The Journal of Fourier Analysis and Applications
Volume 5, Number 1, 1999

Positive Matrix Functions on the Bitorus with Prescribed Fourier Coefficients in a Band

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Communicated by J. William Helton

ABSTRACT. Let S be a band in $\mathbb{Z}^2$ *bordered by two parallel lines that are of equal distance to the origin. Given a positive definite* ℓ^1 *sequence of matrices* $\{c_j\}_{j\in S}$, *we prove that there is a positive definite matrix function f in the Wiener algebra on the bitorus such that the Fourier coefficients* $\widehat{f(k)}$ *equal* c_k *for* $k \in S$. *A parameterization is obtained for the set of all positive extensions f of* $\{c_j\}_{j\in S}$. *We also prove that among all matrix functions with these properties, there exists a distinguished one that maximizes the entropy. A formula is given for this distinguished matrix function. The results are interpreted in the context of spectral estimation of ARMA processes.*

1. Introduction

The extension problem for positive definite functions concerns the problem of finding a positive definite matrix valued function with certain prescribed Fourier coefficients. In this paper we shall mainly be concerned with Wiener algebra functions of two variables, i.e., functions defined on the bitorus with an absolutely summable Fourier expansion. The classical positive extension problem for functions of one variable goes back to the works of Carathéodory, Toeplitz, Fejér, and Riesz in the beginning of this century (see [15] for a full account). Attempts to generalize some of the one-variable results to the case of two or more variables have often lead to negative conclusions. Perhaps one of the most well-known results in this respect is the impossibility of a straightforward generalization of the Riesz–Fejér lemma to functions of two or more variables, which was exposed independently by Calderon and Pepinsky [4] and by Rudin [23]. In both papers it is shown that there exist positive trigonometric polynomials of two variables which are not sums of squared absolute values of polynomials. As a corollary it was shown by duality that the classical trigonometric moment problem does not extend trivially to the two-variable case. We will state a precise result later in the introduction.

One of the main applications of the positive extension problem concerns spectral estimation of stationary processes based on measured correlation coefficients, and the related linear prediction

Math Subject Classifications. 42B99, 47B35, 15A54, 60G35.
Keywords and Phrases. Positive extensions, Toeplitz operators, matrix functions on bitorus, Wiener algebra, band method, entropy, almost periodic functions, ARMA processes.

ISSN 1069-5869

theory. It is in the linear prediction theory literature and the signal processing literature that many positive results for the multivariate positive extension problem may be found (see, e.g., [5] or [18] and the references therein). One of the main breakthroughs in this area was obtained in the work of Helson and Lowdenslager [14], who extended several one-variable linear prediction results to the multivariate case after recognizing that the appropriate definition of "past" is a halfspace in $\mathbf{Z}^d$ (the definition will follow below). Here and elsewhere in the paper we denote by $\mathbf{Z}$ the set of integers, and let $\mathbf{Z}^d = \{(z_1, \cdots, z_d) \ : \ z_j \in \mathbf{Z}\}$. One of the important features of a halfspace in $\mathbf{Z}^d$ is that there exist $d-1$ linearly independent vectors along which (after slightly shifting) the halfspace extends to infinity in both directions. Thus, loosely speaking, only in one direction $\mathbf{Z}^d$ is cut in half by the halfspace. It is in this spirit that we generalize some of the positive extension results to two (or more) variables. Namely, our index sets of prescribed Fourier coefficients is chosen to be a doubly infinite band in $\mathbf{Z}^d$, which also has $d-1$ linearly independent vectors along which the domain extends to infinity in both directions. We shall now state some of our main results in detail.

Let S be a non-empty subset in $\mathbf{Z}^d$, and let $C = \{c_j\}_{j\in S}$ be an ℓ^1 sequence of $n \times n$ matrices, i.e., the sequence satisfies

$$\sum_{j\in S} \|c_j\| < \infty ,$$

where $\|\cdot\|$ is the operator norm (= the largest singular value for matrices). We associate with C a matrix valued function f_C defined on the d-torus $\mathbf{T}^d$, where $\mathbf{T} = \{e^{it} : t \in \mathbf{R}\}$, via

$$f_C(t_1, \ldots, t_d) = \sum_{j=(j_1,\ldots,j_d)\in S} c_j \exp\{i\,(j_1t_1 + \cdots + j_dt_d)\} ,$$

which will be written in self-explanatory shorthand notation

$$f_C(t) = \sum_{j\in S} c_j e^{i\langle j,t\rangle} .$$

Note that we represent functions on $\mathbf{T}^d$ as periodic functions on $\mathbf{R}^d$, which will be more convenient for our purposes. The *positive extension problem* on S can now be stated as follows:

(PEP): *Find, if possible, an $n \times n$ matrix function g on $\mathbf{R}^d$ with the following properties:*

(i) *g admits a representation*

$$g(t) = \sum_{j\in\mathbf{Z}^d} \hat{g}_j e^{i\langle j,t\rangle}, \quad t \in \mathbf{R}^d ,$$

where $\sum_{j\in\mathbf{Z}^d} \|\hat{g}_j\| < \infty$; in other words, g is in the Wiener algebra $\mathcal{W}_d^{n\times n}$;

(ii) *$\hat{g}_j = c_j$ for every $j \in S$;*

(iii) *g is positive definite, i.e., $g(t)$ is positive definite for every $t \in \mathbf{R}^d$.*

A matrix function g with Properties (i) through (iii) is called a *positive extension* of f_C.

For certain sets S necessary conditions for the existence of positive extensions can be given in terms of Toeplitz operators. Assume that S is such that $S = \Delta - \Delta = \{x - y : x, y \in \Delta\}$ for some $\Delta \subseteq \mathbf{Z}^d$. Let $L^2(\mathbf{T}^d)^{N\times 1}$ be the Hilbert space of N-component column vectors with components in the Lebesgue space $L^2(\mathbf{T}^d)$ of square integrable functions on $\mathbf{T}^d$ with respect to the normalized Lebesgue measure. A function $f \in L^2(\mathbf{T}^d)^{n\times 1}$ can be identified with its Fourier series

$$f(t_1, \ldots, t_d) = \sum_{j\in\mathbf{Z}^d} \hat{f}_j e^{i\langle j,t\rangle} ,$$

where $\sum_{j\in\mathbf{Z}^d}\|\hat{f}_j\|^2 < \infty$. Denote by $L^2(\Delta)^{n\times 1}$ the subspace of all such $f \in L^2(\mathbf{T}^d)^{n\times 1}$ that $\hat{f}_j = 0$ for $j \notin \Delta$. We define the Toeplitz operator T_{Δ,f_C} associated with the sequence C, or equivalently the function f_C, by:

$$T_{\Delta,f_C}h = P_\Delta(f_C h), \quad h \in L^2(\Delta)^{n\times 1}, \tag{1.1}$$

where P_Δ is the orthogonal projection onto $L^2(\Delta)^{n\times 1}$. Clearly, T_{Δ,f_C} is a bounded operator on $L^2(\Delta)^{n\times 1}$. The hypothesis $S = \Delta - \Delta$ guarantees that $T_{\Delta,f_C} = T_{\Delta,g}$ for any existing positive extension g of f_C. Consequently, a necessary condition for the existence of a positive extension is that T_{Δ,f_C} is *positive definite* (notation: $T_{\Delta,f_C} > 0$), i.e.,

$$\langle T_{\Delta,f_C}h, h\rangle \geq \epsilon\langle h, h\rangle, \quad h \in L^2(\Delta)^{n\times 1},$$

where $\epsilon > 0$ is independent of h. As alluded to before, this necessary condition is not always sufficient as was shown in [4] and [23]. Therefore, we introduce the following definition. Let $S \subseteq \mathbf{Z}^d$ be such that $S = \Delta - \Delta$ for some non-empty set $\Delta \subseteq \mathbf{Z}^d$. We say that S has the *positive extension property with respect to* Δ if for every ℓ^1 sequence $C = \{c_j\}_{j\in S}$ of $n \times n$ matrices with the property that the Toeplitz operator T_{Δ,f_C} defined on $L^2(\Delta)^{n\times 1}$ by (1.1) is positive definite, the matrix function $f_C(t) = \sum_{j\in S} c_j e^{i\langle j,t\rangle}$ admits a positive extension. Note that if S has the extension property with respect to Δ, it also has the extension property with respect to the sets $-\Delta$ and $m + \Delta$, where $m \in \mathbf{Z}^d$. Furthermore, if $S = \Delta_j - \Delta_j$ for $j = 1, 2$, where $\Delta_1 \subseteq \Delta_2$, then it is easy to see that if S has the positive extension property with respect to Δ_1, it also has the positive extension property with respect to Δ_2 (indeed, one has only to observe that T_{Δ_1,f_C} is a compression of T_{Δ_2,f_C}, and therefore $T_{\Delta_2,f_C} > 0$ implies $T_{\Delta_1,f_C} > 0$). Therefore, the sets Δ which are minimal (by inclusion) subject to $S = \Delta - \Delta$ are of particular interest. Note that in our main result below (Theorem 1) the set $S^+_{r,\nu}$ is indeed minimal in this sense.

With the above terminology we may now reformulate the classical trigonometric moment result in the following form: The sets $\{-p, \ldots, p\} \subseteq \mathbf{Z}$ have the positive extension property with respect to $\{0, \ldots, p\}$. On the other hand, the negative results in [4, 23] may be stated as that for $d \geq 2$ and $N \geq 3$ the set

$$\left\{(s_1, \ldots, s_d) \in \mathbf{Z}^d : |s_j| \leq N,\ j = 1, \ldots, d\right\}$$

does not have the positive extension property with respect to

$$\left\{(s_1, \ldots, s_d) \in \mathbf{Z}^d : 0 \leq s_j \leq N,\ j = 1, \ldots, d\right\}.$$

One of our main results establishes the positive extension property for infinite bands in $\mathbf{Z}^2$ bordered by two parallel lines that are of equal distance to the origin. For $S \subseteq \mathbf{Z}^d$ let $\mathcal{W}_S^{n\times n}$ be the subspace of $\mathcal{W}_d^{n\times n}$ consisting of all $n \times n$ matrix functions f of the form

$$f(t) = \sum_{j\in S} c_j e^{i\langle j,t\rangle}; \quad t = (t_1, \ldots, t_d) \in \mathbf{R}^d.$$

We denote by A^* the conjugate transpose of a matrix A. For a matrix valued function $f(t)$, we let f^* be the matrix function defined by $f^*(t) = (f(t))^*$. The shorthand f^{*-1} is used for $(f^*)^{-1}$.

Theorem 1.
Let

$$S_{r,\nu} = \left\{(k, \ell) \in \mathbf{Z}^2 : |k - r\ell| \leq \nu\right\},$$

where r is a real and ν is a positive number. Then $S_{r,\nu}$ has the positive extension property with respect to

$$S^+_{r,\nu} = \left\{(k, \ell) \in \mathbf{Z}^2 : 0 < k - r\ell \leq \nu \ \textit{or}\ k - r\ell = 0 \ \textit{and}\ k \geq 0\right\}.$$

In fact, if C is a sequence for which $T_{S^+_{r,\nu},f_C}$ is positive definite, then a positive extension of f_C may be found as follows: Let $x(t) = (x_{ij}(t))_{i,j=1}^n \in \mathcal{W}^{n\times n}_{S^+_{r,\nu}}$ and $y(t) = (y_{ij}(t))_{i,j=1}^n \in \mathcal{W}^{n\times n}_{-S^+_{r,\nu}}$ be defined by

$$\left(x_{ij}\right)_{i=1}^n = \left(T_{S^+_{r,\nu},f_C}\right)^{-1}(e_j); \ \left(y_{ij}\right)_{i=1}^n = \left(T_{-S^+_{r,\nu},f_C}\right)^{-1}(e_j),$$

where e_j denotes the j^{th} column of the identity matrix, and let $D(x) = P_{\{(0,0)\}}(x)$ and $D(y) = P_{\{(0,0)\}}(y)$. Then

$$g_0(t) := x(t)^{*-1}D(x)x(t)^{-1} = y(t)^{*-1}D(y)y(t)^{-1}$$

is a positive extension of f_C.

Let $\Lambda \subseteq \mathbf{Z}^d$ be a *halfspace*, i.e., (i) $0 \notin \Lambda$, (ii) for a d-tuple $m \neq 0$, $m \in \mathbf{Z}^d$, we have that $m \in \Lambda$ if and only if $-m \notin \Lambda$ and (iii) $m \in \Lambda$ and $p \in \Lambda$ imply $m + p \in \Lambda$. Note that Λ is a halfspace if and only if $-\Lambda$ is. For a positive definite $g \in \mathcal{W}^{n\times n}_d$, we say that g allows a Λ-*spectral factorization* if we may write

$$g = (I + g_\Lambda)^* \, D_\Lambda(g) \, (I + g_\Lambda) \ , \tag{1.2}$$

where g_Λ, $(I + g_\Lambda)^{-1} - I$ belong to $\mathcal{W}^{n\times n}_\Lambda$, and $D_\Lambda(g) \in \mathbf{C}^{n\times n}$. When this factorization exists, it is unique (see, e.g., (proof of) Lemma II.3.2 in [27]). In the scalar case one may use the results of [7] to show that this factorization exists for all positive definite functions. In the matrix valued case we shall show that for the halfspaces Λ we consider, any positive definite function in $\mathcal{W}^{n\times n}_d$ admits a Λ-spectral factorization. Consequently, in our cases $D_\Lambda(g)$ is well defined for every positive definite g.

If $f \in \mathcal{W}^{n\times n}_{S_{r,\nu}}$ admits positive extensions, then in the set of all positive extensions of f there is a salient extension characterized by the following theorem. For Hermitian matrices A and B we let $A \geq B$ denote the Loewner ordering, i.e., $A \geq B$ denotes that $A - B$ is positive semidefinite.

Theorem 2.

Suppose that $f \in \mathcal{W}^{n\times n}_{S_{r,\nu}}$ is such that $T_{S^+_{r,\nu},f}$ is positive definite, and let

$$\Lambda = \left\{(k, \ell) \in \mathbf{Z}^2 : \ k - r\ell > 0 \ or \ k - r\ell = 0 \ and \ k > 0\right\} .$$

Then there is a unique positive extension g_0 of f with the property that

$$D_\Lambda(g_0) \geq D_\Lambda(g)$$

for every positive extension g of f. Moreover, g_0 is given by Theorem 1, and is the unique positive extension of f with the property that $g_0^{-1} \in \mathcal{W}^{n\times n}_{S_{r,\nu}}$.

Theorem 2 remains valid when one replaces Λ by $-\Lambda$. Note that $D_\Lambda(g_0) = D(x)^{-1}$ and $D_{-\Lambda}(g_0) = D(y)^{-1}$. The extension g_0 is sometimes referred to as the *central extension* of f. For a positive function g we define its *entropy* as

$$\mathcal{E}(g) = \frac{1}{(2\pi)^d}\int_0^{2\pi} \cdots \int_0^{2\pi} \log\left[\det g\,(t_1, \ldots, t_d)\right] dt_1 \ldots dt_d \ . \tag{1.3}$$

Note that $\mathcal{E}(g) = \log\det D_\Lambda(g)$ and log det is strictly concave on the set of positive definite matrices. As a result, we obtain the following corollary of Theorem 2.

Corollary 1.

In the notation of Theorem 2, g_0 is the unique positive extension of f that maximizes the entropy among all positive extensions of f.

Our final main result in the introduction gives a parameterization of the set of all positive extensions of f.

Theorem 3.

Let f_C, x, y, $D(x)$, and $D(y)$ be as in Theorem 1. Put $u(t) = x(t)D(x)^{-\frac{1}{2}}$ and $v(t) = y(t)D(y)^{-\frac{1}{2}}$. Then each positive extension of $f_C(t)$ in $\mathcal{W}_2^{n\times n}$ is given by

$$(u+vg)^{*-1}\left(I-g^*g\right)(u+vg)^{-1}\,,$$

where $g(t)$ is an arbitrary Wiener $n \times n$ matrix function with support in the set $\{(k,\ell) \in \mathbf{Z}^2 : k - r\ell > \nu\}$ and such that

$$\sup_{t\in\mathbf{R}^2} \|g(t)\| < 1\,.$$

This correspondence is one-one.

Similar results may be obtained by replacing $S^{+}_{r,\nu}$ with

$$\left\{(k,\ell)\in\mathbf{Z}^2:\ 0<k-r\ell\leq\nu \ \text{ or } \ k-r\ell=0 \ \text{ and } \ k\leq 0\right\}\,.$$

Since the modifications for this choice are obvious, we shall not provide separate statements and proofs for this variation.

Our paper is organized as follows. In Section 2 we prove the main results in the case that the slope r of the band is irrational. In Section 3 the main results are established in the case that r is rational. The proofs for the irrational and the rational case are completely different. In the first case we employ a new variation of recent results on almost periodic functions obtained in [22], while in the second case we reduce the problem to an operator valued one-variable case. In Section 4 we discuss some extensions of the obtained results to other sets S, in particular, to periodic matrix functions of more than two variables. In Section 5 we present a version of the extension problem for positive measures. Finally, in Section 6 we interpret some of the results in the context of spectral estimation of ARMA processes.

The scalar valued versions of Theorems 1, 2, and 3 have been announced in [2].

2. Proofs of the Main Theorems: The Irrational Case

Our reasoning in this case uses extension results for almost periodic functions and related properties of operators on Besikovitch spaces, discussed earlier in [22]. Let us begin with the necessary background.

Let (AP) be the algebra of complex-valued almost periodic functions on the real line, i.e., the closed subalgebra of $L^\infty(\mathbf{R})$ generated by the functions $e^{i\lambda t}$, $\lambda \in \mathbf{R}$. Recall that for any $f(t) \in (AP)$ the *Fourier series* is defined by the formal sum

$$\sum_{\lambda} f_\lambda e^{i\lambda t}\,, \tag{2.1}$$

where

$$f_\lambda = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} e^{-i\lambda t} f(t)dt, \quad \lambda\in\mathbf{R}\,,$$

and the sum in (2.1) is taken over the set $\sigma(f) = \{\lambda \in \mathbf{R} :\ f_\lambda \neq 0\}$, called the *Fourier spectrum* of $f(t)$. The Fourier spectrum of every $f \in (AP)$ is at most a countable set. The coefficient f_0 in (2.1) is called the *mean* of $f(t) \in (AP)$, and is denoted $M\{f\}$. The *Wiener algebra* (APW) is defined as the set of all $f \in (AP)$ such that the Fourier series of $f(t)$ converges absolutely. For the general theory of almost periodic functions we refer the reader to the books [6, 16, 17].

Denote by $(APW)^{m\times n}$ the set (algebra if $m = n$) of $m \times n$ matrices with entries in (APW). For $\Lambda \subseteq \mathbf{R}$, we denote by $(APW)^{m\times n}_{\Lambda}$ the set of all $f \in (APW)^{m\times n}$ such that $\sigma(f_{ij}) \subseteq \Lambda$ for each matrix component f_{ij} of f.

Introduce a scalar product on (AP) by the formula

$$\langle f, g\rangle = M\left\{fg^*\right\}, \quad f, g \in (AP)\,. \tag{2.2}$$

The completion of (AP) with respect to this scalar product is called the *Besikovitch space* and is denoted (B). Thus, (B) is a Hilbert space.

For a nonempty set $\Lambda \subseteq \mathbf{R}$, define the projection

$$\Pi_\Lambda\left(\sum_{\lambda\in\sigma(f)} f_\lambda e^{i\lambda t}\right) = \sum_{\lambda\in\sigma(f)\cap\Lambda} f_\lambda e^{i\lambda t}\,,$$

where $f \in (APW)$. The projection Π_Λ extends by continuity to the orthogonal projection (also denoted Π_Λ) on (B). We denote by $(B)_\Lambda$ the range of Π_Λ, or, equivalently, the completion of $(APW)_\Lambda$ with respect to the scalar product (2.2). The vector-valued Besikovitch space $(B)^{n\times 1}$ consists of $n \times 1$ columns with components in (B), with the standard Hilbert space structure. Similarly, $(B)^{n\times 1}_\Lambda$ is the Hilbert space of $n \times 1$ columns with components in $(B)_\Lambda$.

For any additive subgroup $\Sigma \subseteq \mathbf{R}$ and $f \in (APW)^{n\times n}_\Sigma$, the *generalized Toeplitz operator*

$$\mathbf{T}(f)_{\Lambda\cap\Sigma} \colon (B)^{n\times 1}_{\Lambda\cap\Sigma} \to (B)^{n\times 1}_{\Lambda\cap\Sigma}$$

is defined by

$$g \mapsto \Pi_\Lambda(fg), \quad g \in (B)^{n\times 1}_{\Lambda\cap\Sigma}\,.$$

In the case $\Sigma = \mathbf{R}$, we use a shorthanded notation $\mathbf{T}(f)_\Lambda$ in place of $\mathbf{T}(f)_{\Lambda\cap\Sigma}$.

A Hilbert space operator $\mathbf{T} : \mathcal{H} \to \mathcal{H}$ is called *positive definite* if there exists an $\epsilon > 0$ such that $\langle \mathbf{T}f, f\rangle \geq \epsilon\|f\|^2$ for all $f \in \mathcal{H}$.

The positive extension theory for almost periodic functions consists of three main theorems below.

Theorem 4.

Let $\Sigma \subseteq \mathbf{R}$ be an additive subgroup, and let $\mu > 0$. For a given function $f \in (APW)^{n\times n}_{[-\mu,\mu]\cap\Sigma}$ the following statements are equivalent:

(i) *f has a positive extension in $(APW)^{n\times n}_\Sigma$, i.e., there exists $h \in (APW)^{n\times n}_\Sigma$ such that $h_\lambda = f_\lambda$ for all $\lambda \in [-\mu, \mu] \cap \Sigma$ and there is an $\epsilon > 0$ such that $h(x) \geq \epsilon I$ for all $x \in \mathbf{R}$;*

(ii) *the generalized Toeplitz operator*

$$\mathbf{T}(f)_{[0,\mu]\cap\Sigma} : (B)^{n\times 1}_{[0,\mu]\cap\Sigma} \to (B)^{n\times 1}_{[0,\mu]\cap\Sigma}$$

is positive definite.

(iii) *the generalized Toeplitz operator*

$$\mathbf{T}(f)_{[-\mu,0]\cap\Sigma} : (B)^{n\times 1}_{[-\mu,0]\cap\Sigma} \to (B)^{n\times 1}_{[-\mu,0]\cap\Sigma}$$

is positive definite.

(iv) *f has a positive extension in $(APW)^{n\times n}$.*

When one (and thus all) of (i) through (iv) is satisfied, then

$$h_0(t) = x^{*-1}(t)M\{x\}x^{-1}(t) = y^{*-1}(t)M\{y\}y^{-1}(t), \quad t \in \mathbf{R} \tag{2.3}$$

is a positive extension of f *in* $(APW)^{n\times n}_{\Sigma}$. *Here,* $x = (x_{ij})^n_{i,j=1} \in (APW)^{n\times n}_{[0,\mu]\cap\Sigma}$ *and* $y = (y_{ij})^n_{i,j=1} \in (APW)^{n\times n}_{[-\mu,0]\cap\Sigma}$ *are given via*

$$\left(x_{ij}\right)^n_{i=1} = \left(\mathbf{T}(f)_{[0,\mu]\cap\Sigma}\right)^{-1}\left(e_j\right); \quad \left(y_{ij}\right)^n_{i=1} = \left(\mathbf{T}(f)_{[-\mu,0]\cap\Sigma}\right)^{-1}\left(e_j\right), \tag{2.4}$$

where e_j *denotes the* j^{th} *column of the* $n \times n$ *identity matrix.*

Let $\Sigma'(\supseteq \Sigma)$ be an additive subgroup of $\mathbf{R}$. Every positive extension h in $(APW)^{n\times n}_{\Sigma'}$ of f $(\in (APW)^{n\times n}_{[-\mu,\mu]\cap\Sigma})$ admits right and left *spectral factorizations*

$$\begin{aligned} h(t) &= h_+^*(t)D_1(h)h_+(t), \quad t \in \mathbf{R}, \\ h(t) &= h_-^*(t)D_2(h)h_-(t), \quad t \in \mathbf{R}, \end{aligned} \tag{2.5}$$

where $h_+^{\pm 1} \in (APW)^{n\times n}_{[0,\infty)\cap\Sigma'}$, $h_-^{\pm 1} \in (APW)^{n\times n}_{(-\infty,0]\cap\Sigma'}$, $M\{h_+\} = M\{h_-\} = I$ and $D_1(h)$, $D_2(h)$ are positive definite constant matrices. For $h = h_0$ formula (2.3) implies that x and y are invertible in $(APW)^{n\times n}_{\Sigma}$, and that $M\{x\}$ and $M\{y\}$ are positive definite matrices. In fact, it turns out that in addition

$$x^{-1} \in (APW)^{n\times n}_{[0,\infty)\cap\Sigma}, \quad y^{-1} \in (APW)^{n\times n}_{(-\infty,0]\cap\Sigma}, \tag{2.6}$$

so that spectral factorizations of h_0 can be obtained from (2.3) by setting $h_+ = M\{x\}x^{-1}$, $h_- = M\{y\}y^{-1}$. Then, of course,

$$D_1(h_0) = M\{x\}^{-1}, \quad D_2(h_0) = M\{y\}^{-1}.$$

The extension h_0 has the following additional properties:

Theorem 5.

Let μ *and* Σ *be as in Theorem 4. Let* $f \in (APW)^{n\times n}_{(-\mu,\mu)\cap\Sigma}$ *be such that one (and thus all) of conditions (i) through (iv) in Theorem 4 is satisfied. Define* h_0 *by (2.3). Then* h_0 *is the unique positive extension in* $(APW)^{n\times n}$ *of* f *with the property that*

$$\sigma\left(h_0^{-1}\right) \subseteq [-\mu, \mu]. \tag{2.7}$$

Moreover, if h *is a positive extension in* $(APW)^{n\times n}$ *of* f, *then*

$$D_1(h_0) \geq D_1(h), \quad D_2(h_0) \geq D_2(h) \tag{2.8}$$

with each equality holding if and only if $h = h_0$.

The third main result in this section concerns a description of all positive extensions in $(APW)^{n\times n}_{\Sigma}$ using the parameter set

$$(CAPW)^{n\times n}_{(\mu,\infty)\cap\Sigma} := \left\{ g \in (APW)^{n\times n}_{(\mu,\infty)\cap\Sigma} : \sup_{t\in\mathbf{R}} \|g(t)\| < 1 \right\}.$$

Theorem 6.

Let f, x, *and* y *be as in Theorem 4, and let*

$$u = xM\{x\}^{-\frac{1}{2}}, \quad v = yM\{y\}^{-\frac{1}{2}}.$$

Then each positive extension in $(APW)_\Sigma^{n\times n}$ *of* f *is of the form*

$$T(g) = (u + vg)^{*-1}\left(I - g^* g\right)(u + vg)^{-1},$$

where $g \in (CAPW)_{(\mu,\infty)\cap\Sigma}^{n\times n}$. *Moreover, the correspondence between* $(CAPW)_{(\mu,\infty)\cap\Sigma}^{n\times n}$ *and the set of positive extensions in* $(APW)_\Sigma^{n\times n}$ *of* f *is one-to-one.*

Theorems 4 through 6 were obtained in [22, Sections 3 and 8] under the additional condition that $\mu \in \Sigma$. The reasoning in [22] was based on the band method (see [8]), and the only statement the proof of which actually used this condition was the following auxiliary result in ([22, Lemma 8.3]).

Lemma 1.

Let $f \in (APW)_{[-\mu,\mu]\cap\Sigma}^{n\times n}$ *be such that* $\mathbf{T}(f)_{[0,\mu]\cap\Sigma} > 0$. *Then there exist unique* $x, y \in (APW)_\Sigma^{n\times n}$ *such that*

(1) $\sigma(x) \subseteq [0,\mu]\cap\Sigma$, $\sigma(y) \subseteq [-\mu,0]\cap\Sigma$;

(2) $\sigma(fx - I) \subseteq ((-\infty,0)\cup(\mu,\infty))\cap\Sigma$,
$\sigma(fy - I) \subseteq ((-\infty,-\mu)\cup(0,\infty))\cap\Sigma$.

Moreover, x *and* y *are invertible in* $(APW)_\Sigma^{n\times n}$ *and*

(3) $\sigma(x^{-1}) \subseteq [0,\infty)\cap\Sigma$, $\sigma(y^{-1}) \subseteq (-\infty,0]\cap\Sigma$;

(4) $M\{x\} > 0$, $M\{y\} > 0$.

We will show that Lemma 1 holds even when $\mu \notin \Sigma$. To this end, recall that the following variation of Lemma 1 (with $\mu \in \Sigma$) is also valid (see [22, Section 10]).

Lemma 2.

Let $f \in (APW)_{(-\mu,\mu)\cap\Sigma}^{n\times n}$ *be such that* $\mathbf{T}(f)_{[0,\mu)\cap\Sigma} > 0$, *and assume that* $\mu \in \Sigma$. *Then there exist unique* $x, y \in (APW)_\Sigma^{n\times n}$ *such that*

(5) $\sigma(x) \subseteq [0,\mu)\cap\Sigma$, $\sigma(y) \subseteq (-\mu,0]\cap\Sigma$;

(6) $\sigma(fx - I) \subseteq ((-\infty,0)\cup[\mu,\infty))\cap\Sigma$,
$\sigma(fy - I) \subseteq ((-\infty,-\mu]\cup(0,\infty))\cap\Sigma$.

Moreover, these x *and* y *are invertible in* $(APW)_\Sigma^{n\times n}$ *and*

(7) $\sigma(x^{-1}) \subseteq [0,\infty)\cap\Sigma$, $\sigma(y^{-1}) \subseteq (-\infty,0]\cap\Sigma$;

(8) $M\{x\} > 0$, $M\{y\} > 0$.

In fact, x *and* y *are given by the formula*

$$\left(x_{ij}\right)_{i=1}^n = \left(\mathbf{T}(f)_{[0,\mu)\cap\Sigma}\right)^{-1}\left(e_j\right);\ \left(y_{ij}\right)_{i=1}^n = \left(\mathbf{T}(f)_{(-\mu,0]\cap\Sigma}\right)^{-1}\left(e_j\right). \tag{2.9}$$

We also need the fact that, under the hypotheses of Lemma 2, x and y are continuous functions of f in the norm $\|f\|_W$:

$$\|f\|_W = \sum_{j\in\mathbf{R}} \left\|f_j\right\|.$$

This is not obvious from (2.9), but follows from alternative formulas for x, y. Namely, the positivity of $\mathbf{T}(f)_{[0,\mu)\cap\Sigma}$ implies (see statement **(A)** in [22, Section 10]) that the matrix function

$$G(t) = \begin{bmatrix} f(t) & e^{-i\mu t} I_n \\ e^{i\mu t} I_n & 0 \end{bmatrix}$$

admits a *canonical APW-factorization* $G = AJA^*$, where $J = \begin{bmatrix} I_n & 0 \\ 0 & -I_n \end{bmatrix}$, and $A = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$ is invertible in $(APW)^{2n\times 2n}_{[0,\infty)}$, with inverse $\begin{bmatrix} p & q \\ r & s \end{bmatrix}$, say. In terms of this factorization,

$$x = \left(\gamma q_0^* + \delta s_0^*\right)\left(s_0^* s_0 - q_0^* q_0\right)^{-1}, \quad y = \left(p^*\alpha_0^* + r^*\beta_0^*\right)\left(\alpha_0\alpha_0^* - \beta_0\beta_0^*\right)^{-1},$$

where $\alpha_0 = M\{\alpha\}$, $\beta_0 = M\{\beta\}$, etc. (It may be observed that $\hat{y}(t) = y(-t)$ plays the role of $x(t)$ for $\hat{f}(t) = f(-t)$.) Since the mapping $G \mapsto A$ is continuous in $(APW)^{2n\times 2n}$ norm [26, Theorem 2], the induced mappings $f \mapsto x$ and $f \mapsto y$ are continuous in $(APW)^{n\times n}$ norm as well.

Proof of Lemma 1 ($\mu \notin \Sigma$). It is well known that either $\Sigma = \tau\mathbf{Z}$ for a certain $\tau > 0$ or Σ is dense in $\mathbf{R}$. In the first case, every $f \in (APW)^{n\times n}_{[-\mu,\mu]\cap\Sigma}$ in fact lies in $(APW)^{n\times n}_{[-\mu_0,\mu_0]\cap\Sigma}$ where $\mu_0 = \lfloor \mu/\tau \rfloor \tau$. Since $\mu_0 \in \Sigma$, Lemma 2 allows construction of x, y satisfying (1), (3), (4), and (in place of (2))

$$\sigma(fx - I) \subseteq ((-\infty, 0) \cup (\mu_0, \infty)) \cap \Sigma, \quad \sigma(fy - I) \subseteq ((-\infty, -\mu_0) \cup (0, \infty)) \cap \Sigma .$$

Since $(\mu_0, \infty) \cap \Sigma = (\mu, \infty) \cap \Sigma$, $(-\infty, -\mu_0) \cap \Sigma = (-\infty, -\mu) \cap \Sigma$, (2) is also satisfied.

Let now Σ be dense in $\mathbf{R}$. Choose $\mu' \in \Sigma \cap (0, \mu)$. From the positivity of $\mathbf{T}(f)_{[0,\mu]\cap\Sigma}$ it follows that $\mathbf{T}(f)_{[0,\mu']\cap\Sigma}$ is also positive, with the same lower bound: $\mathbf{T}(f)_{[0,\mu']\cap\Sigma} \geq \epsilon I$. According to Lemma 8.1 in [22], the latter inequality implies that $\mathbf{T}(f)_{[0,\mu']} \geq \epsilon I$. Since ϵ does not depend on μ', and the latter can be chosen arbitrarily close to μ, it follows from here that $\mathbf{T}(f)_{[0,\mu)} \geq \epsilon I$. The latter condition allows us to apply Lemma 2, with Σ replaced by $\mathbf{R}$.

This leads to the existence of $x, y \in (APW)^{n\times n}$ such that

$$\sigma(x) \subseteq [0, \mu),\ \sigma(y) \subseteq (-\mu, 0],\ \sigma(x^{-1}) \subseteq [0, \infty),\ \sigma(y^{-1}) \subseteq (-\infty, 0],$$
$$\sigma(fx - I) \subseteq (-\infty, 0) \cup [\mu, \infty),\ \sigma(fy - I) \subseteq (-\infty, -\mu] \cup (0, \infty),\ \text{and } M\{x\}, M\{y\} > 0 .$$

These x, y are given by the formula

$$\left(x_{ij}\right)_{i=1}^n = \left(\mathbf{T}(f)_{[0,\mu)}\right)^{-1}(e_j)\,;\ \left(y_{ij}\right)_{i=1}^n = \left(\mathbf{T}(f)_{(-\mu,0]}\right)^{-1}(e_j) . \tag{2.10}$$

Since $e_j \in (B)^{n\times 1}_{[0,\mu]\cap\Sigma}$ and $\mathbf{T}(f)_{[0,\mu)}|(B)^{n\times 1}_{[0,\mu]\cap\Sigma} = \mathbf{T}(f)_{[0,\mu]\cap\Sigma}$, the latter formula for x coincides with (2.9). A similar reasoning works for y. It remains only to show that $\sigma(x^{\pm 1}), \sigma(y^{\pm 1}) \subseteq \Sigma$, because then

$$\sigma(fx - I) \subseteq ((-\infty, 0) \cup [\mu, \infty)) \cap \Sigma = ((-\infty, 0) \cup (\mu, \infty)) \cap \Sigma$$

and

$$\sigma(fy - I) \subseteq ((-\infty, -\mu] \cup (0, \infty)) \cap \Sigma = ((-\infty, -\mu) \cup (0, \infty)) \cap \Sigma .$$

To this end, observe that for every positive operator A and a constant $c > ||A||$,

$$A^{-1} = \frac{1}{c}\sum_{j=0}^{\infty}\left(I - \frac{1}{c}A\right)^j .$$

Applying this observation to the positive operators $\mathbf{T}(f)_{[0,\mu)}, \mathbf{T}(f)_{(-\mu,0]}$ in (2.10), we conclude that $\sigma(x), \sigma(y) \subseteq \Sigma$.

Finally, inclusions $\sigma(x^{-1}), \sigma(y^{-1}) \subseteq \Sigma$ can be proved by using the same idea as in the proof of Lemma 8.3 in [22]. Namely, consider a family $f^{(\alpha)} = \alpha I + (1-\alpha)f$, $0 \leq \alpha \leq 1$. All operators $\mathbf{T}(f^{(\alpha)})_{[0,\mu)}$ are positive, so that for each function $f^{(\alpha)} (\in (APW)^{n\times n}_{[-\mu,\mu]\cap\Sigma})$ we can construct $x^{(\alpha)}$, $y^{(\alpha)}$ the same way as x, y were constructed for f. According to the remark after Lemma 2, $x^{(\alpha)}$ and $y^{(\alpha)}$ are continuous functions of $\alpha \in [0, 1]$ together with $f^{(\alpha)}$, which are invertible in $(APW)^{n\times n}$.

Since $x^{(1)}(= I)$ and $y^{(1)}(= I)$ are invertible in $(APW)_{\Sigma}^{n\times n}$, Theorem 7.2 in [22] implies that $x^{(0)}(= x)$ and $y^{(0)}(= y)$ are also invertible in $(APW)_{\Sigma}^{n\times n}$. □

It follows from a general (i.e., independent of the relation between μ and Σ) statement of Lemma 1 and the band method (as developed in [9, 10, 11], and in [8, Chapter XXXIV]) that Theorems 4 through 6 also hold in this general setting; see in [22, Section 8].

To prove the results stated in the Introduction, in case of irrational r we now set $\Sigma = \mathbf{Z} + r\mathbf{Z}$. Consider the map

$$\mathcal{S} : \mathcal{W}_2^{n\times n} \to (APW)_{\Sigma}^{n\times n}$$

defined by

$$(\mathcal{S}f)(t) = \Sigma c_j e^{i(j_1 - rj_2)t}, \quad t \in \mathbf{R}, \tag{2.11}$$

where

$$f(t_1, t_2) = \sum_{j=(j_1,j_2)\in\mathbf{Z}^2} c_j e^{i(j_1t_1+j_2t_2)}, \quad (t_1, t_2) \in \mathbf{R}^2 .$$

In other words,

$$(\mathcal{S}f)(t) = f(t, -rt) .$$

Since r is irrational, we clearly have

$$(j_1, j_2), (k_1, k_2) \in S_{r,\nu}, \ j_1 - rj_2 = k_1 - rk_2 \Rightarrow (j_1, j_2) = (k_1, k_2) .$$

Therefore, $\mathcal{S}$ is an isometric isomorphism between the Banach algebras $\mathcal{W}_2^{n\times n}$ and $(APW)_{\Sigma}^{n\times n}$. The same formula (2.11) also defines Hilbert space isometric isomorphisms (which we also denote by the same letter $\mathcal{S}$)

$$\mathcal{S} : L^2(S_{r,0})^{n\times 1} \to (B)_{\Sigma}^{n\times 1}; \ \mathcal{S} : L^2\left(S_{r,\nu}^+\right)^{n\times 1} \to (B)_{[0,\nu]\cap\Sigma}^{n\times 1} . \tag{2.12}$$

Application of the isomorphism $\mathcal{S}$ is also known as the *Besikovitch trick*, see [3].

Proof of Theorem 1. Due to the isomorphism (2.12), the operators $T_{S_{r,\nu}^+,f}$ (acting on $L^2(S_{r,\nu}^+)^{n\times 1}$) and $\mathbf{T}(\mathcal{S}f)_{[0,\nu]\cap\Sigma}$ (acting on $(B)_{[0,\nu]\cap\Sigma}^{n\times 1}$) are positive only simultaneously. From here and Theorem 4 it follows that the positivity of $T_{S_{r,\nu}^+,f}$ implies the existence of positive extensions $h \in (APW)_{\Sigma}^{n\times n}$ of the function $\mathcal{S}f$. In other words, there exist functions $h \in (APW)_{\Sigma}^{n\times n}$ such that $h_\lambda = (\mathcal{S}f)_\lambda$ for all $\lambda \in [-\nu, \nu] \cap \Sigma$ and $h(x) \geq \epsilon I$ for all $x \in \mathbf{R}$. Then, of course, $g = \mathcal{S}^{-1}h$ $(\in \mathcal{W}_{S_{r,\nu}}^{n\times n})$ deliver positive extensions of the original function f. In particular, the extension g_0 equals $\mathcal{S}^{-1}h_0$, where h_0 is given by (2.3), (2.4). □

Lemma 3.

Let r be irrational and

$$\Lambda = \left\{(k, \ell) \in \mathbf{Z}^2 : \ k - r\ell > 0\right\} .$$

Then any positive definite $g \in \mathcal{W}_2^{n\times n}$ has a Λ-spectral factorization and a $(-\Lambda)$-spectral factorization.

Proof. Let $g \in \mathcal{W}_2^{n\times n}$ be positive definite. Then $h := \mathcal{S}g$ allows factorizations (2.5). To find the Λ- and $(-\Lambda)$-spectral factorizations of g, put $g_\Lambda = \mathcal{S}^{-1}(h_+)$, $D_\Lambda(g) = D_l(h)$, $g_{-\Lambda} = \mathcal{S}^{-1}(h_-)$, $D_{-\Lambda}(g) = D_2(h)$. □

Proof of Theorem 2. Since r is irrational, the set Λ consists of the pairs $(k, l) \in \mathbf{Z}^2$ such that $k - rl > 0$. Hence, for every positive extension $g(\in \mathcal{W}_2^{n\times n})$ of f, (1.2) yields the right spectral factorization (2.5) of $h = \mathcal{S}g$, with $D_l(h) = D_\Lambda(g)$. According to Theorem 5, $D_l(h) \leq D_l(h_0)$, where h_0 is given by (2.3), (2.4). Hence, $D_\Lambda(g) \le D_\Lambda(g_0)$ for $g_0 = \mathcal{S}^{-1}h_0$. On top of that, h_0 is a unique positive extension of $\mathcal{S}f$ having the property $\sigma(h^{-1}) \subseteq [-\nu, \nu]$, so that g_0 is a unique positive extension of f in $\mathcal{W}_2^{n\times n}$ with the property $g_0^{-1} \in \mathcal{W}_{S_{r,\nu}}^{n\times n}$. □

Proof of Theorem 3. According to Theorem 6, all positive extensions of $\mathcal{S}f$ in $(APW)_\Sigma^{n\times n}$ are of the form

$$(U+VG)^{*-1}\left(I-G^*G\right)(U+VG)^{-1}\,,$$

where $U = xM\{x\}^{-1/2}$, $V = yM\{y\}^{-1/2}$ with x, y given by (2.4), $G \in (APW)^{n\times n}_{(\nu,\infty)\cap\Sigma}$ and $\sup_{t\in\mathbf{R}} \|G(t)\| < 1$. Moreover, this correspondence between the set of positive extensions and the open unit ball of $(APW)^{n\times n}_{(\nu,\infty)\cap\Sigma}$ is one-to-one.

It remains to use a (also one-to-one) correspondence (2.11) between $(APW)_\Sigma^{n\times n}$ and $\mathcal{W}_2^{n\times n}$, and set $u = \mathcal{S}^{-1}U$, $v = \mathcal{S}^{-1}V$, $g = \mathcal{S}^{-1}G$. □

3. Proofs of the Main Theorems: The Rational Case

In this section we assume that r is rational. We prove Theorems 1, 2 and 3 in this case by making use of the band method (see [9, 10, 11] and the books [8, 27]). For this, let $S_{r,\nu}^+$ and Λ be as in the statements of Theorems 1 and 2. Further, let $\mathcal{M} = \mathcal{W}_2^{n\times n}$, and introduce the direct sum decomposition

$$\mathcal{M} = \mathcal{M}_1 + \mathcal{M}_2^0 + \mathcal{M}_d + \mathcal{M}_3^0 + \mathcal{M}_4\,, \tag{3.1}$$

with

$$\mathcal{M}_1 = \mathcal{W}^{n\times n}_{\Lambda\setminus S^+_{r,\nu}},\ \mathcal{M}_2^0 = \mathcal{W}^{n\times n}_{S^+_{r,\nu}\setminus\{(0,0)\}},\ \mathcal{M}_d = \mathbf{C}^{n\times n},\ \mathcal{M}_3^0 = \left(\mathcal{M}_2^0\right)^*,\ \mathcal{M}_4 = (\mathcal{M}_1)^*\ ,$$

where the involution * is given by $F^*(t) = (F(t))^*$. The unit e in the algebra $\mathcal{M}$ is the function that is constant equal to I. Note that decomposition (3.1) defines a band structure on $\mathcal{M}$, as defined in [8, Chapter XXXIV.1]. Let C be a sequence for which $T_{S^+_{r,\nu},f_C}$ is positive definite. In order to apply the band method we need to show that the equations

$$P_2(f_C x) = e, \quad P_3(f_C y) = e \tag{3.2}$$

have solutions $x \in \mathcal{M}_2 := \mathcal{M}_2^0 + \mathcal{M}_d$ and $y \in \mathcal{M}_3 := \mathcal{M}_3^0 + \mathcal{M}_d$ such that $P_d x > 0$, $P_d y > 0$, $x^{-1} \in \mathcal{M}_+ := \mathcal{M}_1 + \mathcal{M}_2^0 + \mathcal{M}_d$, and $y^{-1} \in \mathcal{M}_- := \mathcal{M}_4 + \mathcal{M}_3^0 + \mathcal{M}_d$. Here, P_2, P_3, and P_d denote the projections onto $\mathcal{M}_2$, $\mathcal{M}_3$, and $\mathcal{M}_d$ along their respective complements as suggested by (3.1). It is not hard to see that the functions x and y defined in Theorem 1 satisfy Equation (3.2), and are in fact the unique elements to do so. What is less obvious is that these elements x and y have the desired properties. We shall prove this for x in the next proposition, which makes use of the one variable operator valued results in [9]. The proof for y is similar.

Write $r = \frac{p}{q}$, where p, q are relatively prime integers, and $q > 0$. (If $r = 0$, we take $p = 0$, $q = 1$.) Let $N = \lfloor q\nu \rfloor$, the largest integer that does not exceed $q\nu$. Let also

$$\tilde\Lambda = \left\{(k,\ell) \in \mathbf{Z}^2 : 0 \le qk - p\ell \le N\right\}. \tag{3.3}$$

We start with a lemma.

Lemma 4.

Let $f_C \in \mathcal{W}^{n\times n}_{S_{r,\nu}}$. Then $T_{S^+_{r,\nu},f_C}$ is positive definite if and only if $T_{\tilde\Lambda,f_C}$ is positive definite.

Proof. Since $T_{S^+_{r,\nu},f_C} = P_{S^+_{r,\nu}} T_{\tilde\Lambda,f_C} P_{S^+_{r,\nu}}$, the if part follows immediately.

For the only if part, let us first remark that since p and q are assumed to have no common divisor, we have that

$$I_0 \stackrel{\text{def}}{=} \left\{(k,l) \in \mathbf{Z}^2 : qk - pl = 0\right\} = \{m(p,q) : m \in \mathbf{Z}\} . \tag{3.4}$$

This implies

$$L^2(I_0) = \left\{ \sum_{m=-\infty}^{\infty} c_m e^{impt_1} e^{imqt_2} : \sum_{m=-\infty}^{\infty} |c_m|^2 < \infty \right\} . \tag{3.5}$$

Consider the subspaces

$$V_m = \left\{ e^{-impt_1} e^{-imqt_2} h : h \in L^2\left(S_{r,\nu}^+\right)^{n\times 1} \right\}, \quad m = 0, 1, \cdots .$$

In view of the definition of $S_{r,\nu}^+$ we have $V_{m_1} \subseteq V_{m_2}$ if $m_1 < m_2$. Moreover, using (3.5) we easily see that the union $\cup_{m\geq 0} V_m$ is dense in $L^2(\tilde{\Delta})^{n\times 1}$. For $g = e^{-impt_1}e^{-imqt_2}h$, with $h \in L^2(S_{r,\nu}^+)^{n\times 1}$, $m \geq 0$, we have that

$$\begin{aligned} \left\langle T_{\tilde{\Delta},f_C} g, g \right\rangle &= \left\langle e^{-impt_1} e^{-imqt_2} f_C h, e^{-impt_1} e^{-imqt_2} h \right\rangle \\ &= \langle f_C h, h\rangle = \left\langle T_{S_{r,\nu}^+, f_C} h, h \right\rangle \geq \epsilon ||h||_2^2 = \epsilon ||g||_2^2 , \end{aligned}$$

where ϵ is independent of g. By the earlier mentioned density, it follows that $\langle T_{S_{r,\nu}^+, f_C} g, g\rangle \geq \epsilon ||g||_2^2$, for every $g \in L^2(\tilde{\Delta})^{n\times 1}$. □

We will also need the following result for *operator valued* functions. Let $\mathcal{H}$ be a Hilbert space. Denote by $W_{\mathcal{H}}(\mathbf{T})$ the operator Wiener algebra on the unit circle $\mathbf{T}$, that is, the set of all operator valued functions F on $\mathbf{T}$ of the form

$$F(\lambda) = \sum_{-\infty}^{+\infty} \lambda^j \hat{F}(j), \quad \lambda \in \mathbf{T} ,$$

with $\hat{F}(j) \in \mathcal{L}(\mathcal{H})$ (the algebra of all bounded linear operators on $\mathcal{H}$) and $\sum_j \|\hat{F}(j)\| < \infty$. Also, let $\mathcal{P}$ be a projection acting on $W_{\mathcal{H}}(\mathbf{T})$ according to the formula

$$(\mathcal{P}F)(\lambda) = \sum_{0}^{+\infty} \lambda^j \hat{F}(j) ,$$

and denote $\mathcal{P}W_{\mathcal{H}}(\mathbf{T}) = W_{\mathcal{H}}^+(\mathbf{T})$.

Lemma 5.

Let $F \in W_{\mathcal{H}}(\mathbf{T})$ and $F(\lambda)$ is a positive definite operator on $\mathcal{H}$ for each $\lambda \in \mathbf{T}$. Then there exist a positive definite operator $D(F) \in \mathcal{L}(\mathcal{H})$ and F_+ invertible in $W_{\mathcal{H}}^+(\mathbf{T})$ so that $\hat{F}_+(0) = I$ and

$$F(\lambda) = F_+(\lambda)^* D(F) F_+(\lambda) . \tag{3.6}$$

This representation is unique.

Lemma 5 was obtained in [27, Lemma III.3.1], as a variation of the result in [13] for operator functions close to the identity and their scalar multiples. It is important for our purposes that, according to [13] and [27], F_+ and $D(F)$ in (3.6) are given by the formulas

$$D(F) = G_-(\infty)G_+(0), \quad F_+(\lambda) = G_+(0)^{-1}G_+(\lambda) , \tag{3.7}$$

where

$$G_+(\lambda)^{-1} = I - \mathcal{P}M(\lambda) + \mathcal{P}[M(\lambda)\mathcal{P}M(\lambda)] - \cdots, \quad |\lambda| \le 1, \tag{3.8}$$

$$\beta G_-(\lambda)^{-1} = I - \mathcal{Q}M(\lambda) + \mathcal{Q}[(\mathcal{Q}M(\lambda))M(\lambda)] - \cdots, \quad |\lambda| \ge 1, \tag{3.9}$$

$M(\lambda) = \beta^{-1}F(\lambda) - I$, β is an arbitrary positive constant greater than $\max_{\lambda\in\mathbf{T}} \|F(\lambda)\|_{\mathcal{L}(\mathcal{H})}$, $\mathcal{Q} = I - \mathcal{P}$, and

$$G_-^* G_+ = F . \tag{3.10}$$

In the case of finite dimensional $\mathcal{H}$, factorization (3.10) was established in [12], and its special form (3.6) for positive definite matrices in [25].

Proposition 1.

Let C be a sequence for which $T_{S^+_{r,\nu},f_C}$ is positive definite. Then the unique solution $x \in \mathcal{M}_2$ of Equation (3.2) has the properties that (i) $P_d x > 0$ and (ii) x is invertible in $\mathcal{M}_+$.

Proof. Let

$$I_j = \left\{(k,\ell) \in \mathbf{Z}^2 : qk - p\ell = j\right\}, \ j = 0, \pm 1, \dots .$$

Consider the following alternative decomposition of $\mathcal{M}$:

$$\mathcal{M} = \tilde{\mathcal{M}}_1 + \tilde{\mathcal{M}}_2^0 + \tilde{\mathcal{M}}_d + \tilde{\mathcal{M}}_3^0 + \tilde{\mathcal{M}}_4 , \tag{3.11}$$

with

$$\tilde{\mathcal{M}}_1 = \mathcal{M}_1, \tilde{\mathcal{M}}_2^0 = \oplus_{h=1}^N \mathcal{W}_{I_h}^{n\times n}, \tilde{\mathcal{M}}_d = \mathcal{W}_{I_0}^{n\times n}, \tilde{\mathcal{M}}_3^0 = \left(\tilde{\mathcal{M}}_2^0\right)^*, \tilde{\mathcal{M}}_4 = \left(\tilde{\mathcal{M}}_1\right)^* .$$

Note that decomposition (3.11) also defines a band structure on $\mathcal{M}$. In this context it is natural to consider the Toeplitz operator $T_{\tilde{\Delta},f_C} : L^2(\tilde{\Delta})^{n\times 1} \to L^2(\tilde{\Delta})^{n\times 1}$, which by Lemma 4 is positive definite. Note that (in the self explanatory notation)

$$L^2\left(\tilde{\Delta}\right)^{n\times 1} = \bigoplus_{h=0}^{N} L^2\left(I_h\right)^{n\times 1} . \tag{3.12}$$

With respect to decomposition (3.12) the Toeplitz operator $T_{\tilde{\Delta},f_C}$ has the following matrix form:

$$T_{\tilde{\Delta},f_C} = \begin{bmatrix} M_{f_0} & M_{f_{-1}} & \cdots & M_{f_{-N}} \\ M_{f_1} & M_{f_0} & \cdots & M_{f_{-N+1}} \\ \vdots & \vdots & \ddots & \vdots \\ M_{f_N} & M_{f_{N-1}} & \cdots & M_{f_0} \end{bmatrix},$$

where

$$f_h(t) = \sum_{j\in I_h} c_j e^{i\langle j,t\rangle}; \quad h = 0, \pm 1, \dots, \pm N ,$$

and M_{f_h} denotes the multiplication operator

$$M_{f_h} : L^2\left(I_{h_1}\right)^{n\times 1} \to L^2\left(I_{h_1+h}\right)^{n\times 1}; \ 0 \le h_1, h_1 + h \le N .$$
$$M_{f_h}(g) = f_h g, \quad g \in L^2\left(I_{h_1}\right)^{n\times 1} .$$

To put our problem in the context of Section III.3 in [27] or Section II.1 in [9], we use the unitary operator

$$\Phi : L_2\left(I_1\right)^{n\times 1} \to L_2\left(I_0\right)^{n\times 1}$$

defined by

$$(\Phi f)(t) = e^{-i(k^{(1)}t_1+\ell^{(1)}t_2)} f(t), \quad t = (t_1, t_2) \in \mathbf{R}^2 ,$$

where $k^{(1)}, \ell^{(1)}$ are fixed integers such that $qk^{(1)} - p\ell^{(1)} = 1$. Then $\Phi^h : L_2(I_{h_1})^{n\times 1} \rightarrow L_2(I_{h_1-h})^{n\times 1}$. Introduce $\Psi = \operatorname{diag}(\Phi^h)_{i=0}^n$. By Theorem II.1.2 and its proof in [9], the operator polynomial $X(\lambda) = X_0 + X_1\lambda + \ldots + X_N\lambda^N$, $\lambda \in \mathbf{C}$, where

$$\Psi T_{\tilde{\Delta},f_C} \Psi^* \begin{bmatrix} X_0 \\ X_1 \\ \vdots \\ X_N \end{bmatrix} = \begin{bmatrix} I \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{3.13}$$

has the properties that $X_0 > 0$ and $X(\lambda)$ is invertible for all $|\lambda| \leq 1$. Consequently, $X(\lambda)$ is invertible for all $|\lambda| < 1 + \epsilon$ for some positive ϵ, and the inverse has an expansion $X(\lambda)^{-1} = \sum_{k=0}^{\infty} Y_k \lambda^k$, $|\lambda| < 1 + \epsilon$.

To make the connection with the solution x that we are seeking, first of all observe that by using the inequality $\left\langle T_{\tilde{\Delta},f_C} u, u\right\rangle \geq \epsilon \langle u, u\rangle$, where $\epsilon > 0$ is independent of u, for vectors u of the form

$$u = \begin{bmatrix} u_1 \\ u_2 e^{i(k^{(1)}t_1+\ell^{(1)}t_2)} \\ \vdots \\ u_N e^{i(k^{(N)}t_1+\ell^{(N)}t_2)} \end{bmatrix} ; \; u_j \in \mathbf{C}^n; \; qk^{(j)} - p\ell^{(j)} = j ,$$

that the matrix function

$$F(t) = \begin{bmatrix} f_0(t) & \cdots & f_{-N}(t) \\ \vdots & & \vdots \\ f_N(t) & \cdots & f_0(t) \end{bmatrix}$$

is positive definite for all $t \in \mathbf{R}^2$. It follows (one can use induction and Schur complements to verify) that $F(t)^{-1}$ is a matrix function whose entries are in the Wiener algebra $\mathcal{W}_2^{n\times n}$. Write

$$F(t)^{-1} = \begin{bmatrix} f_{00}(t) & \cdots & f_{0N}(t) \\ \vdots & & \vdots \\ f_{N0}(t) & \cdots & f_{NN}(t) \end{bmatrix} .$$

Here $f_{ij} \in \mathcal{W}_{I_{i-j}}^{n\times n}$. Clearly, $f_{00}(t)$ is positive definite for all $t \in \mathbf{R}^2$. Using (3.4), we may identify f_{00} with an $n \times n$ matrix valued function on the unit circle $\mathbf{T}$. According to Lemma 5, it can therefore be factorized in the form

$$f_{00}(t) = \left(I + f_{00}^+(t)\right)^* D \left(I + f_{00}^+(t)\right) , \tag{3.14}$$

where $D \in \mathbf{C}^{n\times n}$ is positive definite and $f_{00}^+(t)$ and $(I + f_{00}^+(t))^{-1} - I$ lie in $\mathcal{W}_{I_0^+}^{n\times n}$, where $I_0^+ = I_0 \cap \Lambda$. Note now that $X(\lambda) = \sum_{k=0}^{N} \Phi^k M_{f_{k0}} \lambda^k$. Because of the special form of the coefficients of X we get that $X(\lambda)^{-1}$ is of the form

$$X(\lambda)^{-1} = \sum_{k=0}^{\infty} \Phi^k M_{y_k} \lambda^k , \tag{3.15}$$

where $y_k \in \mathcal{W}_{I_k}^{n\times n}$. Put now $\tilde{x}(t) = \sum_{k=0}^{N} f_{k0}(t)$. Note that (3.13) implies that $P_{\tilde{\Delta}}(f_C\tilde{x}) = I$, and moreover, (3.15) yields that $\tilde{x}(t)^{-1} = \sum_{k=0}^{\infty} y_k(t) \in \mathcal{W}_{\Lambda\cup I_0}^{n\times n}$. Let now $x(t) = (I + f_{00}^+(t))^{*-1}(f_{00}(t) + \ldots + f_{N0}(t))$. It is now easy to check that x has the desired properties. □

Proposition 2.
Let $r \in \mathbf{Q}$ and

$$\Lambda = \left\{(k,\ell) \in \mathbf{Z}^2 : \ k - r\ell > 0 \ \ or \ \ k - r\ell = 0 \ \ and \ \ k > 0\right\} .$$

Then every positive definite $g \in \mathcal{W}_2^{n\times n}$ has a Λ- and a $(-\Lambda)$-spectral factorization.

Proof. Let $g \in \mathcal{W}_2^{n\times n}$ be positive definite, and let I_j be as in the proof of Proposition 1. Consider the multiplication operator M_g on $L^2(\mathbf{Z}^2)^{n\times 1}$. With respect to the decomposition $L^2(\mathbf{Z}^2)^{n\times 1} = \oplus_{h\in\mathbf{Z}} L^2(I_h)^{n\times 1}$ the operator M_g has the form $M_g = (M_{g^{(k-j)}})_{k,j\in\mathbf{Z}}$, where $g^{(k)}(t) = \sum_{j\in I_k} g_j e^{i<j,t>}$ and g_j, for $j \in \mathbf{Z}^2$, are the Fourier coefficients of g. Consider now

$$S(\lambda) = \sum_{k\in\mathbf{Z}} M_{g^{(k)}} \Phi^k \lambda^k, \quad \lambda \in \mathbf{T}, \tag{3.16}$$

with Φ defined as in the proof of Proposition 1. This is an operator valued function acting on $\mathcal{H} = L^2(I_0)^{n\times 1}$ and positive definite simultaneously with M_g. According to Lemma 5, $S(\lambda)$ admits a factorization

$$S(\lambda) = S_+^*(\lambda) D(S) S_+(\lambda) , \tag{3.17}$$

where S_+ and $D(S)$ are found via (3.7), (3.8), and (3.9) with F replaced by S. Induction shows that each term in the right-hand side of (3.8) and (3.9) has the form $\sum_j M_{q_j}\Phi^j\lambda^j$ with q_j having nonzero Fourier coefficients only in I_j. Hence, $G_\mp^{-1}$ (defined in (3.9) with F replaced by S) inherit the same structure. Due to (3.16), and (3.10), this is also true for G_- $(= SG_+^{-1})$ and G_+ $(= G_-^{-1}S)$. In particular, $G_-(\infty)^{\pm 1}$ and $G_+(0)^{\pm 1}$ have the form M_{q_0}. From here and (3.7) it follows that $D(S) = M_{q_0}$ and $S_+(\lambda) = I + \sum_{j\geq 1} M_{q_j}\Phi^j\lambda^j$. Setting $\lambda = e^{i(k^{(1)}t_1 + l^{(1)}t_2)}$, we obtain from (3.17):

$$g = \sum_{k\in\mathbf{Z}} g^{(k)} = \left(I + \sum_{j\geq 1} q_j\right)^* q_0 \left(I + \sum_{j\geq 1} q_j\right) .$$

Now from the positive definiteness of $D(S) = M_{q_0}$ we conclude that

$$q_0(t) = \sum_{m\in\mathbf{Z}} \xi_m e^{i(pt_1+qt_2)m}$$

also is positive definite for all $t \in \mathbf{R}^2$. Letting $\lambda = e^{i(pt_1+qt_2)}$, we see that $\tilde{q}_0(\lambda) = \sum_{m\in\mathbf{Z}} \xi_m\lambda^m$ is positive definite for all $\lambda \in \mathbf{T}$. Applying Lemma 5 again (this time, in its finite dimensional version) we may write

$$\tilde{q}_0(\lambda) = \left(I + \tilde{q}_0^+(\lambda)\right)^* D(q_0) \left(I + \tilde{q}_0^+(\lambda)\right) ,$$

where $D(q_0) \in \mathbf{C}^{n\times n}$ is positive definite and $\tilde{q}_0^+(\lambda)$, $(I + \tilde{q}_0^+(\lambda))^{-1} - I$ have Fourier expansions of the form $\sum_{j\geq 1} c_j\lambda^j$. Plugging back $e^{i(pt_1+qt_2)}$ in place of λ, we see that

$$q_0(t) = \left(I + q_0^+(t)\right)^* D(q_0) \left(I + q_0^+(t)\right)$$

with q_0^+, $(I + q_0^+)^{-1} - I$ lying in $\mathcal{W}_{I_0^+}^{n\times n}$ for $I_0^+ = I_0 \cap \Lambda$. Let now $g_\Lambda = (I + q_0^+)(I + \sum_{j\geq 1} q_j)$ and $D_\Lambda(g) = D(q_0)$. This gives the Λ-spectral factorization. For $-\Lambda$ a similar reasoning applies. □

Proof of Theorems 1, 2, and 3 (the case of rational *r*). This is a direct application of Theorems XXXIV.1.1, XXXIV.1.2, XXXIV.1.3 (Theorem 1), XXXIV.4.2 (Theorem 2), and

XXXIV.2.1 (Theorem 3) in [8]. That the notions of positive definiteness here and in [8, Chapter XXXIV.1], are the same, follow from the existence of factorization (1.2) for positive functions as shown in Proposition 2. For the unital C*-algebra $\mathcal{R}$ from [8, Chapter XXXIV.1], we choose $(L^\infty(\mathbf{T}^d))^{n\times n}$. With this choice it is easy to see that Axioms (A) in [8, Chapter XXXIV.2], and (C1) and (C2) in [8], Chapter XXXIV.4, are satisfied. □

4. Some Variations and Generalizations

Along with the set $S_{r,\nu}$, one may also consider

$$S^0_{r,\nu} = \left\{(k,l) \in \mathbf{Z}^2 : |k - rl| < \nu\right\} . \tag{4.1}$$

We have the following variation of Theorems 1, 2, and 3.

Theorem 7.

The set $S^0_{r,\nu}$ defined by (4.1) has the positive extension property, with respect to

$$S^{+0}_{r,\nu} = \left\{(k,l) \in \mathbf{Z}^2 : 0 < k - rl < \nu \ \text{or} \ k - rl = 0 \ \text{and} \ k \geq 0\right\} .$$

If $T_{S^{+0}_{r,\nu}, f}$ is positive definite, then all positive extensions of f in $\mathcal{W}_2^{n\times n}$ are given by

$$(u + vg)^{*-1}\left(I - g^* g\right)(u + vg)^{-1} , \tag{4.2}$$

where $g(t)$ is an arbitrary Wiener $n \times n$ matrix function with support in the set $\{(k,\ell) \in \mathbf{Z}^2 : k - r\ell \geq \nu\}$ and such that

$$\sup_{t\in\mathbf{R}^2} \|g(t)\| < 1 ,$$

$u(t) = x(t)P_{\{(0,0)\}}(x)^{-1/2}$, $v(t) = y(t)P_{\{(0,0)\}}(y)^{-1/2}$, and x, y are defined by

$$\left(x_{ij}\right)_{i=1}^n = \left(T_{S^{+0}_{r,\nu}, fc}\right)^{-1}(e_j) ; \ \left(y_{ij}\right)_{i=1}^n = \left(T_{-S^{+0}_{r,\nu}, fc}\right)^{-1}(e_j) .$$

This correspondence is one-to-one.

The extension g_0 [corresponding to $g = 0$ in (4.2)] is a unique positive extension of f with the property $g_0^{-1} \in \mathcal{W}^{n\times n}_{S^{+0}_{r,\nu}}$. Moreover, g_0 is a unique positive extension of f maximizing $D_\Lambda(g)$, where

$$\Lambda = \left\{(k,\ell) \in \mathbf{Z}^2 : \ k - r\ell > 0 \ \text{or} \ k - r\ell = 0 \ \text{and} \ k > 0\right\} .$$

For the case of irrational r and $r \notin \Sigma$, or for the case of rational r and noninteger $q\nu$, the sets $S^0_{r,\nu}$ and $S_{r,\nu}$ coincide. Hence, in these cases Theorem 7 does not contain any additional information when compared with Theorems 1, 2, and 3. For the case of irrational r and $\nu \in \Sigma$ the proof of Theorem 7 is based on the "point excluding variations" of Theorems 4 through 6 (see Section 10 in [22]). For the case of rational r and integer $q\nu$, one can use the same proofs as in Section 3 but with N substituted by $N - 1$.

Theorems 1, 2, and 3 can be generalized to certain sets of $\mathbf{Z}^d$ (with $d > 2$) for which the Besikovitch trick can be used to reduce the problem to the corresponding positive extension problem for almost periodic matrix functions of one variable.

Theorem 8.

Let $r = (r_1, \ldots, r_d)$ be a d-tuple of real numbers which are linearly independent over the field of rational numbers. Then for every positive number ν, the sets

$$S = \left\{ j = (j_1, \ldots, j_d) \in \mathbf{Z}^d \, : \, |\langle j, r \rangle| \le \nu \right\} ,$$

and

$$S^o = \left\{ j = (j_1, \ldots, j_d) \in \mathbf{Z}^d \, : \, |\langle j, r \rangle| < \nu \right\}$$

have the positive extension property, with respect to

$$\Delta = \left\{ j = (j_1, \ldots, j_d) \in \mathbf{Z}^d \, : \, 0 \le \langle j, r \rangle \le \nu \right\}$$

for S, and analogously for S^o.

The formula for all positive extensions [analogous to (4.2)] is valid under the hypotheses of Theorem 8, as well as the description of the positive extension g_0 with maximal $D_\Lambda(g_0)$ analogous to the description of g_0 in Theorem 7.

The proof is analogous to that of Theorems 1, 2, and 3 for the irrational case, by using the isometric isomorphism

$$\mathcal{S} : \mathcal{W}_d^{n\times n} \to (APW)_\Lambda^{n\times n} ,$$

where $\Lambda = r_1\mathbf{Z} + r_2\mathbf{Z} + \cdots + r_d\mathbf{Z}$ is defined by

$$(\mathcal{S}f)(t) = \sum_{j\in \mathbf{Z}^d} c_j \exp\{(r_1 j_1 + \cdots + r_d j_d)\, t\}, \quad j = (j_1, \ldots, j_d)$$

for

$$f(t_1, \ldots, t_d) = \Sigma c_j \exp\{(j_1 t_1 + \cdots + j_d t_d)\} \in \mathcal{W}_d^{n\times n} .$$

Finally, consider the multidimensional generalization of the rational case:

Theorem 9.

Let $r = (r_1, \ldots, r_d)$ be a nonzero d-tuple of rational numbers. Then for every positive number ν, the sets

$$S = \left\{ j = (j_1, \ldots, j_d) \in \mathbf{Z}^d \, : \, |\langle j, r \rangle| \le \nu \right\} ,$$

and

$$U = \left\{ j = (j_1, \ldots, j_d) \in \mathbf{Z}^d \, : \, |\langle j, r \rangle| < \nu \right\}$$

have the positive extension property with respect to

$$\left\{ j = (j_1, \ldots, j_d) \in \mathbf{Z}^d \, : \, 0 < \langle j, r \rangle \le \nu \ \text{ or } \ 0 = \langle j, r \rangle \ \text{ and } \ j_1 \ge 0 \right\}$$

for S, and analogously for U.

Again, the description of all positive extensions, and the characterizations of the positive extension that maximizes $D_\Lambda(g)$ are valid in the context of Theorem 9.

The proof is analogous to that of the rational case of Theorems 1, 2, and 3 (see Section 3): Denote $r_i = \frac{p_i}{q_i}$, where p_i and $q_i > 0$ are relatively prime integers (put $p_i = 0$, $q_i = 1$ when $r_i = 0$). Denote $Q = \operatorname{lcm}(q_1, \ldots, q_d)$, and $T = \gcd(\frac{Qp_1}{q_1}, \ldots, \frac{Qp_d}{q_d}) > 0$. Let now $N = \lfloor \frac{Q}{T}\nu \rfloor$, and

$$I_j = \left\{ (k_1, \ldots, k_d) \in \mathbf{Z}^d : \sum_{m=1}^{d} k_m \frac{p_m Q}{q_m T} = j \right\}, \ j = 0, \pm 1, \ldots .$$

Now proceed as in Section 3.

Thus, in the d-dimensional case we have proved the positive extension property of the sets

$$\left\{j = (j_1, \ldots, j_d) \in \mathbf{Z}^d : |\langle j, r\rangle| \leq \nu\right\}, \quad \left\{j = (j_1, \ldots, j_d) \in \mathbf{Z}^d : |\langle j, r\rangle| < \nu\right\}, \tag{4.3}$$

provided that the dimension of the vector space spanned by $r_1, \cdots, r_d$ (where $r = (r_1, \cdots, r_d)$) over the field of rationals is either 1 or d. In the intermediate cases, when this dimension is strictly between 1 and d, it is an open question whether the sets (4.3) have the positive extension property.

5. Another Version of the Positive Extension Property

In this section we present a variation of the positive extension property. Here, we allow the sequence C to be just bounded; on the other hand, the positive extensions are sought in the set of measures. Only scalar functions will be considered in this section.

Let us give the precise definitions. For a positive Borel measure μ on $\mathbf{T}^d$, we define the *moments* of μ by

$$c_k(\mu) = \int_{\mathbf{T}^d} x_1^{k_1} x_2^{k_2} \cdots x_d^{k_d} \, d\mu(x), \quad x = (x_1, \ldots, x_d) \in \mathbf{T}^d,$$

where $k = (k_1, k_2, \ldots, k_d) \in \mathbf{Z}^d$. The set $\{c_k(\mu)\}_{k \in \mathbf{Z}^d}$ is clearly bounded; it is also *positive (definite)* in the sense that

$$\sum_{k,\ell \in K} c_{k-\ell}(\mu) h_k \bar{h}_\ell \geq 0 \tag{5.1}$$

for every finite set $\{h_k\}_{k \in K}$ of complex numbers. The verification of (6.1) is immediate: for a continuous function

$$h(t) = \sum_{k \in K} h_k e^{i\langle k, t\rangle}, \quad t \in \mathbf{R}^d,$$

the left-hand side of (6.1) coincides with $\int_{\mathbf{T}^d} h(x)\overline{h(x)} d\mu(x)$, where $x = \left(e^{ik_1t_1}, \ldots, e^{ik_dt_d}\right)$.

We can now formulate the positive extension problem for measures. Let be given a set $S \subseteq \mathbf{Z}^d$ and a sequence $C = \{c_j\}_{j \in S}$ of complex numbers such that $\sup_{j \in S} |c_j| < \infty$ (ℓ^∞ sequence).

(PEPM): *Find, if possible, a positive Borel measure μ on $\mathbf{T}^d$ such that $c_j = c_j(\mu)$ for all $j \in S$.*

If $S = \Delta - \Delta$ for some $\Delta \subseteq \mathbf{Z}^d$, then a necessary condition that (PEPM) admits a solution μ is that $\{c_j\}_{j \in S}$ is *positive on S*, i.e.,

$$\sum_{k,\ell \in K} c_{k-\ell} h_k \bar{h}_\ell \geq 0 \tag{5.2}$$

for every finite set $K \subseteq \Delta$ and every set of complex numbers $\{h_k\}_{k \in K}$. If this necessary condition is also sufficient for existence of a solution μ of (PEPM), then we say that S has *positive measure extension property with respect to Δ.*

For example, $\mathbf{Z}^d$ has (PEPM). This is a consequence of Bochner's Theorem and the fact that $\mathbf{T}^d$ is the character group of $\mathbf{Z}^d$. Sasvári proved [24] that vertical bands $\{(k, l) \in \mathbf{Z}^2 : |k| \leq \nu\}$ have the positive moment extension property. Our main theorem in this section is the following generalization of Sasvári's result. It should be noted that there is a simple trick to go from the case when $r = 0$ to the case when $r \in \mathbf{Q}$ (see, e.g., Lemma 1.2.6 in [18]).

Theorem 10.

(a) *The sets*

$$S_{r,\nu} = \left\{(k,\ell) \in \mathbf{Z}^2 : \ |k - r\ell| \le \nu\right\} ,$$

and

$$U_{r,\nu} = \left\{(k,\ell) \in \mathbf{Z}^2 : \ |k - r\ell| < \nu\right\} ,$$

where r is a real number, and ν is a positive number, have the positive measure extension property, with respect to

$$\left\{(k,\ell) \in \mathbf{Z}^2 : \ 0 < k - r\ell \le \nu \ \text{or} \ k - r\ell = 0 \ \text{and} \ k \ge 0\right\} ,$$

and

$$\left\{(k,\ell) \in \mathbf{Z}^2 : \ 0 < k - r\ell < \nu \ \text{or} \ k - r\ell = 0 \ \text{and} \ k \ge 0\right\} ,$$

respectively.

(b) *The sets*

$$\left\{j = (j_1, \dots, j_d) \in \mathbf{Z}^d : |\langle j, r\rangle| \le \nu\right\}$$

and

$$\left\{j = (j_1, \dots, j_d) \in \mathbf{Z}^d : \ |\langle j, r\rangle| < \nu\right\} ,$$

where $r = (r_1, \dots, j_d)$ is a d-tuple of real numbers that are either all rational or linearly independent over the rationals, have the positive measure extension property, with respect to the sets

$$\left\{j = (j_1, \dots, j_d) \in \mathbf{Z}^d : \ 0 < \langle j, r\rangle \le \nu \ \text{or} \ 0 = \langle j, r\rangle \ \text{and} \ j_1 \ge 0\right\}$$

and

$$\left\{j = (j_1, \dots, j_d) \in \mathbf{Z}^d : \ 0 < \langle j, r\rangle < \nu \ \text{or} \ 0 = \langle j, r\rangle \ \text{and} \ j_1 \ge 0\right\} ,$$

respectively.

Lemma 6.

Let $F \in \mathcal{W}^{n\times n}_{S_{r,\nu}}$ be such that $F(t)$ is positive definite for every $t \in \mathbf{R}^2$. Then there exist $G,\ \tilde{G} \in W^{n\times n}_{S^+_{r,\nu}}$ such that

$$F = GG^* = \tilde{G}^*\tilde{G} .$$

Proof. This lemma is easily proved using the factorization (1.2). Indeed, since F is positive, by Propositions 3 and 2 we may factorize F as $(I+H)D(I+H)^*$, where H and $(I+H)^{-1} - I$ belong to $\mathcal{W}^{n\times n}_{\Lambda}$ and $D \in \mathbf{C}^{n\times n}$. In fact, since $F \in \mathcal{W}^{n\times n}_{S_{r,\nu}}$, we get that $H = F(I+H)^{*-1}D^{-1} - I \in \mathcal{W}^{n\times n}_{(S_{r,\nu}-\Lambda)\cap\Lambda} \subseteq \mathcal{W}^{n\times n}_{S^+_{r,\nu}}$. □

Proof of Theorem 10. We prove only part (a) for $S_{r,\nu}$. All other parts of the theorem can be proved similarly. Let a bounded positive sequence $C = \{c_j\}_{j\in S_{r,\nu}}$ be given. Define a functional ϕ_C on $\mathcal{W}_{S_{r,\nu}}$ by

$$\phi_C(h) = \sum_{j\in S_{r,\nu}} c_j h_j , \tag{5.3}$$

where

$$h(t) = \sum_{j\in S_{r,\nu}} h_j e^{i\langle j,t\rangle} \in \mathcal{W}_{S_{r,\nu}} .$$

Clearly, $\phi_C(h)$ is a linear functional on $\mathcal{W}_{S_{r,\nu}}$, and

$$|\phi_C(h)| \leq \sup_{j\in\mathbf{Z}^d} |c_j| \sum_{j\in\mathbf{Z}^d} |h_j| \,.$$

Let $h \in \mathcal{W}_{S_{r,\nu}}$ be such that $h(t) \geq 0$ for all $t \in \mathbf{R}^2$ and the support $\{j \in S_{r,\nu} : h_j \neq 0\}$ of h is finite.

By Lemma 6 (or since we are in the scalar case, we can also use [7]), h admits the factorization

$$h = GG^* \,,$$

where $G \in \mathcal{W}_{S^+_{r,\nu}}$. Writing $G(t) = \sum_{j\in S^+_{r,\nu}} g_j e^{i\langle j,t\rangle}$, we have

$$\phi_C(h) = \sum_{j\in\mathbf{Z}^d} c_j \sum_{k,k-j\in S^+_{r,\nu}} g_k \bar{g}_{k-j} = \sum_{k,\ell\in S^+_{r,\nu}} c_{k-\ell} g_k \bar{g}_\ell \,.$$

Thus, $\phi_C(h) \geq 0$ by the positivity of C. Since the set of functions with finite support is dense in $\mathcal{W}$ (in the norm $\|\cdot\|_{\mathcal{W}}$), using (5.3) we obtain by continuity that $\phi_C(h) \geq 0$ for every $h \in \mathcal{W}_{S_{r,\nu}}$ such that $h(t) \geq 0$ for all t. $\mathcal{W}_{S_{r,\nu}}$ is an operator system in the unital C^*-algebra $C(\mathbf{T}^d)$ of all continuous functions on $\mathbf{T}^d$ (see, e.g., [20] for definition and properties of operator systems.) By Krein's Theorem (see [21, p. 156] or [20, p. 23]), ϕ_C can be extended to a (necessarily bounded) positive functional $\tilde{\phi}_C$ on $C(\mathbf{T}^2)$. By the Riesz Representation Theorem,

$$\tilde{\phi}_C(h) = \int_{\mathbf{T}^d} h\,d\mu, \quad h \in C\left(\mathbf{T}^2\right) \,, \tag{5.4}$$

where μ is a positive Borel measure on $\mathbf{T}^2$. The measure μ satisfies the requirements of the theorem. □

Our final remark connects with Toeplitz operators. Let $S = \Delta - \Delta$, and let a sequence $C = \{c_j\}_{j\in S}$ be given such that $\sum_{j\in S} |c_j| < \infty$. Then the Toeplitz operator defined by $T_{f_C} h = P_\Delta(f_C h)$, $h \in L^2(\Delta)$ is positive semidefinite if and only if C is positive in the sense of (5.2). Indeed, let $h \in L^2(\Delta)$ have only finitely many non-zero Fourier coefficients:

$$h(t) = \sum_{k\in K} h_k e^{i\langle j,t\rangle} \,,$$

where $K \subset \Delta$ is a finite set. Then

$$\langle T_{f_C} h, h\rangle = \langle f_C h, h\rangle = \left\langle \sum_{j\in S} c_j e^{i\langle j,t\rangle} \sum_{k\in K} h_k e^{i\langle k,t\rangle}, \sum_{p\in K} h_p e^{i\langle p,t\rangle} \right\rangle = \sum c_j \bar{h}_p h_k \,,$$

where the sum is over all triples $k \in \Delta$, $p \in \Delta$, and $j \in S$ such that $-p + j + k = 0$, which proves our claim.

6. Application to Spectral Estimation for ARMA Processes

The identification problem for wide sense stationary autoregressive moving averages (ARMA) stochastic processes is a classical signal processing problem. In this section we consider (wide sense) stationary processes $X_{m,n}$ depending on two discrete variables defined on a fixed probability space $(\Omega, \mathcal{A}, P)$. We shall assume that the random variables $X_{m,n}$ are *centered*, i.e., their mean $E(X_{m,n})$

equals zero. Recall that the space $L^2(\Omega, \mathcal{A}, P)$ of square integrable random variables endowed with the inner product

$$\langle X, Y\rangle := E\left(Y^* X\right)$$

is a Hilbert space. A sequence $X = (X_{m,n})_{(m,n)\in \mathbf{Z}^2}$ is called a *stationary process* on $\mathbf{Z}^2$ if for $m, n, k, \ell \in \mathbf{Z}$ we have that

$$E\left(X^*_{m,n} X_{k,\ell}\right) = E\left(X^*_{m+p,n+q} X_{k+p,\ell+q}\right) =: R_X\left(m-k, n-\ell\right), \quad \text{for all } p, q \in \mathbf{Z}\,.$$

It is known that the function R_X, termed the *covariance function* of X, defines a *positive semi-definite function* on $\mathbf{Z}^2$, i.e.,

$$\sum_{i,j=1}^{k} \alpha_i \bar{\alpha}_j R_X\left(r_i - r_j, s_i - s_j\right) \geq 0\,,$$

for all $k \in \mathbf{N}$, $\alpha_1, \ldots, \alpha_k \in \mathbf{C}, r_1, \ldots, r_k, s_1, \ldots, s_k \in \mathbf{Z}$. The theorem of Herglotz, Bochner, and Weil on positive definite functions states that for such a function R_X there is a non-negative measure μ_X defined for Borel sets on the torus $\{(u, v) : u, v \in [0, 2\pi]\}$ such that

$$R_X(r, s) = \int e^{-i(ru+sv)} d\mu_X(u, v)\,,$$

for all integers r and s. The measure μ_X is referred to as the *spectral distribution measure* of the process X. The *spectral density* $f_X(u, v)$ of the process X is the density of the absolutely continuous part of μ_X, i.e., the absolutely continuous part of μ_X equals

$$f_X(u, v)\frac{dudv}{4\pi^2}\,.$$

The classical signal processing question of estimation concerns finding one possible spectral distribution measure of processes X, or the determination of all such measures, based on observations of the covariance function $R_X(r, s)$ of X over a limited region ($(r, s) \in S$, say). When processes with the prescribed observations indeed exist, one may in addition ask for the one(s) with the worst possible prediction error. Helson and Lowdenslager [14] have shown that, with the definition of the past being a halfspace Λ, this prediction error is given by the entropy $\mathcal{E}(f_X)$ of f_X. When this prediction error is > 0 (or, equivalently, $\int \log f_X > -\infty$; a so called *non-deterministic process*) and μ_X is absolutely continuous, the stationary process may be represented as a *moving average*. For this, represent f_X (nonnegative and summable on the bitorus) with respect to the halfspace Λ as a square

$$f_X(u, v) = \left| b_{00} + \sum_{(m,n)\in\Lambda} b_{m,n} e^{-i(mu+nv)} \right|^2, \; b_{0,0} \neq 0, \sum \left|b_{m,n}\right|^2 \leq \infty\,,$$

the existence of which is guaranteed since $\int \log f_X > -\infty$ (see [14, Theorem 3]). Then X_{p_0,q_0}, where (p_0, q_0) is the successor (when it exists) of (0, 0) in the ordering induced by Λ, may be written as

$$X_{p_0,q_0} = b_{0,0}\xi_{0,0} + \sum_{(m,n)\in\Lambda} b_{m,n}\xi_{m,n}\,,$$

where $\xi_{m,n}$ are orthogonal stochastic processes. If, in addition, we may write

$$f_X(u, v) = \left| \frac{e^{\lambda/2}}{1 - H(u, v)} \right|^2,$$

where $H(u, v) = \sum_{(m,n)\in\Lambda} H_{m,n} e^{i(mu+nv)}$, with $\sum \|H_{m,n}\| < \infty$, then we have that

$$X_{0,0} - \sum_{(m,n)\in\Lambda} H_{m,n} X_{m,n} = e^{\lambda/2}\xi_{0,0} , \tag{6.1}$$

which means that the process is *autoregressive*. The set $\{(m, n) \in \Lambda : H_{m,n} \neq 0\}$ is sometimes called the *support* of the autoregressive process X. We refer to Reference [1] as a general reference on ARMA processes and related issues.

The results stated in the introduction may now be interpreted as follows.

Theorem 11.
Let

$$S_{r,\nu} = \left\{(k, \ell) \in \mathbf{Z}^2 : \ |k - r\ell| \leq \nu\right\} ,$$

where r is a real and ν is a positive number, and

$$S_{r,\nu}^+ = \left\{(k, \ell) \in \mathbf{Z}^2 : \ 0 < k - r\ell \leq \nu \ \text{or} \ k - r\ell = 0 \ \text{and} \ k \geq 0\right\} .$$

Given are complex numbers $R(s, t)$, $(s, t) \in S_{r,\nu}$. There exist wide sense stationary processes X such that its covariance function R_X satisfies

$$R_X(s, t) = R(s, t), \quad (s, t) \in S_{r,\nu} , \tag{6.2}$$

if and only if $\{R(s, t) : (s, t) \in S_{r,\nu}\}$ is a positive sequence on $S_{r,\nu}^+$. In case the slope r is rational, and $\{R(s, t) : (s, t) \in S_{r,\nu}\}$ is absolutely summable and strictly positive on $S_{r,\nu}^+$ (i.e., $T_{S_{r,\nu}^+, f_R} > 0$), then X may be chosen to be an ARMA process with support in $S_{r,\nu}^+ \setminus \{(0, 0)\}$. The autoregressive representation (6.1) of an ARMA process with this support is unique and may be found by taking $x(t)$ as in Theorem 1 (with the sequence C replaced by R), and letting

$$H(u, v) = I - x(u, v)D(x)^{-1} ,$$

and $\lambda = -\log D(x)$. The processes with these autoregressive representations are also the processes with the maximal prediction error among all processes satisfying (6.2).

Analogous interpretations can be given to the results stated in Section 4.

Let us remark that in [19] the domain

$$S_{M,N} = \{(n, m) : n = -N, m \geq -M \ \text{ or } \ -N + 1 \leq n \leq N - 1 \ \text{ or } \ n = N, m \leq M\}$$

is considered. In that paper reflection coefficients are developed as well as a 2-D Levinson algorithm. It should be noted that this domain is also conducive to using the band method; however, it is an open problem on how to obtain the appropriate analog of Proposition 1.

Acknowledgments

The authors wish to thank Professor I. Gohberg for suggesting the use of the Besikovitch trick to them, Professor A. Seghier for acquainting them with References [4] and [23], and Professor G. Castro for acquainting them with Reference [1]. In addition, we would like to thank Professor Ph. Loubaton for his comments, especially regarding the signal processing application. The research of MB was partially supported by GSU Grant 97-024. The research of LR and HJW was partially supported by the NSF Grant DMS-9500924. The research of IMS was partially supported by the NSF Grant DMS-9401848. The research of HJW was also partially supported by a Faculty Research Assignment Grant from the College of William and Mary.

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Received August 11, 1997

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