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NEW ACTIONS UPON OLD OBJECTS: A NEW ONTOLOGICAL PERSPECTIVE ON FUNCTIONS

ABSTRACT. We argue in this paper that the systematic use of special software in instruction has a profound impact on the notion of function as an abstract entity to be constructed. We argue that through the medium of the computer, the objects in the graphical, tabular and algebraic settings can change their essence and thus become objects of a new kind we call *representatives*. Actions on representatives which naturally arise in this framework induce an ontological shift. A taxonomy of the skills involved in the learning of the concept of function through these new ontological lenses is presented, as well as software, and problem solving tasks that embody the same ontological perspective. Within the framework of a teaching experiment, students' acquisition of many of the identified skills was investigated by means of a questionnaire and interviews during computer supported problem solving sessions. The most salient results of the study indicate that a majority of students were able (1) to cope with partial data about functions (e.g., problems of interpolation and arbitrariness), (2) to recognize invariants (i. e., properties of functions) while coordinating actions among representatives from different settings, and (3) to recognize invariants while creating and comparing different representatives from the same setting.

1. INTRODUCTION

The mechanisms that underlie the growth of concepts taught in schools are often very different from the mechanisms accompanying concepts growing outside of school. Resnick and Greeno (Resnick and Greeno, 1990; Resnick, 1992; Greeno, 1991) have articulated a theory which posits that important segments of mathematical knowledge have their origins in everyday experience with quantities of physical material and discourse about that material. According to this view, concept acquisition is intimately linked to actions on objects and conservation of invariants under actions. In contrast, school learned concepts are often acquired through analogical and metaphorical processes, in which the learner needs to map a well known concept onto a new (target) knowledge (for example learning about electric circuits using a water flow model; see Gentner and Gentner, 1983). In the case of mathematics, there is often no *source knowledge* that students can map onto high level concepts, and learning is reduced to mapping between several notation systems signifying the same abstract object (Kaput, 1992).

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The concept of mathematical function does not escape this pattern. The relevant symbolic notational systems are the tabular, the graphical and the algebraic settings.[1] By means of them students make descriptions and predictions, and solve problems about functions. In a comprehensive review of the literature about functions, Leinhardt, Zaslavsky and Stein (1990) surveyed research on functions and graphs. Some interesting general phenomena emerged. First, research focusing on students' difficulties shows persistent problems in linking information from different settings, indicating that different entities are signified, and that mental representations corresponding to each of the settings have their own developmental trajectory. In other words, research shows that students' knowledge is compartmentalized. Second, in many instructional research studies, students used notational systems which were not constructed but given to them; when students constructed their own graph, formula, or table, the construction was often technically so demanding that modifications such as changing scales or adding new data were avoided.

In this study, we created an entire functions curriculum, and implemented it in several classes. We took the three classical settings (graphical, tabular and algebraic) as given and designed activities which stressed the relationships between them. As pointed out by Douady (1986), such activities induce interplays between settings ("jeux de cadres"), which allow for changes of setting according to students' progress and the evolution of their concept of function (see also Artigue, 1990). The activities required students to construct new objects belonging to the settings, called representatives, and to manipulate, compare, and transform these objects. A central aim in the design of the activities was the discovery of invariants under actions, i.e. functional properties. Thus, no new notational system was created, but new objects, new actions, and new links among the objects and actions of different settings could be created and were aimed to allow students to construct properties of functions at a level which integrated between the graphical, tabular, and algebraic settings.

The environment we used to achieve this goal, the *Triple Representation Model* (TRM) is a computer microworld that eliminates the "technical load" from tasks on functions and stresses the use of concurrent dynamic settings. It thus shares common properties with other environments such as the *Function Analyzer* (Schwartz and Yerushalmy, 1989) or the *Function Probe* (Confrey, 1991). In each of these systems, settings are linked: operations undertaken in one setting affect the others. A design characteristic specific of TRM is that it not only stresses parallels between settings but also compels students to actively construct the links between them.

In this paper, we investigate two questions of a very different nature. First, we explain why, and how microworlds such as TRM can redefine the function concept. The second question is an experimental one: what is the nature of knowledge acquired by students who manipulate the objects of a microworld about functions? The approach we adopted in this research differs from the theory-driven studies undertaken by Dubinsky and Harel (1992) and Sfard (1992); their approach was to articulate a theory (based in both cases on ideas used in other content domains by Thompson, 1985, and Douady, 1986), and apply it to the notion of function. In their studies, experimental research was to some extent accessory; it played the role of checking out the theory. In our approach, the theory developed from, in conjunction with and simultaneously with the experimental classroom implementation of the curriculum. Comparing the theoretical positions is not the goal of this paper; for this purpose, it would have been necessary to undertake a fine-grained developmental study.

Our scope is limited to a non-developmental analysis of cognitive constructs of individuals. We attempt to describe the status of functional representations for students, how they connected them, and whether they comprehended them as signifying the same abstract entity. We ignored "situations", confining ourselves to the kinds of settings and actions that computerized environments such as TRM could generate. In spite of these limitations, a new view of the nature of functional thinking emerged; the results of the experiment proved this new view to be realistic.

2. SETTINGS AND REPRESENTATIVES

2.1. *Ambiguity*

Graphs, tables, and formulae are often treated as if they characterized a function unambiguously. A table with a constant rate of change is supposed to describe a linear function. Similarly, any straight line segment is considered to be the graph of a linear function. And the formula 1/x is considered to be the function given by $f(x) = 1/x$ in the domain $x \neq 0$. However, such plain formulations hide several problems: For the formula 1/x, the domain has not been specified. Graphs and tables are generally partial and partiality generates ambiguity: For example, the table

x	-1	0	1
y	-1	0	1

matches the functions y = x, $y = x^n$ (n odd) and $y = \sin(\pi x/2)$; and the graphs of $y = x(x + |x|)/2$ or $y = x^2$ look the same on any partial domain contained in $x \geq 0$.

Another problem causes graphs to be ambiguous. Graphs are concrete realizations which are equivocal in the sense that there is a limit to the accuracy of their drawing. For example, in most coordinate systems the graphs of $f(x) = 1/(1 + x)$ and $g(x) = 1 - x + x^2$ appear identical in the bounds $-0.2 < x < 0.2$, $0 < y < 1$. Worse, the graphs of an arbitrary function f(x) appears identical to the graph of $h(x) = 10^{0.0001x} \cdot f(x)$ in a much larger domain. (See Bertin, 1968 and Hajri, 1986 for additional examples). In order to distinguish the two kinds of graphical ambiguity, the former one will be called partiality and the latter one equivocity.

Finally, many algebraic representatives are also ambiguous, for two possible reasons: One is the frequent failure to specify a domain for a function; the other is that there is no unique formula representing a function. For example, the formulae $y = 4x - 12$ and $y = 4(x - 3)$ define the same linear function; similarly, the expressions $|x|$ and $\sqrt{(x^2)}$ define identical objects; whether or not $x + 3$ and $\frac{x^2+x-6}{x-2}$ do, depends on the role which is assigned to the domain of a function and exemplifies the ambiguity arising from this source. Algebraic ambiguities have received far more attention than the tabular or graphical ones; for example, Kieran (1989) discriminates between the different forms that an object can take in the algebraic setting.

2.2. *Curricula and Ambiguities*

All these distinctions may seem subtle, if not pedantic. Why care about the accuracy of a drawing, or about the fact that what looks like a straight line is perhaps not? And indeed, curricula and teachers usually avoid dealing explicitly with ambiguities due to partiality and equivocity. Instead, they tacitly use conventions; for example, graphs and tables are used in spite of being partial if they exhibit as many properties of the function as are deemed sufficient; hereby sufficiency cannot be defined absolutely but may depend on such extraneous factors as curricular goals, grade and ability level, and even on the particular problem under consideration. A graph might be deemed sufficient if it displays domains of increase, extrema, points of inflection and zeros. Similarly, the problem of equivocity is circumvented by using conventions. For example, a smooth graph is assumed to preserve its smoothness when undergoing magnification (see, however, the local straightness approach by Tall, 1991).

In contrast to graphs and tables, algebraic ambiguities are dealt with extensively in most school curricula: students manipulate algebraic expressions in order to find out whether two algebraic formulae define the same function or not.

The fact that educators dodge most problems of ambiguity causes surprising effects: graphs, tables, and formulae representing the same function are grasped as separate static entities, as mathematical objects in their own right, instead of as distinct representatives of a single object; thus students do not experience the need to analyze a graph, because they consider it as displaying all the properties one needs to know. Instruction often concentrates simply on translation skills between these separate objects; and these skills tend to become mere technicalities for the students. As a consequence, they do not have the opportunity to appreciate the overall structure of the notion of function (Schoenfeld, Smith and Arcavi, 1993). It is then not surprising that research has shown that most translation skills taught in schools are difficult, including interpretation of graphs, translation between graphical and algebraic settings, or fitting an algebraic expression to a table of values (e.g., Leinhardt, Zaslavsky and Stein, 1990). In summary, instead of being grasped as an object through the different settings, functions are apprehended as formal entities, the only possible actions on them being algebraic. Avoiding the problem of ambiguity gives rise to compartmentalization of knowledge about functions.

We suggest that ambiguity problems are avoided in standard curricula because students do not have the tools to cope with them. Reordering tables of values according to various criteria is tedious, at best. Drawing a single graph of a function is so time-consuming that drawing several graphs at different scales for comparison purposes cannot be made a standard activity. Moreover, the technical load when drawing graphs is such that beginning students cannot be expected to simultaneously think at the higher level needed to decide which bounds and scales are appropriate to a given task. Instead of becoming the objects of transforming actions, graphs and tables are thus given or constructed once and for all. As Kaput (1992) notices, they are *display notation systems* as opposed to the algebraic setting which is an *action notation system* in which the student can compute values or transform formulae.

This situation is compounded by the fact that in principle, graphs are themselves abstract. Mathematicians distinguish between the graph which is an abstract entity, and the concrete realization of this graph: For mathematicians (and for all the students who learned in the 60's!) the graph of a function is the set of all the points (x, f(x)) in a Cartesian plane. Therefore, the abstract mathematical graph is disconnected from any concrete embodiment: no system of coordinates, and no units. Throughout this paper, the term "graph" will designate a concrete realization of the abstract graph.

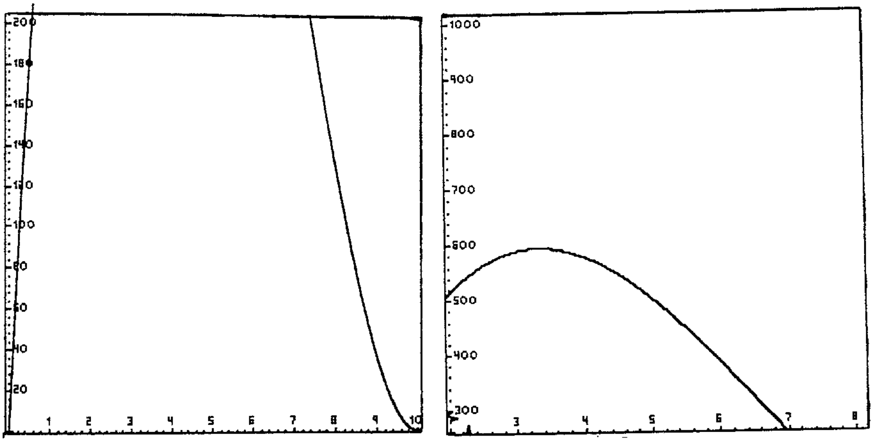


Fig. 1. Two graphical representatives of $f(x) = x(20 - 2x)^2$.

As mentioned in the introduction, concept acquisition is intimately linked to actions. The lack of such actions in certain settings implies that properties may not be seen as invariant through several settings. Such properties may, for the learner, become features of formal objects (graphs, formulae or tables) rather than properties of the concept itself. This is why the above compartmentalization of students' knowledge occurs.

This situation caused us to take a theoretical position which makes explicit the difference between a setting, say the graphical one, and the objects which correspond to a given function in this setting, i. e. the possible graphs of that function; these objects will be called *representatives*. Our position is closely linked to the profound impact which systematic use of specifically designed software can have on the notion of function as an abstract entity to be constructed. We argue that the computer medium replaces each of the canonical objects of the settings by a variety of representatives which may be created by the student, and that the problems of ambiguity arising in traditional instruction can be tackled by means of the actions made possible by the software. This ontological shift allows to consider all settings as action settings where the actions are on representatives. In the following subsection, we shall first discuss the meaning of the term representative and then state what we mean by action on representatives.

2.3. *Representatives of Functions*

In the graphical setting a representative is obtained by choosing a viewing window characterized by the bounds of the x-values and of the y-values. For example, Fig. 1 shows two representatives of the function f(x) =

$x(20 - 2x)^2$. It is important (though obvious) to note that new representatives of a graph can be obtained simply by changing the units of the axes.

In the algebraic setting, specific formulae with or without domain specification are representatives. Tabular representatives are simply all the possible tables obtained by choosing a set of x-values with corresponding y-values. An additional type of representative is intermediate between the tabular and the algebraic: computer environments allow one to automatically generate a set of x values with corresponding values of $y = f(x)$. The resulting representative resembles a table of values, but the fact that the values appear automatically and in the form f(x) confers to this kind of representative a dynamic, algebraic character. We will call these representatives of the *search type*.

We distinguish two kinds of actions. The first kind changes the function itself. Such actions allow one to define and study an auxiliary function in order to solve a problem. Examples include shift and stretch transformations, derivatives in function discussions, and squaring, say, to replace the search for extrema of f(x) by the search for extrema of the simpler function $g(x) = (f(x))^2$. Such activity is typical in more advanced mathematics and therefore this kind of action is rarely included in introductory courses about functions.

The second kind of action does not alter the function but only its representatives. A typical aim is to generate a representative which shows a particular property of the function. Such actions are made possible in typical functions software by means of a set of operations such as scaling, rearranging a table according to a particular criterion such as decreasing x-values, or refining the step of a search type representative (Schwartz and Yerushalmy, 1989). By means of such actions on representatives, students can discover properties which are invariant under change of representative and which are thus characteristic of the function itself. Realizing the existence of such invariant properties is likely to give the student access to the concept of function at a level transcending that of representatives.

3. A BRIEF TAXONOMY OF FUNCTION SKILLS

As has been explained in Section 2, whenever one solves a problem about a function, one is in fact dealing with (acting on, operating on, transforming, ...) one or several representatives of that function. Solving any such problem whatever, will require "thinking about functions". The study of "thinking about" is usually dealt with in terms of "thinking skills". This is not because of the wish to equate the knowledge of a particular concept

The function f satisfies $f(1) = 3$ $f(1.1) = 3.10$ $f(1.2) = 3.30$ $f(1.3) = 3.60$ $f(1.4) = 4$ $f(1.5) = 4.50$ $f(1.6) = 5$ $f(1.7) = 5.40$ $f(1.8) = 5.70$ $f(1.9) = 5.90$ $f(2) = 6$

Which of the following graphs is the 'best fit' for f?

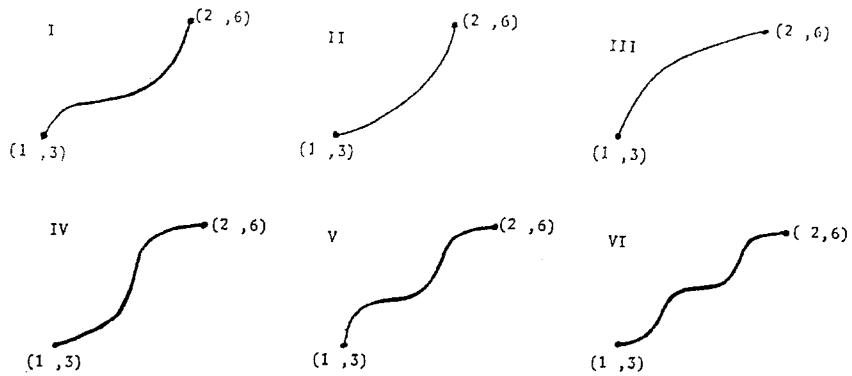


Fig. 2. Q_3: Translating from numerical to graphical information.

with the knowledge of a list of skills, but because a taxonomy of skills constitutes a syntax for dealing with these concepts (Astington and Olson, 1990). While these skills may express, imply or require understanding about the (abstract) function, they will always be carried out on representatives of the function. Skills may link comparable actions on different representatives and compare features of representatives which originate in the same properties of the (abstract) function; skills may infer properties of the (abstract) function from representational information. In the present section, we list and classify the skills which are central for an understanding of the function concept in those terms. This classification is not all inclusive, but most of the skills which are not considered here, are assumed to be held by all the students, for example the fact that an element of the domain of a function has only one image.

3.1. *Overview of Skills*

To begin with, let us consider a problem that will be designated as Q_3 later on (see Fig. 2). There are several ways of solving this problem. But in whichever way one solves it, one needs to translate the numerical information which is given in an algebraic notation into graphical information which will be given in terms of points or slopes. Moreover, one needs in some way or other to deal with the fact that the given numerical information is partial – it contains only eleven points out of a continuum. When integrating all moves towards the solution, one needs to read abstract functional properties (in this case increase and rate of increase) from information given in the numerical or algebraic setting.

Similar analysis of large sets of problems led to the following classes of skills:

S1: Be able to cope with the fact that representational information is partial.
S2: Be able to link between representatives belonging to different settings.
S3: Be able to carry out transformations between representatives within the same setting. This includes mainly dynamic transformations on graphs.

Each of these classes of skills can be performed at different levels, from technical manipulation to the recognition of abstract properties of function. For example, S2 can stand for (i) simple translation of information such as points over (ii) integration of information from several settings to (iii) "seeing" one setting "in" or "through" a representative from another one. In the following subsections, these classes of skills are concretized by listing specific pertinent skills and illustrating them by means of problems in whose solution they are useful.

Remark: As with any classification, this one is not completely clean. It will be seen below that one skill (linking parts of a graph) will be classified into two classes because in one way or another it belongs to either. A few skills could not be classified at all; but it was gratifying to find out, and it confirmed our belief in the usefulness of the proposed classification, that the few skills which could not be classified were either too general (*general problem solving skills*) or too trivial (*remembering the shapes of certain graphs*) to be important at the level of this discussion.

3.2. *Partial Information*

Representatives, to whatever setting they belong, generally present partial information. "Advanced" representatives, in particular formulae, may at least implicitly contain most of the possible information about a function. For example, if a mathematics teacher in a calculus course refers to "the function $f(x) = 1/\sqrt{(x^2 - 4)}$", the well educated student might infer that f(x) is a real valued function defined on the set of reals greater than 2 and smaller than -2 which assigns to any x in this domain a real number y given by the relationship $y = f(x)$. From this, tables of values and graphical settings of various kinds for the same function can be constructed. On the other hand, information about a function given in graphical form is necessarily always partial because of the choice of a viewing window and limited precision of any graphing tools. Similar remarks apply to tables of values. If it is assumed that a rounded picture of the function concept includes representatives from such settings, that in many cases they give

Given the function $f(x) = -2x^3$ Which of the following graphs cannot be a part of the graph of f?

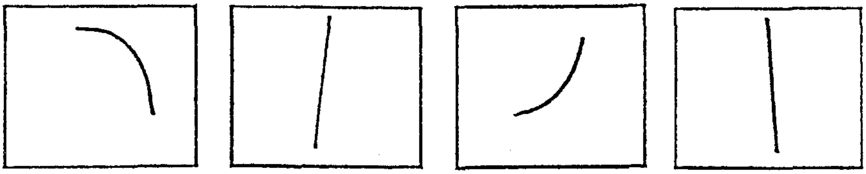


Fig. 3. Q_2: Recognizing whether graphs have the same or different concavity properties.

The graphs below represent a function f in three sub-domains. Draw the graph of the function in one single coordinate system.

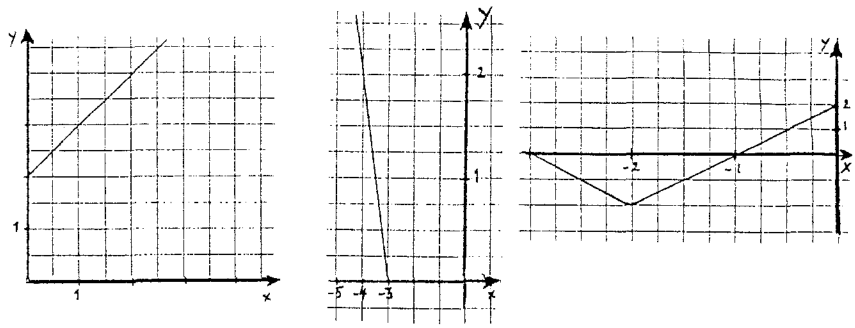


Fig. 4. Q_{1a}: Integrating several partial graphs of a function into a single graph.

a required global picture of the function and that they are sometimes the only available choice, beginning students need to learn how to deal with the partiality.

S1a: PARTIAL DATA: To be able to recognize discrete numerical and/or graphical information about preimage-image pairs (graphed, tabulated, or listed as in Q_3) as belonging to a continuum of data points. Being able to infer properties such as increase of the function from this discrete information. Such recognition is closely linked to skill S1b.

S1b: INTERPOLATION, particularly, interpolation between points of a graph; often, interpolation needs to be sufficiently smooth to take concavity into account. This is the case, for example, in Q_2 displayed in Fig. 3: here, it is necessary to recognize whether two graphs have the same or different concavity properties.

S1c: PARTIAL GRAPH: To be able to recognize and use the fact that any representative from the graphical setting has properties that derive from the abstract mathematical graph. Question Q_{1a} in Fig. 4 illustrates this skill.

The graph of the function f passes through the points shown in the figure.

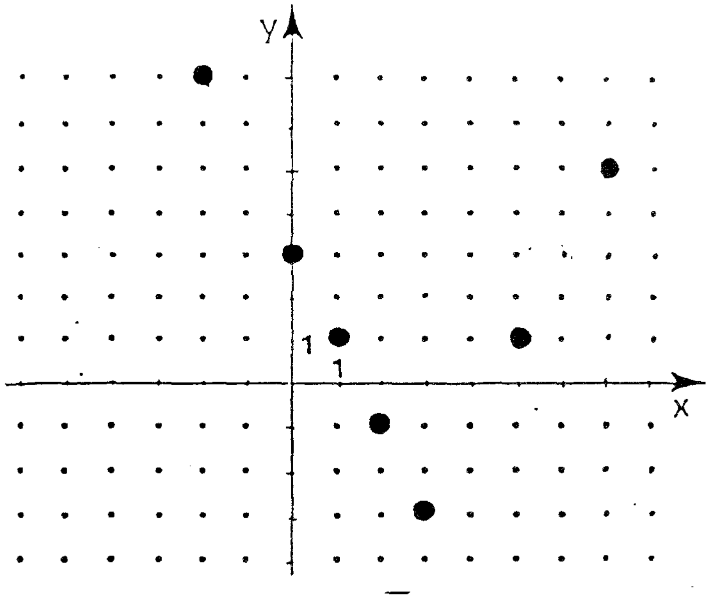


Fig. 5. Q_{4a}: Using the arbitrariness of functions.

1. Which of the following values of x gives the minimum of the function?

(a) x = -3 (b) x = -2 (c) x = 3 = (d) Other (e) Undecidable.

Explain your answer.

2. The minimum satisfies which of the following conditions?

(a) $f(x) = \square$ (b) $f(x) \leq \square$ (c) $f(x) \geq \square$ (d) Other (e) Undecidable.

Explain your answer.

S1d: LINKING PARTS OF A GRAPH: To be able to integrate into a single graph of a function several partial graphs of that function from different, possibly partially overlapping domains and with possibly different scales (see for example Q_{1a}).

S1e: ARBITRARINESS: To be conscious of the fact that a function is "arbitrary" and to use this in order to think in a flexible manner about several/many/all possible interpolations (see skill S1b), even if the given data suggest a very specific interpolation. For example, in Question Q_{4a} in Fig. 5, the student needs to guard against what is suggested as natural, namely a linear interpolation.

3.3. *Links between Different Settings*

The skills of transferring information between different settings have been widely discussed in the literature and we will therefore limit their description to the minimum; however, in view of the curricular goals described above, such transfer skills are considered as prerequisites to skills which go beyond transfer and lead to integration of information into a single, unified concept image (Tall and Vinner, 1981). These integrating and unifying skills are included here under the same headings with the transfer skills because only in specifically designed and well controlled situations have we been able to make a founded judgment on whether a student is simply transferring information or also integrating it. Research results from such a situation have been presented in Schwarz and Dreyfus (1993) and will be recapitulated below (Section 5.3).

S2a: LINKING BETWEEN GRAPHICAL AND NUMERICAL INFORMATION: This includes, for example, the ability to move points from a table or list into a graph, such as in the problem described above (Q_3). But it also includes the ability to induce numerical information using qualitative properties of graphs.

S2b: LINKING BETWEEN NUMERICAL AND ALGEBRAIC INFORMATION: Realize that any algebraic rule is the representative of a set of (numerical) preimage-image pairs and be able to reason within this framework; in other words: be able to handle functional properties as relationships between these ordered pairs. A more advanced aspect of this skill is *symbolization from numerical information*; e.g. find a quadratic function with vertex at (0, 5) which goes through (5, 0).

S2c: LINKING BETWEEN ALGEBRAIC AND GRAPHICAL INFORMATION: This includes graphing on the basis of algebraic information, including many levels from point plotting to calculus based function discussion. An example from Q_5 is, whether $f(x) = 2 + 1/x$ is increasing in $x \geq 0$? This question requires a very high level of algebraic reasoning. The reverse direction, symbolization, includes dealing with graphically given data by means of symbols such as the inequalities in Q_{4a}; more advanced symbolization skills have been investigated by Ruthven (1990).

3.4. *Transformations of Representatives*

In view of what has been explained in Section 2, establishing links between different representatives from the same setting becomes equally important as establishing links between representatives belonging to different settings. In most cases, this concerns dynamically transforming (parts of) graphs into each other; but it also includes transformations within the algebraic setting (such as simplifications), which will be omitted from consideration here and transformations within the numerical settings such as the following skill:

S3a: REORDERING TABLE: Understand that order in a table is irrelevant and be able to reorder a table, e.g. according to increasing values of the independent variable.

S3b: SCALING: This is the prototype link within the graphical setting. It is the ability to recognize and to carry out, at least intuitively, a stretch (or shrink) transformation on one or both axes of a graph; in other words: to construct the transform of a given graph under a different (linear) scaling. This is a visual, analytic rather than an algebraic skill. Q_{1a} directly addresses this skill: the behavior of slope under scale transformations is considered; Q_2 requires an understanding of the fact that the absolute value of slope can be changed by scaling, but not its sign, and that properties such as smoothness, and concavity are qualitatively preserved under scaling, but not the values of slope and curvature.

S3c: LINKING PARTS OF A GRAPH: (Note that the same skill is listed in Subsection 3.2 as skill S1d; it belongs to both classes.) The ability to integrate into a single graph of a function several partial graphs of that function from different domains and with possibly different scales (see for example Q_{1a}).

S3d: TRANSFORMING FUNCTIONS[2]: The ability to create new functions from given ones by specified rules such as shifts and reflections. These transformations were hardly used in this research and the skill is listed primarily in order to provide an outlook onto where a classification of more advanced function skills could lead.

3.5. *Complex Function Skills and Abstraction*

The skills discussed hitherto are high level skills in the sense that they are complex: they concern a network of mutually interconnected pieces of information. They involve in most cases several representatives, often from different settings; they go beyond simple functional skills such as plugging a value into a formula, or reading coordinates in a graph: they establish and use connections between the ways in which different representatives exhibit the same functional properties; they are used to mentally add information to one representative that can only be gleaned from another one or to integrate different properties of the same function on the basis of representatives expressing different parts of that function.

For example, the skill S1d (linking parts of a graph) requires the establishment of links between different partial graphs of the same function; hereby, the scales on the x-axes of the two graphs may be different, the scales on the y-axes may be different, and the two ratios between the respective units on the y- and x- axes may be different so that even if the two windows overlap (which they may) the corresponding representatives may look qualitatively different. The student thus needs to conceptually connect and carry out comparisons between the different scales, as well as corresponding sub-domains (on the x-axes) and corresponding representatives.

Conceivably, such a global complex skill can be taken apart and analyzed into its constituent skills; a start of such a decomposition has been carried out in the previous paragraph. However, learning about these components will not usually bring a student very far in acquiring the global skill because very often, the components make sense only within the larger framework of the global skill.[3] In designing learning activities, we thus considered the skills globally from the top down, rather than by trying to help students to construct them out of constituent skills. A leading idea of this top down approach was to help the students generate links between functional settings by stressing the parallels between them. We argue that the skills listed above support the formation of such links and thus the ability to handle abstract properties of functions. For example, the ability to recognize the property that a function increases from any of its graphical or numerical representatives can be based on skill S1a; it is much harder to recognize increase from an algebraic representative (without differentiation) directly. However, we will see that, by linking algebraic, numeric, and graphical representatives (S2b and S2c), it is possible to acquire a well developed sense of the concept of increase.

Similarly to the property of increase, the general property of "linearity" is generally easier to recognize in the graphical and the algebraic settings, than in the numerical one. Here also, knowing linearity stems from studying partial data, from constructing links among representatives from different settings, and transforming representatives from the same setting into each other.

4. MULTIREPRESENTATIONAL SOFTWARE FOR FUNCTION SKILLS

Because of the globality and the complexity of the skills at which we were aiming, we decided that open ended problem solving activities should be a central feature of the curriculum; sample activities will be given below, in Sections 4.2 and 5.1. The curriculum has been described in more detail elsewhere (Schwarz, Dreyfus and Bruckheimer, 1990; Schwarz and Bruckheimer, 1990). Here we limit ourselves to those aspects that are relevant for the experimental part of the research.

The curriculum was built around a software environment for the following four reasons: first, because appropriately designed software environments support problem solving (Dreyfus, 1991); secondly because in a software environment it can be made easy for students to generate representatives; one role of the software is thus to be a factory for representatives – and a very flexible factory at that; the third reason is that the software environment can make it easy to move between representatives, whether the move is within one setting such as in a rescaling operation or between different settings. Finally, as will be seen in the following subsection, a software environment can be designed in such a way as to stress the parallels between settings.

4.1. *The Triple Representation Model Environment*

In the present research TRM played the role of a tool that supports students in using representatives from several settings to solve function problems and, at the same time, allowed the researchers to identify the setting in which the students act, the representatives they are using, and the progress they make towards solving the problem. Thus it was both, a learning tool and a research tool; many design decisions were taken because of their (presumed) cognitive effectiveness; others were introduced in order to adapt TRM to the research needs; often, the two aims resonated with each other.

TRM is structured into three distinct settings: *Table* (later abbreviated as T), *Graph* (G) and *Algebra* (A). Only one TRM setting is active at any given time. There are two ways in which the student can, from the active setting, access the passive settings: Reading from and switching to another setting. This significant design decision was taken for didactic reasons and because of our research aims. It forces the student to make a conscious choice, in which setting to work; this choice, in turn, enables the researcher to identify in which setting the student acts, at any given moment. Within each setting, TRM is structured operationally. Next, the TRM operations relevant for this paper are described separately for each setting.

Table. The most important operation of the tabular setting is *FindImage*. The *FindImage* operation can be used if a function has previously been defined by means of an equation, in the algebraic setting. *FindImage* can be used to display the y-values corresponding to given x-values in a table. The table thus produced is a particular representative of f.

Graph. The most important operation of the graphical setting is *Draw*. Like FindImage, *Draw* is used to generate representatives of a function which has been defined algebraically. The representatives generated by *Draw* are graphical. Figure 1a shows the graph produced by $y = f(x) = x(20 - 2x)^2$, after the student has specified the bounds $0 \leq x \leq 10$, and $0 \leq y \leq 200$. Obviously, the graph may look quite differently after a change of the viewing window. Figure 1b shows a different graphical representative of the same function. Another operation, *Plot*, allows one to read and plot points on this graph.

Algebra. The operation in the algebraic setting which corresponds to *FindImage* (in T) and *Draw* (in G) is *Compute*. In fact, *Compute* is identical to *FindImage*, except for the manner it is presented on the screen. In *Compute*, the x-value is entered into the blank space of the form "f( ) = " and TRM produces the corresponding y-value after the equality sign.

Another important operation in the algebraic setting is *Define*. *Define* allows one to specify a function by means of a formula and a domain.

A crucial operation in the algebraic setting is *Search*. It produces representatives of the search type (see Section 2.1). It allows one to check for which x-values within a given interval a certain algebraic condition is satisfied. For example, a student who requires an answer to the question for which values of x, in the interval $0 \leq x \leq 10$, the condition $f(x) = x(20 - 2x)^2 > 500$ holds, may specify the numbers 0 and 10 for the interval boundaries, 0.2 for the stepsize and the condition ≥ 500. TRM then selects and displays all values of x at which the condition is satisfied. *Search* thus produces a representative from which it is possible to infer the answer. *Search* can be made very powerful through sophisticated conditions; for example a condition of the type $f(x) > f(x + \varepsilon)$ allows one to check where f is decreasing.

The design of TRM systematically stresses parallels between the three functional settings: For example, zooming in on the viewing window $0 \leq x \leq 10$, $0 \leq y \leq 500$ is parallel to searching for those (x, y) – pairs which satisfy the condition $f(x) \leq 500$, where the search is carried out from $x = 0$ to $x = 10$. Similarly, changing the stepsize of the Search condition is a form of scaling. Like other multirepresentational function software, TRM makes the construction and comparison of many representatives an easy matter. It was, however, not the software environment but the problem solving activities in the curriculum, i. e. the "jeux de cadres" (Douady, 1986), which systematically drew students' attention to the fact that they were dealing with representatives and made them explicitly confront this fact. For example, Q_2 (in Section 3.2) specifically addresses the issue of the variety of graphical representatives of the same function; but similar situations arose naturally during more extensive problem solving activities such as when one student obtained the graphical representative in Figure 1b, rather than the one in 1a, while he was supposed to look for the maximum of the function in the interval $0 \leq x \leq 10$. In the next subsection some typical activities with TRM are presented.

4.2. *Typical Activities with TRM*

The overall goal of the TRM-based curriculum was to tackle the problem of ambiguity and to strive for unified integrated images of functions by creating links between representatives within and among settings. The TRM curriculum creates situations in which representatives need to be constructed, compared, transformed, and coordinated, i. e. problem situations which are solved by what we call an arithmetic of representatives. Situations included: (i) activities in which the student is asked to construct a given representative of a function; (ii) activities for which only some of the settings and operations of TRM were made accessible, the student being invited to reach conclusions under these restrictions; (iii) problem-solving activities requiring the use and comparison of representatives from different settings; (iv) activities in which several representatives are in conflict, the student having to explain why the conflict is apparent only.

A typical activity exemplifying (i) can be generated by asking students to construct, in the graphical setting, the representative appearing in Figure 1a, for the function $f(x) = x(20 - 2x)^2$.

For solving the following problem, only algebraic tools were made available:

Find the minimum of the function f,

$$f\colon \{x \text{ natural}, 0 \leq x \leq 100\} \rightarrow \Re,\ f(x) = x^5 - 5x^4 - 9x^3.$$

In spite of the lack of non-algebraic tools, this problem can be solved with TRM, e. g. by choosing judicious Search conditions. This activity thus illustrates (ii).

The Box Problem (described in Section 5.1) is an example of an activity of type (iii) which invites students to coordinate actions from several settings.

Finally, the following problem illustrates activities of type (iv):

Students are asked to produce three representatives of the function:

$$f\colon \{x \mid -10 \leq x \leq 10\} \rightarrow \Re,\ f(x) = 72x^3 - 54x^2 + 13x - 1$$

and to conclude something about its domain of increase. The representatives are:

a) a Search representative with step 1,

b) a Search representative with step 0.2,
c) a graph in the domain $0 < x < 1$.

The Search representatives show an apparent conflict which is solved by the graphical representative. This example shows that a function satisfying a condition such as $f(x) < f(x + \varepsilon)$ does not necessarily increase within the interval $[x, x + \varepsilon]$. In this activity, the problem of ambiguity is tackled directly by the arithmetic of representatives.

5. THE EXPERIMENT

The experiment described in this section aimed to investigate whether the skills listed in Section 3 were acquired during the experiment. We already indicated that the term "skills" is misleading, especially when one comes to deal with the three general classes of skills pertaining to an arithmetic of representatives. The whole picture of the effect of TRM on students will uncover a conceptual shift, that is linked to the new ontology of the concept of function defined in Section 2.

5.1. *The Tools of the Research*

In this subsection, the tools used to assess the effects of the software are described. These are the Box problem, the Rectangles task, and a Questionnaire. The Box problem and the Rectangles task were given to TRM students, whereas the Questionnaire was used as a comparative tool between experimental and control students.

5.1.1. *The Box problem*

The BOX problem is a maximum problem which is ordinarily given in introductory calculus courses. Software sustaining the development of an arithmetic of representatives makes this problem accessible to junior high school students: *An open box is constructed by removing a square from each corner of a 20 by 20 cm square sheet of tin and folding up the sides. Find the largest possible volume of such a box with an accuracy of* 10^{-4}.

This problem can be solved in many different ways. One could, for example, first compute the volume of the box for several dimensions of the corner square. Thus, if the edge of the corner square is chosen to be $x = 7$, the volume will be $y = 7(20 - 2 \cdot 7)(20 - 2 \cdot 7) = 252$. One could construct a table with different values of x and y, and draw a Cartesian graph in which these (x, y)-values are plotted. One could then evaluate the maximum by linking the points of the graph by linear segments, and by "reading" from the graph the maximum, with a low accuracy. A more

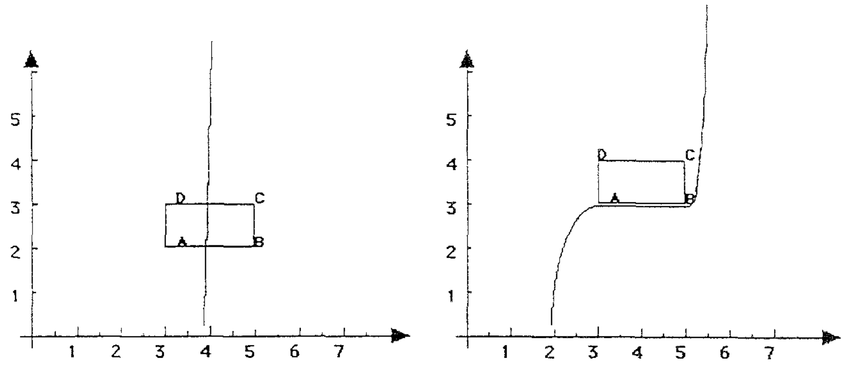


Fig. 6. The graphs of the two undisclosed functions for the Rectangle task.

systematic and accurate way to reach the solution is to express the volume of the box algebraically. The formula is $y = f(x) = x(20 - 2x)^2$. The maximum can then be obtained by searching for the domains of increase and decrease. A formal condition for increase is of the type $f(x + \varepsilon) > f(x)$; a formal condition for decrease is of the type $f(x + \varepsilon) < f(x)$. Students may automatize such symbolic methods, without developing a real sense for what they are doing. Such a sense is more easily developed with graphical and tabular methods. As worded in Section 1, the Box problem enables the application of "interplay between settings" (jeux de cadres), namely the algebraic, graphical, and tabular settings, and subsequently, it provides good opportunities to investigate the nature of students' links among settings, i.e. the skill class S2, discussed in Section 3.3.

5.1.2. *The Rectangles task*

In the Rectangles task, an undisclosed continuous function is chosen by TRM; the student is presented with a rectangle and asked to find out whether the graph of the function passes through the rectangle. "Compute" is the only operation made available to the students for this task. Two such tasks were presented to the students: in one, the graph does cut the rectangle but at a very steep angle; in the other, the graph does not pass through the rectangle but follows it closely. Figure 6 displays the graphs of the two undisclosed functions together with the respective rectangles.

In some way, these functions are likely to create a conflict: students need to deal with the question whether they may rely on (linear) interpolation in order to predict values of images. In the first task, linear interpolation leads to points in the given rectangle; on the other hand, the second task is more problematic; while interpolation is a good strategy, it does not give a definite answer to the student who understands the arbitrary character of functions. In short, the Rectangles task was tailored to give information about the partiality class of skills, especially "partial data", "interpolation" and "arbitrariness".

5.1.3. *The Questionnaire*

The questionnaire was designed to study most of the skills fostered by TRM, and to compare experimental and control students. A crucial fact is that none of the questions needed familiarity with TRM. The TRM curriculum was problem based (see Section 4), and thus did not foster the intensive learning of specific skills. Consequently, the Questionnaire tasks were transfer tasks for both groups. Questions Q_{1a}, Q_2, Q_3, Q_{4a}, and Q_5 described above are five among the eleven questions constituting it. The partiality class of skills (S1) is appropriate to solve Q_{1a}, Q_3, and Q_{4a}; the ability to link between settings (S2) is relevant for solving Q_3, Q_{4a}, and Q_5; and the transformation skills (S3) are useful for Q_{1a}, and Q_2. Thus, there is no one-to-one correspondence between skills and questions, but since students were asked to justify their answers, it was often possible to discern how the different skills were used to solve the questions.

5.2. *Global Evaluation*

Even though the questions asked in the Questionnaire did not contain any direct allusion to the software, it is important to put limitations to its validity as a comparative tool: Experimental students worked in pairs; the role of the teacher was less directive than in the control classes; and the commitment to a new, computer-based curriculum may have positively affected extrinsic motivation. Nevertheless, such an "en bloc" approach is valuable because it enabled us to develop an explanatory frame linked to the nature of the tools used by the students. The comparison was mainly based on a qualitative analysis of answers given to the Questionnaire. The Box problem, and the Rectangles task that were given to TRM students only, were used to characterize some of their cognitive processes.

5.2.1. *The research design*

Three 9^{th} grade classes used the TRM curriculum, Exp1 (n = 30), Exp2 (n = 23), and Exp3 (n = 25). An achievement test was given to the three experimental classes and to several other classes, in order to choose from them control classes which would match the experimental ones. The achievement test checks the prerequisite arithmetic, algebraic and graphical skills. Its reliability is high ($\alpha_{Cronbach} = 0.85$, n = 956) and it discriminates well between different ability levels. The test results led to the choice of control class C1 matching Exp1, C2 matching Exp2, and C3 matching Exp3, in each case with a slight advantage for the control class.

TABLE I

Mean scores of the experimental and the control groups on the questionnaire

Question	Skills	Expl n = 14 + 16	C1 n = 11 + 12	Exp2 n = 11 + 12	C2 n = 11 + 10	Exp3 n = 12 + 13	C3 n = 20 + 18
Q_{1a}	S_1 S_3	0.49	0.31	0.53	0.35	0.82	0.57
Q_{1b}	S_1 S_3	0.54	0.34	0.58	0.40	0.97	0.64
Q_2	S_2 S_3	0.43	0.40	0.40	0.56	0.68	0.53
Q_3	S_1 S_2	0.69	0.30	0.76	0.29	0.81	0.74
Q_{4a}	S_1 S_2	0.64	0.47	0.68	0.49	0.72	0.52
Q_{4b}	S_1 S_2	0.56	0.31	0.58	0.39	0.85	0.57
Q_5	S_2	0.61	0.41	0.70	0.41	0.71	0.69
Q_6	S_2	0.67	0.35	0.61	0.41	0.64	0.69
Q_7	S_2 S_3	0.52	0.16	0.63	0.14	0.72	0.31
Q_{8a}	S_1 S_2	0.71	0.35	0.86	0.43	0.86	0.58
Q_{8b}	S_1	0.55	0.00	0.79	0.05	0.64	0.18

The control classes were taught a current Israeli functions curriculum over the same 12 week period which the experimental classes worked with TRM. Most of the activities in this curriculum were about interpretation within settings, or translation between settings, and only a few were about modeling, constructing, or acting on functions. The Rectangles task was given to Exp1 and Exp2 as an intermediate test during the experiment; each student's performance was recorded in a dribble file. After the experimental period, the Questionnaire was administered to all control and experimental classes. Finally, about half of the TRM students (n = 43) were chosen randomly but in equal numbers from each of the experimental classes and given the Box problem to solve with TRM. The data of all 43 students were collected in dribble files. In order to cope with the problem of interpretability of students' actions, we asked them to use as few operations as they could to find the solution.

5.2.2. *Global comparative results*

Table I contains the mean scores for the experimental and control classes for the eleven questions of the questionnaire. Some questions (those with subscript) were asked of half of the class only, hence the two-part value for n at the head of each column. The scores for each question were ranged between 0 and 1.

A quick examination of Table I shows that the achievement of students in Expl and Exp2 is substantially higher than that of C1 and C2 (except for Q_2). Achievement in Exp3 is the highest, and higher than that of C3 for all except Q_6. Only a small part of the analysis is reported in the remainder of this section (see Schwarz 1989, for more details). This analysis is organized by classes of skills pertaining to the arithmetic of representatives: "partiality" (S1), "links" (S2) and "transformations" (S3).

5.3. *Partiality Skills*

Table I shows that all the questions for which partiality skills were useful uncovered very substantial differences between experimental and control students: The results of Q_{1a} show that TRM students succeed to link several partial graphs, whereas this activity was difficult for the control students; similarly, Q_{1b} showed that TRM students recognized through one graphical representative properties that were hidden in another one. Results on Q_{8a} and Q_{8b} show the superiority of TRM students in inferring graphical continuous information from discrete numerical data.

From Q_3, and Q_{4a}, it appears that TRM students are able to interpolate the general shape of a graph from discrete numerical data. The results for Q_{4a} are eloquent: 22% of the TRM students gave the full answer, 20% gave the answer "x undecidable, f(x) undecidable", and 45% the answer x = 3, f(x) = −3". For the control students, the distribution is very different: 2%, 10%, and 59% respectively. The correct answer, "x undecidable and f(x) ≤ -3" suggests that the student knows that a function with an infinite domain cannot be determined by a finite number of images (arbitrariness). The answer "x undecidable, f(x) undecidable", though not completely correct, suggests that the student grasps that the graph can pass under the point (3, −3) *arbitrarily*. Question $Q_{4b}$was similar to Q_{4a}, except that a specific quadratic formula was given as the rule of the function; its minimum was close but not equal to the lowest point shown in the graph. The answers to this question were similar to those to Q_{4a} with an interesting variant for TRM students: while many of them answered that it was impossible to decide about the exact location of the minimum, others hypothesized its location by using trial-and-error methods and checked their hypotheses by computing a number of images in the neighborhood of the lowest point they found. Such a behavior shows an ability to correctly interpolate values: on the basis of the given partial data, it is reasonable to "try a graph" whose minimum is near to the lowest among the given points. These answers suggest that experimental students were able to infer global properties of functions (here the meaning of minimum, that is "for any x in the neighborhood of the minimum, the function is bigger than for the minimum") from discrete data.

In summary, the scores and the distribution of answers to the Questionnaire strongly suggest that experimental students handled skills from the "partiality" class well. They could interpolate between points of graphs (for example, Q_{4a} and Q_{4b}). They were able to recognize that any repre-

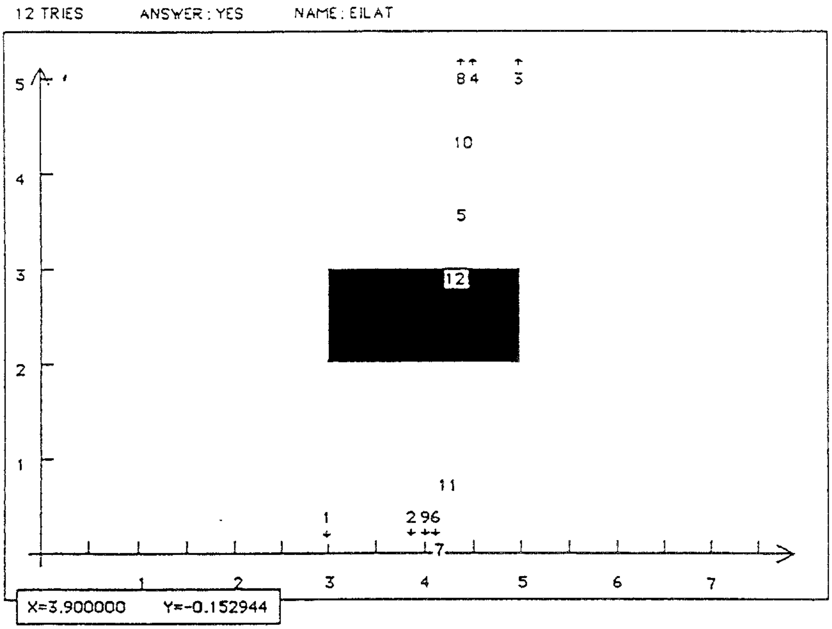


Fig. 7. Eilat's trials for the first Rectangles task.

sentative from the graphical setting has "intrinsic" properties (Q_2). They could integrate into a single graph of a function several partial graphs of that function from different, possibly partially overlapping domains and with possibly different scales (Q_{1a}). Finally, they were conscious of the arbitrariness of a function, and used in a flexible way of several/many/all possible interpolations, even if the given data suggested a specific interpolation (Q_{4a}).

The Rectangles tasks give richer information about this class of skills. Figure 7 shows graphically how Eilat solved the first of these tasks using twelve Compute operations (see Figure 6 for the undisclosed function). Her first trial, for x = 3, yields a negative y-value, far below the rectangle which is positioned between y = 2 and y = 3. She then tries x = 4, and obtains another negative y-value. For x = 5 (third trial), on the other hand, she obtains a very high value for y, far above the y = 3 bound of the rectangle. Her following trial is x = 4.4, presumably because she observed that for x = 4 the y-value is nearer to the rectangle than for x = 5. In the following trials, Eilat attempts to "frame" a value which falls into the rectangle. For example, trial 4 is between 2 and 3, 5 is between 2 and 4, 6 is between 2 and 5, etc.. This process of framing is not flawless; for example, trial 8 is not between 7 and 5; but the trend is for the solution process to converge quickly and efficiently. During this process of framing, Eilat appears to

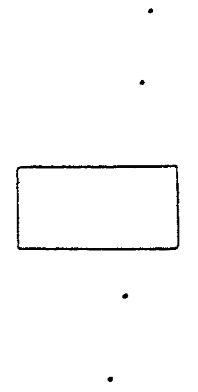
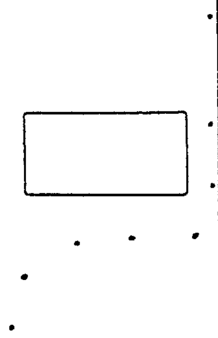
Class	 Yes(not certain)	 No(not certain)	Mean no. of steps
Exp1 $N = 14$	6 (3)	13 (4)	12
Exp2 $N = 17$	12 (3)	17 (9)	8

Fig. 8. Results for the Rectangles tasks.

interpolate, most of the time successfully, the probable behavior of the function from the discrete data she already uncovered.

The rectangle for the second task is $\{3 \leq x \leq 5\} \times \{3 \leq y \leq 4\}$. Eilat's first trial was x = 1, and the second x = 4. The y-value for x = 4 is very close to 3, (the lower bound of the rectangle); Eilat then tried x = 5, which also yielded a value very close to but still below the rectangle. All following trials were made for relevant values of x (within the x-bounds of the given rectangle). After nine trials, she stated that the function does not pass through the rectangle, although, when asked about whether she is sure of it, she said "I think it does not pass, but I am not certain". It seems that she did not completely exclude the possibility that the function might pass through the rectangle.

In summary, Eilat constantly interpolates, although she is aware that such a strategy does not assure one to determine the behavior of the function with absolute certainty (arbitrariness). Figure 8 shows that Eilat's solution was typical in the sense that the results for the Rectangles task were excellent for most of the TRM students.

TABLE II

Justifications of two clauses for question 5

Class	a	b	c	d	wrong
Exp1	52%	17%	0%	5%	26%
Exp2	59%	0%	7%	0%	34%
C3	30%	28%	13%	5%	23%

5.4. *Links Skills*

Table I shows that the experimental students achieved substantially better results than the control students on the questions which related to links between settings (Q_3, Q_{4a}, Q_{4b}, Q_5, Q_6,Q_7 and Q_{8a}). As an example, we focus here on the analysis of Q_3.

Students' justifications to Q_3 were categorized into four groups:

(a) Correct verbal justification without graph, such as: "First the function increases moderately, and then its climbs quickly, and then its rise slows down again".
(b) Construction of a graph and correct comparison with the six presented graphs.
(c) Computation of the rates of change and comparison with the graphs.
(d) Correct answer without justification.

Table II compares the classes Exp1 and Exp2 to C3 with respect to the justifications for their answers to Q_3. We chose to compare Exp1 and Exp2 to C3, because C3 was the best control class (pre-test) whereas Exp1 and Exp2 achieved less than Exp 3 on the pretest. Therefore, the data in Table II which show an advantage of the experimental classes over C3 is highly significant. The most salient fact from Table II is that the students from Exp1 and Exp2 generally used neither graphs nor rates of change to find that the fourth graph matches the series of values. In contrast, students from C3 used graphs and rates of change very often.

In other terms, Table II shows that many TRM students fell into Category (a), i.e. they did not need any new information to interpret the graph in numerical terms, or to interpret the numerical values in graphical terms. Category (a) could be interpreted in two different ways.

(i) The student compares the graph and the numerical data.
(ii) The student sees the graph through the numerical data.

TABLE III
Eilat's operations during the BOX problem

	Operation			
1	A define			
2	G Draw	$0 \leq x \leq 10$	$0 \leq y \leq 200$	
3	G Draw	$2 \leq x \leq 8$	$260 \leq y \leq 1000$	
4	G Draw	$3 \leq x \leq 4$	$550 \leq y \leq 610$	
5	A Search	From 3.285 To 3.378	Step 0.0001	If $f(x) > 593$
6	A Search	From 3.3 To 3.34	Step 0.001	If $f(x) > 592.58$
7	A Search	From 3.316 To 3.36	Step 0.0001	If $f(x) > 592.59259$
8	A Search	From 3.336 To 3.36	Step 0.001	If $f(x) > 592.59259$
9	A Search	From 3.382 To 3.334	Step 0.0001	If $f(x) > f(x + 0.0001)$
10	A Search	From 3.3322 To 3.3334	Step 10^{-5}	If $f(x) > f(x + 0.00001)$
11	A Search	From 3.3332 To 3.3334	Step 10^{-5}	If $f(x) > f(x + 0.00001)$

Both are manifestations of skill S2a – linkage between graphical and numerical settings; they differ, however, in the level of depth of the skill (see Section 3.1): (i) shows coordination among the settings, while (ii) indicates a (unified) level in which information from different settings is integrated. From students' written answers, it was impossible to discern between these two interpretations. However, both go far beyond simple translation of information and point out meaningful coordination (i) or integration (ii).

Just like the Rectangles task provided richer information on the S1 skills, so the Box problem provides richer information on the S2 skills. This task created a problem space in which passages between settings were natural moves. We will show that the analysis of students' solutions to the Box task corroborates the analysis of Q_3; in many cases, we could also show when students coordinated between representatives from different settings, and when they could integrate (or see through) representatives.

In order to illustrate how, we discuss Eilat's solution of the Box problem in detail. Table III shows the sequence of her eleven operations. Eilat started in the algebraic setting and defined the function modeling the volume of the open box. She then moved to the graphical setting and generated three representatives of the function. Figure 1a shows her first graph; it was truncated. The next one she created had the appropriate bounds $2 \leq x \leq 8$, $260 \leq y \leq 1000$ (Figure 1b). Finally, she passed back to the algebraic setting and used seven Search operations to gradually refine her knowledge about the position of the maximum. These operations used various conditions: In operation 5, $f(x) > 593$ (a value she presumably read from the graph during the fourth operation); in operation 6, $f(x) > 592.58$ (the maximal value which appeared on the screen while running the previous Search); in operations 7 and 8, $f(x) > 592.59259$ (the maximal value while running Search for the sixth operation; and from the seventh operation on, $f(x) > f(x + \varepsilon)$ for different ε within ever smaller x-intervals.

Intuitively, it appears from these operations that Eilat consistently took advantage of what she learned from previous operations to decide about the next one. There are no "dead moments". Eilat seems to constantly link among the different representatives she creates. These representatives are graphical or numerical-algebraic (with the Search operation); thus, the links express all skills of class S2. This intuitive impression about Eilat's solution path can be made precise, and this work has been carried out elsewhere (Schwarz and Dreyfus, 1993). Here is a summary of the methodology and results from that work:

The approach was computational. Two numerical indices were computed on the basis of student's TRM operations while solving the Box problem. The main result of the paper was to show that the indices can be reliably interpreted in terms of

(i) the significance the students were able to give to the information they received as feedback to their operations from TRM, and
(ii) the extent and level at which they were able to pass information between, and coordinate or integrate information from different settings.

Eilat's Indices were high, meaning that she correctly interpreted the feedback she received, and that she could coordinate between the different representatives she created, in order to progress rapidly in the solution of the problem.

It is quite difficult to know whether Eilat integrated the properties of all the representatives at a unified level or whether she was only able to coordinate between the different representatives. For several students such as Eilat, the alternative between integration and coordination was undecidable on the basis of the data. Other students, however, solved the Box problem in a way which could clearly be interpreted as integration between settings. Such was the case of Ayelet.

Table IV shows Ayelet's operations. During the entire solution process, Ayelet remained in the algebraic setting and used Search operations only.

TABLE IV

Ayelet's operations on the Box problem

	Operation			
1	A Define			
2	A Search	From 0 To 10	Step 1	If f(x) > 0
3	A Search	From 3 To 4	Step 0.1	If f(x) > 588
4	A Search	From 3.20 To 3.30	Step 0.01	If f(x) > f(x + 0.01)
5	A Search	From 3.30 To 3.40	Step 0.01	If f(x) > f(x + 0.01)
6	A Search	From 3.33 To 3.34	Step 0.001	If f(x) > f(x + 0.001)
7	A Search	From 3.333 To 3.334	Step 0.0001	If f(x) < f(x + 0.0001)

Her solution process converged very quickly. All her actions indicate that she took optimal advantage of TRM's feedback to decide on further operations. For example, in Operation 2, she wrote the condition "From 0 to 10 Step 1, If $f(x) > 0$" and apparently noticed that the largest value of f is obtained when x equals 3 and f(x) equals 588. Thus she was able, in the following operation, to use the condition "$f(x) > 588$" for x between 3 and 4. Her remaining Search conditions were of the form $f(x) > f(x + \varepsilon)$, checking where the function decreases. She finally concluded that the maximum lies between 3.3333 and 3.3334, because the decrease condition is fulfilled for $x \geq 3.3334$.

In conclusion, Ayelet was very skillful at interpreting algebraic operations to find the solution. Since she worked exclusively in the algebraic mode, it is impossible to draw conclusions about her ability to pass information to and from other settings on the basis of her operations alone. However, the interviewer profited of three occasions (Operations 2, 5, and 7) during which a Search condition was "running", to present Ayelet with a graph that showed a rise, followed by a drop and ask her in which interval the Search operation ran; on all three occasions Ayelet was able to answer this question without hesitation by pointing to the appropriate portion of the graph. It seems probable that she used graphical representatives mentally but felt no need to physically construct one, i. e. that Ayelet integrated representatives from all settings into a unified image, at a high conceptual level. For her, deciding on conditions of the kind $f(x + 0.0001) > f(x)$, was a way to interpret her graphical image of the function. She could interpret the numerical data "running" during the Search operation in terms of properties such as increase.

Eilat and Ayelet were typical among the 43 TRM students who solved the Box problem. We categorized these students according to the values of their indices into four categories. This classification showed that many of the students' solutions are similar to Ayelet's or Eilat's: seven students used the algebraic setting exclusively (like Ayelet); they created various Search conditions to find the maximum; sixteen students used several settings, consistently took advantage of TRM feedback to their previous operations in the same or in another setting, and their progress towards the solution was rapid (like Eilat's). In other words, more than half of the TRM students who solved the Box problem fully coordinated and/or integrated the information gleaned from various representatives they created. This is a very high achievement for grade 9 students who had just learned about functions for two hours per week during twelve weeks. Moreover, for most of the remaining 20 students, coordination was flawed at the beginning of the solution process only. With few exceptions, the number of operations did not exceed fifteen. In summary, the Box problem solutions showed that in a problem solving situation, most TRM students were able to link between representatives of the same and of different settings, that many of them did so consistently, and that some of them used the representatives at a unified level where properties of the function were the objects of their actions (see Schwarz & Dreyfus, 1993, for details).

5.5. *Transformation Skills*

Although in some of the activities during the experimental teaching phase transformation skills were useful, this class of skills was somewhat less systematically stressed than the two previous ones. Transformations of representatives, especially scaling, was used as a tool but did not itself constitute the center of attention for any of the activities; transformations of functions (translations, stretches) were not dealt with at all. Moreover, the research design did not include an interview situation set up to investigate transformation skills in a manner similar to the one in which the Rectangles problem was used to investigate partiality skills and the Box problem was used to investigate linkage skills. The analysis of transformation skills therefore has to be based on the questionnaire results and some observations from the teaching phase.

The results on transformation questions in the questionnaire (Q_{1a}, Q_{1b}, Q_2, and Q_7) clearly favor the experimental students, with one exception (Q_2) where performance was about equal. Q_{1a} and Q_2 concerned transformations of the graphical setting (Skills S3b and S3c). Q_{1a} demanded from the student to figure out how graphs look like after undergoing a scaling action. In Q_{1b}, representatives with different units had to be created (again scaling) and assembled (linking different parts of a graph). Similarly, Q_2 could serve as paradigm of a graphical rescaling task. Surprisingly, experimental students performed clearly less well on Q_2 than on Q_1.

We could not clarify the reasons for this "failure" beyond any doubt since we only have the answers from the written questionnaire. However, there are a number of observations concerning transformation skills which contribute to clarify the picture:

(i) Familiarity of the control students with the third power function would explain the results: In fact, it is known from another study (Eisenberg and Dreyfus, 1994) that function transformations are far more accessible when they concern familiar functions than when they concern unfamiliar ones. The third power, is one of the most convenient elementary examples (beyond linear and quadratic functions).
(ii) The types of mistakes of the experimental students on Question 2 tended to be different from those of the control students. Control students tended to disqualify the linear graphs (II and IV). Experimental students often accepted II and IV. It thus appears that control students were convinced that the function was not linear, whereas many of the experimental students were misled by their realization that "any" function can locally look like a linear function. Therefore, even the wrong results of the experimental students show that they do not see the graph of a function statically but can imagine it undergoing a "metamorphosis".
(iii) It may be inappropriate to check transformation skills in a paper-and-pencil test because such skills are inherently linked to dynamic changes.

The experimental results show that the experimental students have a better command of transformation skills than the control students but the difference between the groups is somewhat less striking than for the other two classes of skills.

6. CONCLUSIONS

The most salient results of this research indicate that working with the TRM environment led most of the experimental students (1) to cope with partial data about functions (e.g., problems of interpolation and arbitrariness), (2) to recognize invariants (properties of functions) while coordinating actions among representatives pertaining to different settings, (3) to recognize invariants while creating and comparing representatives pertaining to the same setting.

As described in Sections 3 and 4, the series of activities around TRM created an appropriate environment for enabling students to apprehend the conservation of objects under manipulations. Ambiguity pertaining to settings was lifted: students did not deal with given graphs or tables in which information is partial or equivocal; instead, they could reach information by acting on concrete objects, the representatives, because these became of an *extensive* nature: Graphical representatives could be distorted, cut, or zoomed on; formulae could be expanded to numerical lists of preimage-image pairs that fulfill algebraic conditions; tabular representatives could be created, or ordered. Moreover, the TRM activities fostered the creation of links among settings. As a consequence, properties such as linearity, or maximum, emerged as the invariants conserved under actions on representatives. Students acquired the ability to recognize in one representative many invariants and to compare two representatives in order to decide whether they arose from the same abstract function.

Studying the acquisition of functional thinking with multirepresentational software necessitates the tracing of dynamic processes. This endeavor is delicate. Several assessment tools were created to give an answer to the ensuing methodological problems: the Rectangles task and the Box problem gave some indications about how students cope with problems of arbitrariness or of partial data in "real time", and showed that some students integrated information collected from the creation of several representatives. However, other kinds of assessment tools need to be developed to encompass the complexity of functional thinking in multirepresentational environments.

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NOTES

1. The term "setting" may sound somewhat unusual. A more common term is "representation". However, we have systematically avoided the term representation, because it is misleading: Functions are represented not by representations, but by particular formulae, graphs, tables, etc.. We thank Pat Thompson for making this point clear to us.
   In spite of avoiding the term "representation" otherwise, we kept the original name "Triple Representation Model" to designate the software used in this study.

2. Transformations should be well distinguished mathematically from rescaling: In rescaling the function is preserved but its graph may look different because it is presented in a different(ly scaled) coordinate system; in transformations, however, the function itself is changed, i. e. a new function is generated from the given one (Eisenberg and Dreyfus, 1994).
3. In almost no case, a specific skill can be claimed to be necessary for answering a question. Whether or not a student has a specific skill can therefore never be judged on the basis of the performance of the student on a single question, and far less so if that question was presented in a questionnaire only. How to identify skills anyway will be addressed in Section 5.

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