11}}{a},$$

$$k_1{}^2k_2{}^2 + k_2{}^2k_3{}^2 + k_3{}^2K_1{}^2 = \frac{a_{21}}{a},$$

$$k_1{}^2k_2{}^2k_3{}^2 = \frac{a_{31}}{a}.$$

One of the roots, say $k_1{}^2$, has a simple expression given by

$$k_1{}^2 = \mathrm{Pr}\,s. \tag{21}$$

The other two roots $k_2{}^2$ and $k_3{}^2$ satisfy the relation

$$k_2{}^2 + k_3{}^2 = \frac{a \cdot b \cdot s + \left(s + \frac{1}{K}\right) + b\alpha^2}{a}, \tag{22.1}$$

$$k_2{}^2k_3{}^2 = \frac{bs\left(s + \frac{1}{K}\right)}{a}. \tag{22.2}$$

The MacLaurin series expansion of $\exp\,[A(s)\,y]$ is given by

$$\exp\,[A(s)\,y] = \sum_{n=0}^{\infty} \frac{[A(s)\,y]^n}{n!}.$$

Using the Cayley-Hamilton theorem, the infinite series can be truncated to the following form

$$\exp\left[A(s)\,y\right] = L(s, y) = a_0 I + a_1 A + a_2 A^2 + a_3 A^3 + a_4 A^4 + a_5 A^5, \tag{23}$$

where I is the unit matrix of order 6 and $a_0 - a_5$ are some parameters depending on s and y. The characteristic roots $\pm k_1$, $\pm k_2$, and $\pm k_3$ of the matrix A must satisfy the equations

$$\exp(k_1 y) = a_0 + a_1 k_1 + a_2 k_1^2 + a_3 k_1^3 + a_4 k_1^4 + a_5 k_1^5,$$

$$\exp(-k_1 y) = a_0 - a_1 k_1 + a_2 k_1^2 - a_3 k_1^3 + a_4 k_1^4 - a_5 k_1^5,$$

$$\exp(k_2 y) = a_0 + a_1 k_2 + a_2 k_2^2 + a_3 k_2^3 + a_4 k_2^4 + a_5 k_2^5,$$

$$\exp(-k_2 y) = a_0 - a_1 k_2 + a_2 k_2^2 - a_3 k_2^3 + a_4 k_2^4 - a_5 k_2^5,$$

$$\exp(k_3 y) = a_0 + a_1 k_3 + a_2 k_3^2 + a_3 k_3^3 + a_4 k_3^4 + a_5 k_3^5,$$

$$\exp(-k_3 y) = a_0 - a_1 k_3 + a_2 k_3^2 - a_3 k_3^3 + a_4 k_3^4 - a_5 k_3^5.$$

The solution of this system of linear equations is given by

$$\begin{aligned}
a_0 &= -F(k_2^2 k_3^2 C_1 + k_1^2 k_3^2 C_2 + k_2^2 k_1^2 C_3),\\
a_1 &= -F(k_2^2 k_3^2 S_1 + k_3^2 k_1^2 S_2 + k_1^2 k_2^2 S_3),\\
a_2 &= F[(k_2^2 + k_3^2)\,C_1 + (k_3^2 + k_1^2)\,C_2 + (k_1^2 + k_2^2)\,C_3],\\
a_3 &= F[(k_2^2 + k_3^2)\,S_1 + (k_3^2 + k_1^2)\,S_2 + (k_1^2 + k_2^2)\,S_3],\\
a_4 &= -F(C_1 + C_2 + C_3),\\
a_5 &= -F(S_1 + S_2 + S_3),
\end{aligned} \tag{24}$$

where

$$F = \frac{1}{(k_1^2 - k_2^2)\,(k_2^2 - k_3^2)\,(k_3^2 - k_1^2)},$$

$$C_1 = (k_2^2 - k_3^2)\cosh(k_1 y), \qquad S_1 = \frac{(k_2^2 - k_3^2)}{k_1}\sinh(k_1 y),$$

$$C_2 = (k_3^2 - k_1^2)\cosh(k_2 y), \qquad S_2 = \frac{(k_3^2 - k_1^2)}{k_2}\sinh(k_2 y),$$

$$C_3 = (k_1^2 - k_2^2)\cosh(k_3 y), \qquad S_3 = \frac{(k_1^2 - k_2^2)}{k_3}\sinh(k_3 y).$$

Substituting for the parameters $a_0 - a_5$ from Eq. (24) into Eq. (23) and computing A^2, A^3, A^4, and A^5, we get the elements ($L_{i,j}$, $i, j = 1, 2, 3, 4, 5, 6$) of the matrix $L(s, y)$ to be

$$L_{11} = F(k_1^2 - k_2^2)\,(k_3^2 - k_1^2)\,C_1,$$

$$L_{12} = L_{13} = 0,$$

$$L_{14} = F(k_1^2 - k_2^2)\,(k_3^2 - k_1^2)\,S_1,$$

$$L_{15} = L_{16} = 0,$$

$$L_{21} = \frac{G}{a} F[(k_1{}^2 - bs)\, C_1 + (k_2{}^2 - bs)\, C_2 + (k_3{}^2 - bs)\, C_3],$$

$$L_{22} = \frac{F}{a}\left[(k_1{}^2 - k_2{}^2)\left(\left(s + \frac{1}{K}\right) - ak_3{}^2\right) C_2 + (k_1{}^2 - k_3{}^2)\left(\left(s + \frac{1}{K}\right) - ak_2{}^2\right) C_3\right],$$

$$L_{23} = \frac{bs\alpha^2}{a} F[(k_2{}^2 - k_1{}^2)\, S_2 + (k_3{}^2 - k_1{}^2)\, S_3],$$

$$L_{24} = \frac{G}{a} F[(k_1{}^2 - bs)\, S_1 + (k_2{}^2 - bs)\, S_2 + (k_3{}^2 - bs)\, S_3],$$

$$L_{25} = F[(k_2{}^2 - bs)\,(k_1{}^2 - k_2{}^2)\, S_2 + (k_3{}^2 - bs)\,(k_1{}^2 - k_3{}^2)\, S_3],$$

$$L_{26} = \frac{\alpha^2}{a} F[(k_2{}^2 - k_1{}^2)\, C_2 + (k_3{}^2 - k_1{}^2)\, C_3],$$

$$L_{31} = -\frac{G\, b}{a} F[k_1{}^2 S_1 + k_2{}^2 S_2 + k_3{}^2 S_3],$$

$$L_{32} = \frac{b}{a}\left(s + \frac{1}{K}\right) F[(k_2{}^2 - k_1{}^2)\, S_2 + (k_3{}^2 - k_1{}^2)\, S_3], \tag{25}$$

$$L_{33} = F[(bs - k_3{}^2)\,(k_1{}^2 - k_2{}^2)\, C_2 + (bs - k_2{}^2)\,(k_1{}^2 - k_3{}^2)\, C_3],$$

$$L_{34} = -\frac{G}{a} bF[C_1 + C_2 + C_3],$$

$$L_{35} = bF[(k_2{}^2 - k_1{}^2)\, C_2 + (k_3{}^2 - k_1{}^2)\, C_3],$$

$$L_{36} = \frac{F}{a}\left[\left(ak_2{}^2 - \left(s + \frac{1}{K}\right)\right)(k_1{}^2 - k_2{}^2)\, S_2 + \left(ak_3{}^2 - \left(s + \frac{1}{K}\right)\right)(k_1{}^2 - k_3{}^2)\, S_3\right],$$

$$L_{41} = -Fk_1{}^2(k_1{}^2 - k_2{}^2)\,(k_1{}^2 - k_3{}^2)\, S_1,$$

$$L_{42} = L_{43} = 0,$$

$$L_{44} = -F(k_1{}^2 - k_2{}^2)\,(k_1{}^2 - k_3{}^2)\, C_1,$$

$$L_{45} = L_{46} = 0,$$

$$L_{51} = \frac{G}{a} F[k_1{}^2(k_1{}^2 - bs)\, S_1 + k_2{}^2(k_2{}^2 - bs)\, S_2 + k_3{}^2(k_3{}^2 - bs)\, S_3],$$

$$L_{52} = \frac{F}{a}\left(s + \frac{1}{K}\right)[(K_1{}^2 - k_2{}^2)\,(k_2{}^2 - bs)\, S_2 + (k_1{}^2 - k_3{}^2)\,(k_3{}^2 - bs)\, S_3],$$

$$L_{53} = \frac{s}{a}\alpha^2 L_{35},$$

$$L_{54} = L_{21},$$

$$L_{55} = F[(k_1{}^2 - k_2{}^2)\,(k_2{}^2 - bs)\, C_2 + (k_1{}^2 - k_3{}^2)\,(k_3{}^2 - bs)\, C_3],$$

$$L_{56} = \frac{\alpha^2}{a} F[k_2{}^2(k_2{}^2 - k_1{}^2)\, S_2 + k_3{}^2(k_3{}^2 - k_1{}^2)\, S_3],$$

$$L_{61} = -\frac{G}{a} bF[k_1{}^2c_1 + k_2{}^2c_2 + k_3{}^2c_3],$$

$$L_{62} = \frac{b}{\alpha^2}\left(s + \frac{1}{K}\right) L_{26},$$

$$L_{63} = bsL_{36},$$

$$L_{64} = L_{31},$$

$$L_{65} = \frac{ba}{\alpha^2} L_{56},$$

$$L_{66} = \frac{F}{a}\left[(k_1{}^2 - k_2{}^2)\left(ak_2{}^2 - \left(s + \frac{1}{K}\right)\right) C_2 + (k_1{}^2 - k_3{}^2)\left(ak_3{}^2 - \left(s + \frac{1}{K}\right)\right) C_3\right].$$

It should be noted here that we have used Eqs. (21) and (22) repeatedly in order to write these entries in the simplest possible form. It should also be noted that this is a formal expression for the matrix exponential. In the physical problem $0 \leqq y < \infty$, we should suppress the positive exponentials which are unbounded at infinity. Thus we should replace each $\sinh(ky)$ by $-1/2\exp(-ky)$ and each $\cosh(ky)$ by $1/2\exp(-ky)$.

It is now possible to solve a broad class of problems of magnetohydrodynamic free convection flow in the Laplace transform domain.

3 Applications

Problem 1: thermal shock problem

We shall consider the free convection flow of an incompressible viscoelastic fluid in the presence of a magnetic field occupying a semi-infinite region $y \geqq 0$ of the space bounded by an infinite vertical plate $y = 0$, with the condition

$$u(0, t) = 0, \qquad H_x(0, t) = 0. \tag{26}$$

We assume that a thermal shock of the form

$$\theta(0, t) = \theta_0 H(t) \tag{27}$$

is applied to the plate at time $t = 0$, where θ_0 is a constant and $H(t)$ is Heaviside unit step function. All initial conditions are assumed to be zero.

We now apply the state space approach described above to this problem. The three components of the transformed initial state vector $\bar{V}(0, s)$ are known, namely

$$\bar{\theta}(0, s) = \frac{\theta_0}{s}, \tag{28}$$

$$\bar{u}(0, s) = 0, \tag{29}$$

$$\bar{H}(0, s) = 0. \tag{30}$$

In order to obtain the remaining three components $\bar{\theta}'(0, s)$, $\bar{u}'(0, s)$ and $\bar{H}'(0, s)$, we substitute $\mathbf{y = 0}$ into Eqs. (25) and (19), to obtain the following linear system of equations in the unknowns $\bar{\theta}'(0, s)$, $\bar{u}'(0, s)$ and $\bar{H}'(0, s)$

$$\bar{\theta}'(0, s) = L_{41} \frac{\theta_0}{s} + L_{44}\bar{\theta}'$$

$$\bar{u}'(0, s) = L_{51} \frac{\theta_0}{s} + L_{54}\bar{\theta}' + L_{55}\bar{u}' + L_{56}\bar{H}' \tag{31}$$

$$\bar{H}'(0, s) = L_{61} \frac{\theta_0}{s} + L_{62}\bar{\theta}' + L_{65}\bar{u}' + L_{66}\bar{H}'.$$

Solving system (31), we arrive at

$$\bar{\theta}'(0, s) = -\frac{k_1}{s} \theta_0, \tag{32}$$

$$\bar{u}'(0, s) = -\frac{\mathrm{G}\,\theta_0}{\Gamma} [b_1 k_1 A_1 + b_2 k_2 A_2 + b_3 k_3 A_3], \tag{33}$$

$$\bar{H}'(0, s) = -\frac{b\,\mathrm{G}\,\theta_0}{\Gamma} [k_1{}^2 A + k_2{}^2 A_2 + k_3{}^2 A_3], \tag{34}$$

where

$$b_1 = (k_1{}^2 - bs), \quad b_2 = (k_2{}^2 - bs),$$

$$b_3 = (k_3{}^2 - bs), \quad A_1 = (b_3 k_2 - b_2 k_3),$$

$$A_2 = (b_1 k_3 - b_3 k_1), \quad A_3 = (b_2 k_1 - b_1 k_2),$$

$$\Gamma = s(b_2 k_3 - b_3 k_2)(k_1{}^2 - k_2{}^2)(k_1{}^2 - k_3{}^2).$$

Equations (21) and (22) were used again to simplify the forms (32), (33), and (34).

Finally substituting the above value into Eqs. (19), we obtain the solution of the problem in the transformed domain as

$$\bar{\theta}(y, s) = \frac{\theta_0}{s} \exp(-k_1 y), \tag{35}$$

$$\bar{u}(y, s) = \frac{\mathrm{G}\,\theta_0}{\Gamma} [A_1 b_1 \exp(-k_1 y) + A_2 b_2 \exp(-k_2 y) + A_3 b_3 \exp(-k_3 y)], \tag{36}$$

$$\bar{H}(y, s) = \frac{b\,\mathrm{G}\,\theta_0}{\Gamma} [k_1 A_1 \exp(-k_1 y) + k_2 A_2 \exp(-k_2 y) + k_3 A_3 \exp(-k_3 y)]. \tag{37}$$

Problem 2: the flow between two parallel fixed plates

Consider an incompressible viscoelastic fluid in the presence of a magnetic field occupying the region $0 \leqq y \leqq h$ bounded by two vertical fixed plates. The mechanical boundary conditions can be written as

$$u(0, s) = 0, \quad \text{or} \quad \bar{u}(0, s) = 0, \tag{38}$$

$$u(h,t)=0,\quad \text{or}\quad \bar{u}(h,s)=0, \tag{39}$$

$$H_x(0,t)=0,\quad \text{or}\quad \bar{H}_x(0,s)=0, \tag{40}$$

$$H_x(h,t)=0,\quad \text{or}\quad \bar{H}_x(h,s)=0. \tag{41}$$

The thermal boundary conditions are assumed to be

$$\theta(0,t)=\theta_0 H(t),\quad \text{or}\quad \bar{\theta}(0,s)=\frac{\theta_0}{s}, \tag{42}$$

$$\theta'(h,t)=0,\quad \text{or}\quad \bar{\theta}'(h,s)=0. \tag{43}$$

Condition (42) means that the plate $y=0$ is acted on by a constant thermal shock at time $t=0$, while condition (43) signifies that the plate $y=h$ is thermally insulated.

Equations (38), (40), and (42) give three components of the initial state vector $\bar{V}(0,s)$. To obtain the remaining three components, we use (19) between $y=0$ and $y=h$ to obtain the following three simultaneous linear equations

$$\begin{aligned}
0&=L_{41}(h,s)\,\frac{\theta_0}{s}+L_{44}(h,s)\,\bar{\theta}'(0,s),\\
0&=L_{21}(h,s)\,\frac{\theta_0}{s}+L_{24}(h,s)\,\bar{\theta}'(0,s)+L_{25}(h,s)\,\bar{u}'(0,s)+L_{26}(h,s)\,\bar{H}'(0,s),\\
0&=L_{31}(h,s)\,\frac{\theta_0}{s}+L_{34}(h,s)\,\bar{\theta}'(0,s)+L_{35}(h,s)\,\bar{u}'(0,s)+L_{36}(h,s)\,\bar{H}_x'(0,s).
\end{aligned} \tag{44}$$

The solution of these equations gives

$$\bar{\theta}'(0,s)=-\frac{\theta_0}{s}\,k_1\tanh(k_1h), \tag{45}$$

$$\begin{aligned}
\bar{u}'(0,s)=F_1[&\Gamma_1k_1b_1\sinh k_1h-B_1b_2k_2\sinh k_2h-B_2b_2k_2\cosh k_2h\\
&+B_4k_2b_2-B_5k_3b_3\sinh k_3h-B_6k_3b_3\cosh k_3h+B_8k_3b_3],
\end{aligned} \tag{46}$$

$$\begin{aligned}
\bar{H}_x'(0,s)=bF_1[&\Gamma_1k_1{}^2\cosh k_1h-B_1k_2{}^2\cosh k_2h-B_2k_2{}^2\sinh k_2h-B_3k_2{}^2\\
&-B_5k_3{}^2\cosh k_3h-B_6k_3{}^2\sinh k_3h-B_7k_3{}^2],
\end{aligned} \tag{47}$$

where

$$F_1=F\,\frac{\mathrm{G}\,\theta_0(k_2{}^2-k_3{}^2)}{\Gamma_1 s\cosh k_1h},$$

$$\Gamma_1=2b_2b_3k_2k_3(\cosh k_2h\cosh k_3h-1)-(b_2{}^2k_3{}^2+b_3{}^2k_2{}^2)\sinh k_2h\sinh k_3h,$$

$$B_1=b_3k_2[b_1k_3(\cosh k_1h\cosh k_3h-1)-b_3k_1\sinh k_1h\sinh k_3h],$$

$$B_2=b_2k_3[b_3k_1\sinh k_1h\cosh k_3h-b_1k_3\sinh k_3h\cosh k_1h],$$

$$B_3=b_1b_3k_2k_3(\cosh k_3h-\cosh k_1h),$$

$B_4 = b_2 k_3 [b_3 k_1 \sinh k_1 h - b_1 k_3 \sinh k_3 h],$

$B_5 = b_2 k_3 [b_1 k_2 (\cosh k_1 h \cosh k_2 h - 1) - b_2 k_1 \sinh k_1 h \sinh k_2 h],$

$B_6 = b_3 k_2 [b_2 k_1 \sinh k_1 h \cosh k_2 h - b_1 k_2 \sinh k_2 h \cosh k_1 h],$

$B_7 = b_1 b_2 k_2 k_3 (\cosh k_2 h - \cosh k_1 h),$

$B_8 = b_3 k_2 [b_2 k_1 \sinh k_1 h - b_1 k_2 \sinh k_2 h].$

Substituting the above values into the right-hand side of (19) and performing the matrix multiplications, we finally obtain the solution of the problem in the Laplace transform domain as

$$\bar{\theta}(y, s) = \frac{\theta_0 \cosh k_1(h - y)}{s \cosh k_1 h}, \tag{48}$$

$$\begin{aligned} \bar{u}(y, s) = F_1[&B_1 b_2 \cosh k_2(h - y) + B_2 b_2 \sinh k_2(h - y) + b_2 B_3 \cosh k_2 y \\ &+ B_4 b_2 \sinh k_2 y + B_5 b_3 \cosh k_3(h - y) + B_6 b_3 \sinh k_3(h - y) \\ &+ B_7 b_3 \cosh k_3 y + B_8 b_3 \sinh k_3 y - \Gamma_1 b_1 \cosh k_1(h - y)], \end{aligned} \tag{49}$$

$$\begin{aligned} \bar{H}_x(y, s) = bF_1[&k_2 B_1 \sinh k_2(h - y) + k_2 B_2 \cosh k_2(h - y) - k_2 B_3 \sinh k_2 y \\ &- k_2 B_4 \cosh k_2 y + k_3 B_5 \sinh k_3(h - y) + k_3 B_6 \cosh k_3(h - y) \\ &- k_3 B_7 \sinh k_3 y - k_3 B_8 \cosh k_3 y - k_1 \Gamma_1 \sinh k_1(h - y)]. \end{aligned} \tag{50}$$

Problem 3: plane distribution of heat sources

We shall consider an incompressible viscoelastic fluid in the presence of a magnetic field occupying the region $y \geqq 0$ whose state depends only on the space variables y and time variables t. We also assume that there is a plane distribution of continuous heat sources located at the plate $y = 0$. The intensity of the heat sources is taken as

$$Q(y, t) = Q_0 H(t)\, \delta(y),$$

where Q_0 is a constant and $\delta(y)$ is Dirac's delta function. Taking Laplace transform, we obtain

$$\bar{Q}(y, s) = \frac{Q_0}{s}\, \delta(y).$$

We shall now proceed to find the solution of the problem in the right half-space $y \geqq 0$ using (18).

Substituting for $\bar{Q}$ in the expression for B and inserting the result in the right-hand side of (18), we get upon using the integral properties of the Dirac's delta function,

$$\bar{V}(y, s) = L(y, s)\, [\bar{V}(0, s) + H(s)] \tag{51}$$

where

$$H(s) = -\frac{Q_0}{2s}\begin{bmatrix} \frac{1}{2k_1} \\ \frac{\mathrm{G}\,F}{2a}\left(b_1\frac{k_2{}^2 - k_3{}^2}{k_1} + b_2\frac{k_3{}^2 - k_1{}^2}{k_2} + b_3\frac{k_1{}^2 - k_2{}^2}{k_3}\right) \\ 0 \\ \frac{1}{2} \\ \frac{\mathrm{G}\,F}{2a}\left(b_1(k_2 - k_3{}^2) + b_2(k_3 - k_1{}^2) + b_3(k_1{}^2 - k_3{}^2)\right) \\ \frac{b\,\mathrm{G}\,F}{2a}\left(k_1k_2(k_2 - k_1) + k_1k_3(k_1 - k_3) + k_2k_3(k_3 - k_2)\right) \end{bmatrix}.$$

Equation (51) expresses the solution of the problem in the Laplace transform domain for $y \geqq 0$ in terms of the vector $H(s)$ representing the applied heat source and the vector $\bar{V}$ representing the conditions at the plate $y = 0$. To evaluate the components of the vector, we note that it follows from the construction of the problem that

$$u(0, t) = 0 \quad \text{or} \quad \bar{u}(0, s) = 0, \tag{52}$$

$$H_x(0, t) = 0 \quad \text{or} \quad \bar{H}_x(0, s) = 0. \tag{53}$$

Gauss's divergence theorem will now be used to obtain the thermal condition at the plane source. We consider a cylinder of unit base whose axis is perpendicular to the plane source of heat and whose bases lie on opposite sides of it. Taking the limit as the height of the cylinder tends to zero and noting that there is no heat flux through the lateral surface, we get

$$q(0, t) = \frac{Q_0}{2}H(t) \quad \text{or} \quad \bar{q}(0, s) = \frac{Q_0}{2s}. \tag{54}$$

Using Fourier's law of heat condition in the non-dimensional form, namely $q = -\partial\theta/\partial y$, we obtain the condition

$$\bar{\theta}'(0, s) = -\frac{Q_0}{2s}. \tag{55}$$

Equations (52), (53) and (55) give three components of the vector $\bar{V}(0, s)$. To obtain the remaining three components, we substitute $y = 0$ on both sides of (51) obtaining a system of linear equations whose solution gives

$$\bar{\theta}(0, s) = \frac{Q_0}{2k_1 s}, \tag{56}$$

$$\bar{u}'(0, s) = -\frac{\mathrm{G}\,Q_0}{2k_1\Gamma}[b_1k_1A_1 + b_2k_2A_2 + b_3k_3A_3], \tag{57}$$

$$\bar{H}_x'(0, s) = -\frac{b\,\mathrm{G}\,Q_0}{2k_1\Gamma}[k_1{}^2A_1 + k_2{}^2A_2 + k_3{}^2A_3]. \tag{58}$$

As before, we have suppressed the positive exponential terms appearing in the entries of $L(y, s)$.

Substituting the above value into the right-hand side of (51), we obtain

$$\bar{\theta}(y, s) = \frac{Q_0}{2k_1 s} \exp(-k_1 y), \tag{59}$$

$$\bar{u}(y, s) = \frac{Q_0\,\mathrm{G}}{2k_1 \Gamma} [b_1 A_1 \exp(-k_1 y) + b_2 A_2 \exp(-k_2 y) + b_3 A_3 \exp(-k_3 y)], \tag{60}$$

$$\bar{H}_x(y, s) = \frac{b\,\mathrm{G}\,Q_0}{2k_1 \Gamma} [k_1 A_1 \exp(-k_1 y) + k_2 A_2 \exp(-k_2 y) + k_3 A_3 \exp(-k_3 y)]. \tag{61}$$

4 Inversion of the Laplace transform

In order to invert the Laplace transforms in the above equations we shall use a numerical technique based on Fourier expansion of functions.

Let $g(t)$ be the Laplace transform of a given function $g(t)$. The inversion formula of Laplace transform states that

$$g(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{st} \bar{g}(s)\, ds,$$

where c is an arbitrary constant greater than all real parts of the singularities of $g(t)$. Taking $s = c + iy$, we get

$$g(t) = \frac{e^{ct}}{2\pi} \int_{-\infty}^{\infty} e^{ity} \bar{g}(c + iy).$$

This integral can be approximated by

$$g(t) = \frac{e^{ct}}{2\pi} \sum_{k=-\infty}^{\infty} e^{ikt\Delta y} \bar{g}(c + ik\Delta y)\, \Delta y.$$

Taking $\Delta y = \pi/t_1$, we obtain

$$g(t) = \frac{e^{ct}}{t_1} \left[\frac{1}{2} \bar{g}(c) + \mathrm{Re} \left(\sum_{k=1}^{\infty} e^{ik\pi t/t_1} \bar{g}(c + ik\pi/t_1) \right) \right].$$

For numerical purposes this is approximated by the function

$$g_N(t) = \frac{e^{ct}}{t_1} \left[\frac{1}{2} \bar{g}(c) + \mathrm{Re} \left(\sum_{k=1}^{N} e^{ik\pi t/t_1} \bar{g}(c + ik\pi/t_1) \right) \right], \tag{62}$$

where N is a sufficiently large integer chosen such that

$$e^{ct}\, \mathrm{Re}\, [e^{iN\pi t/t_1} \bar{g}(c + iN\pi/t_1)] < \varepsilon,$$

and ε is a preselected small positive number that corresponds to the degree of accuracy to be achieved. Formula (62) is the numerical inversion formula valid for $2t_1 \geqq t \geqq 0$ [23]. In particular, we choose $t = t_1$, obtaining

$$g_N(t) = \frac{e^{ct}}{t_1} \left[\frac{1}{2} \bar{g}(c) + \mathrm{Re} \left(\sum_{k=1}^{N} (-1)^k\, \bar{g}(c + ik\pi/t) \right) \right]. \tag{63}$$

5 Numerical results

In this paper the state space approach is adopted for the solution of one-dimensional problems of a viscoelastic fluid of hydromagnetic free convection boundary layer flow past an infinite vertical plate. The technique is applied to a thermal shock problem and to a problem for the flow between two parallel fixed plates both without heat sources. Also a problem with a distribution of heat sources is considered. The inversion of the Laplace transforms is carried out using a numerical approach.

The computations were carried out for three values of the elastic parameter k_0, namely, $k_0 = 0.0$, $k_0 = 0.2$ and $k_0 = 0.4$, where $\Pr = 0.71$ (which corresponds to air) for a value of time $t = 10$. Formula (4.2) was used to invert the Laplace transforms in equations (35), (36), (37), (48), (49), (50), (59), (60), and (61) giving the functions $\theta(y, t)$, $u(y, t)$ and $H_x(y, t)$ for problems (1)–(3). The velocity distribution for each problem is illustrated in Figs. 1–6.

The important phenomenon observed in all computations is that the velocity increase with the variable coordinate y up to a maximum value and then its decreases again.

6 Concluding remarks

The importance of state space analysis is recognized in fields where the time behavior of any physical process is of interest.

The state space approach is more general than the classical Laplace and Fourier transform techniques. Consequently, state space is applicable to all systems that can be analyzed by integral transforms in time, and is applicable to many systems for which transform theory breaks down [25].

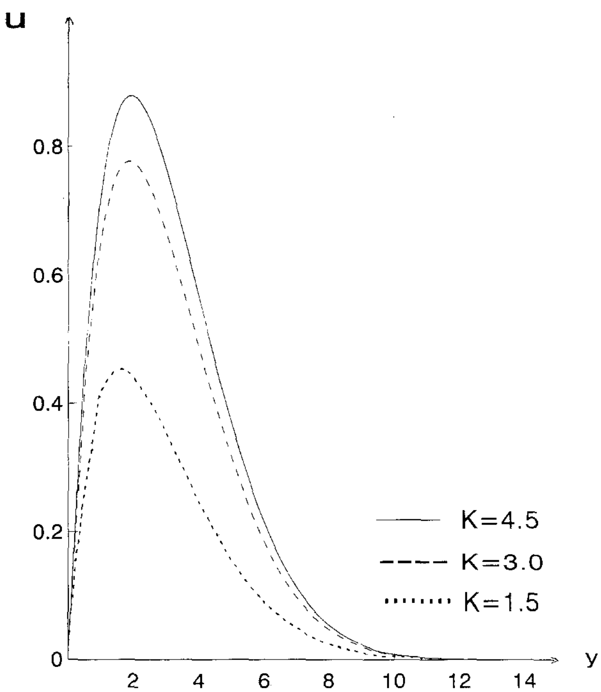

Fig. 1

Owing to the complicated nature of the governing equations for the unsteady magnetohydrodynamic free convection flow, few attempts have been made to solve problems in this field [4]–[7]. These attempts utilized approximate methods valid for only a specific range of some parameters.

In this work, the method of direct integration by means of the matrix exponential, which is a standard approach in modern control theory and developed in detail in many texts such as

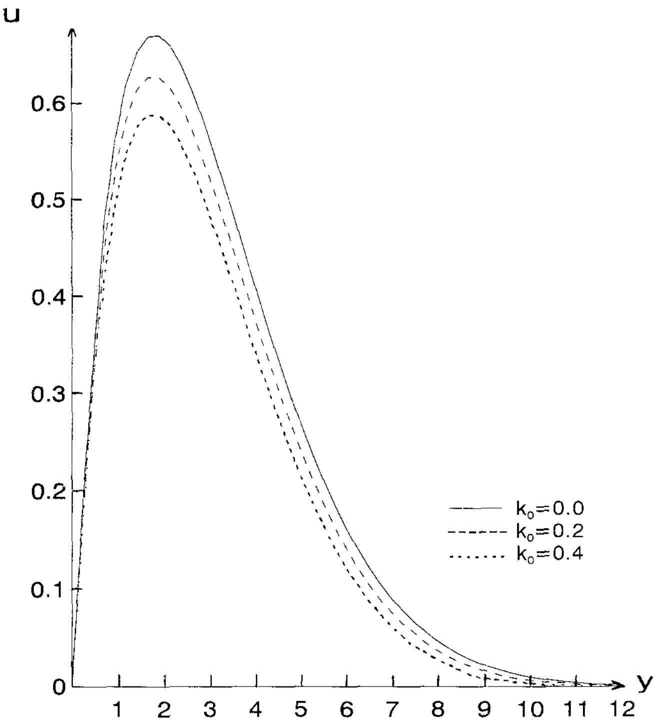

Fig. 2

Figs. 1, 2. Velocity distribution of problem 1 for different values of K, $k_0 = 0.2$ **(Fig. 1)** and k_0, $K = 2$ **(Fig. 2)**

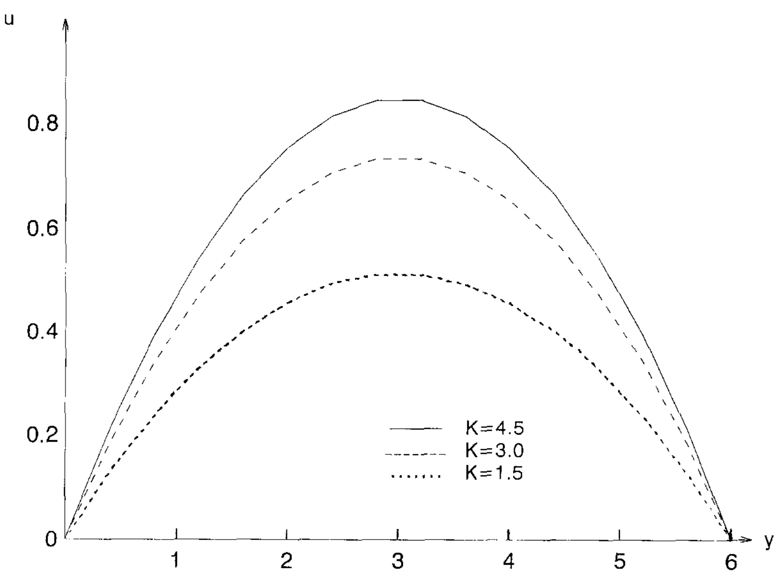

Fig. 3

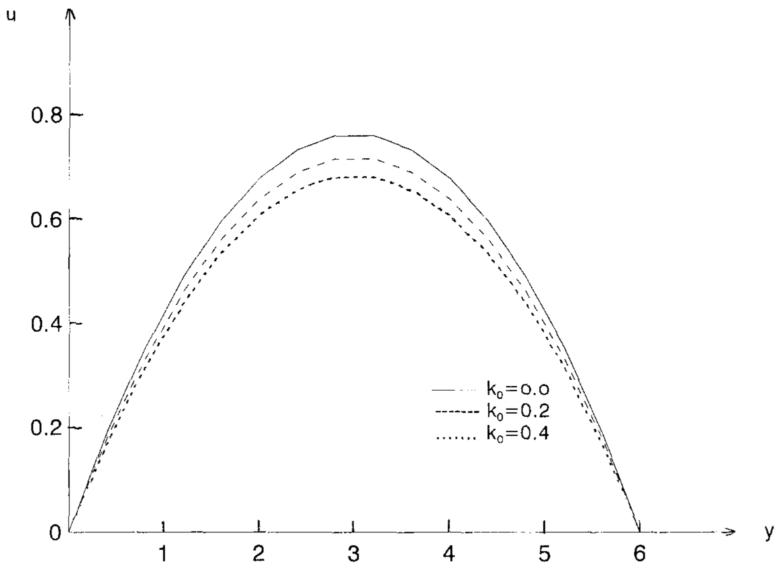

Fig. 4

Figs. 3, 4. Velocity distribution of problem 2 for different values of K, $k_0 = 0.2$ (**Fig. 3**) and k_0, $K = 2$ (**Fig. 4**)

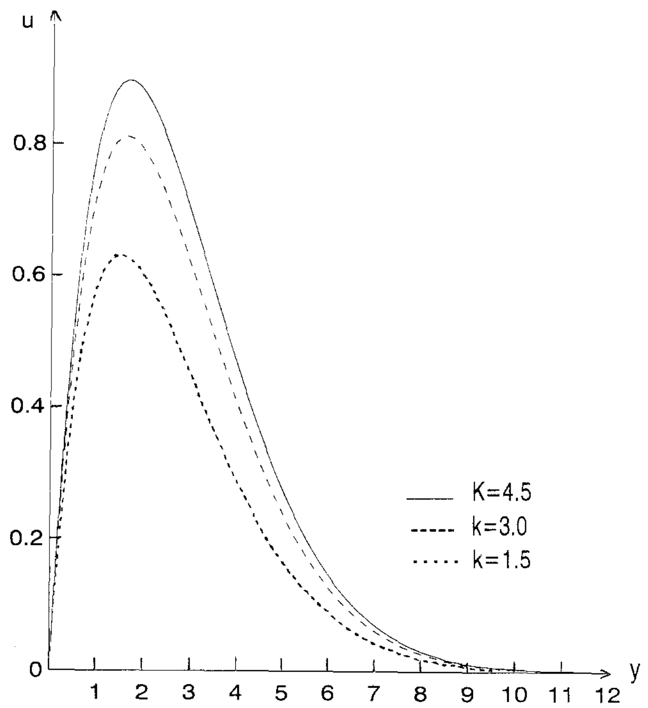

Fig. 5

Ogata [26], is introduced in the field of magnetohydrodynamic and applied to three specific problems in which the temperature, velocity and magnetic field are coupled. This method gives exact solutions in the Laplace transform domain without any assumed restrictions on either the applied magnetic field or the velocity, temperature distributions and viscoelastic parameter.

The method used in the present work is applicable to a wide range of problems. It can be applied to problems which are described by the linearized Navier-Stokes equations. The same approach was used quite successfully in dealing with problems in thermoelasticity theory [27], [28].

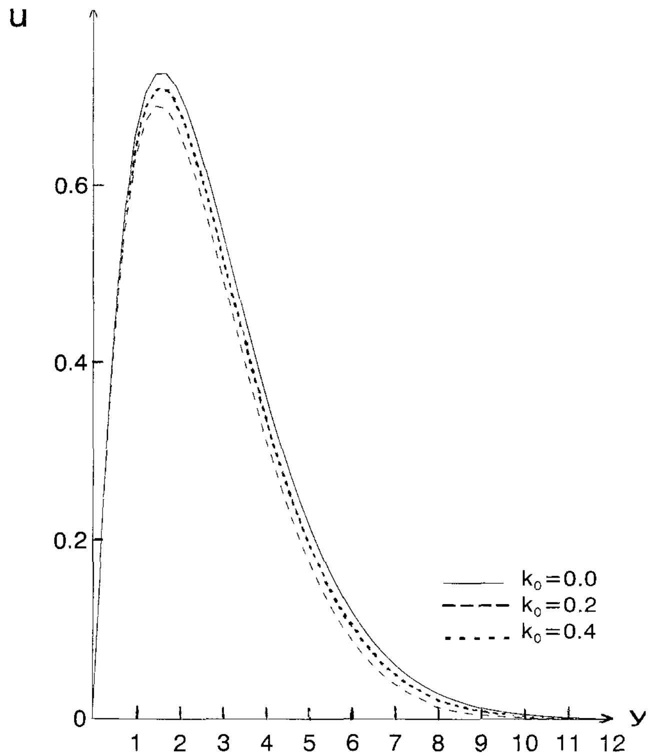

Fig. 6

Figs. 5, 6. Velocity distribution of problem 3 for different values of K, $k_0 = 0.2$ **(Fig. 5)** and k_0, $K = 2$ **(Fig. 6)**

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