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SELF LEARNING OF NEGATIVE NUMBER CONCEPTS BY LOWER DIVISION ELEMENTARY STUDENTS THROUGH SOLVING COMPUTER-PROVIDED NUMERICAL PROBLEMS[1]

ABSTRACT. Research has identified difficulties in students' understanding of concepts of either signed or negative numbers and in operations on these numbers. The present study examines the feasibility of teaching certain negative number concepts and procedures to students of a much younger age than is presently done in schools. The method suggested employs the computer for promoting autonomous learning processes through solving challenging problems that are adapted to students' aptitudes, using the number line as an intuitive model. Two fourth grade classes served as the treatment and no-treatment groups. The findings support prior evidence that students have pre-instructional intuitions and informal knowledge of negative numbers and can perform simple operations on them. Such knowledge and intuitions show for high achievers to a much larger extent than for low achievers. Students' related misconceptions are also identified. Pre- and post-treatment tests and interviews reveal that students who received the treatment gained significantly more than those in the no-treatment group regarding all but one of the concepts and procedures of the negative numbers and on the overall score on the test. Low achievers gained at least as much as the high achievers, indicating that the method used here of adjusting the level of challenge to students' aptitude works well. Performing operations on negative numbers proves to be particularly difficult for the lower-achieving students.

School students were found in research to have difficulties in understanding concepts of either negative or signed numbers and in performing procedures with them (e.g., Janvier, 1983). Murray (1985) administered written tests to high-school students (ninth and tenth graders) with at least one full year of formal instruction in signed numbers. These tests produced intermediate results, with low success rates for some of the cases. Interviews with these students identified deeply rooted and widely held misconceptions. Murray also compared the performance on written tests of ninth graders who had received formal instruction on signed numbers with that of eighth graders just before they started such instruction. Findings suggests that formal instruction had had only little effect on the success rate in some of the cases.

The main sources for these difficulties, as suggested by several researchers, are: (a) The conflict between the practical meaning of magnitude

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or quantity associated with numbers in early arithmetic teaching, and the concept of negative numbers (Fischbein, 1987; Hefendehl-Hebeker, 1991); (b) The conflict between the two different meanings (an operation and a direction) of the sign "–" as in (–1) – (+2) (Janvier, 1985; Carraher, 1990); and (c) The absence of a good, intuitive, familiar model which would consistently satisfy all the algebraic properties of signed numbers (Glaeser, 1981, quoted in Fischbein, 1987).

These difficulties have historical roots. For example, the conflict between the negative numbers and the notion of magnitude or quantity, accompanied the history of these numbers from the very beginning. "For many centuries, attempts have been made to attribute to the negative numbers some kind of a practical behavioral validity in order to solve the problem of their legitimacy" (Fischbein, 1987; p. 98). Two other historical difficulties were the perception of zero as absolute zero with nothing "below" it and, consequently, the lack of notion of a uniform number line. The preferred model was then the divided number line, composed of two distinct oppositely-oriented half lines (Hefendehl-Hebeker, 1991). The intellectual hurdles that blocked the understanding of negative numbers throughout its historical evolution may also throw a light on the intellectual hurdles that block the understanding of present-day students (ibid).

Only in the 19th century, negative numbers were given their current interpretation. They are being perceived now as an extension of the (positive +0) number system with the condition that the definite fundamental formulas and rules of computations that are valid in the system of positive (+0) numbers remain valid in the extended number system. In this approach, the concept of number is introduced in a purely formal manner without content meaning or consideration of the notion of magnitude. However, the separation of the construction of the number system from content does not mean that the system is detached from content meaning. Numbers are used to specify and describe real situations (ibid).

STUDENTS' PRE-INSTRUCTIONAL INTUITIONS OF NEGATIVE NUMBERS

There are research indications that children do develop intuitions related to negative numbers and to operating on them, prior to formal instruction. Murray (1985) found that middle and high schools students had pre-instructional intuitions and informal knowledge of negative numbers and a capacity to correctly perform certain computations on these numbers. Moreover, intuitions and informal knowledge of this type appeared even at a younger age – at the elementary-school level. Aze (1989) found that most fourth and fifth graders that he studied had come across "minus" numbers somewhere and that these did not present any apparent difficulty in explorations of the number line "to the left of zero", even when simple calculations were involved. He found that some third and fourth graders even successfully performed mental calculations involving fairly large numbers, both positive and negative. Peled *et al.* (1989) showed that in third and fifth grades, before formal learning of negative numbers, up to half of the children were able to solve many of the operation problems involving these numbers, and almost all fifth graders knew that –4 is larger than –6.

THE BASIS FOR TEACHING NEGATIVE NUMBERS AT THE ELEMENTARY SCHOOL IN OUR STUDY

Murray (1985) concluded from the spontaneous strategies employed by fourth through seventh graders and the ease with which they learned addition with negative numbers, that certain concepts and procedures with negative numbers can already be introduced at the elementary school level, much earlier that it is done in the regular school curriculum. Teachers, however, were much more conservative. A survey among elementary school teachers yielded that many considered it inappropriate to set questions on these numbers and only rarely were negatives mentioned in primary-school mathematics guidelines (ibid).

On the basis of the findings and conclusions described so far, we examined the feasibility of introducing certain concepts of, and procedures with, negative numbers to elementary-school students. The rationale was, first, to lay a foundation for learning the more difficult concepts and procedures at an older age, at middle or high school, and second, that students' intuitions at this age should be reinforced and corrected to prevent the development of misconceptions at a later age, as is shown to happen in Murray's (1985) study.

Introducing negative numbers to elementary students, as planned for our study, neccessitates choosing from the whole scope of negative number concepts and operations those appropriate for this age. That is, to find out which of the copious models of negative numbers are the most appropriate for use by students of a relatively young age; and which of the concepts and procedures related to negative numbers is appropriate for teaching these students. A third question examined here is whether students can acquire these concepts and procedures through a special method of instruction based on computer-supported self learning that we present below. The last topic investigated in our study is the differences between students of high- and low-aptitude for learning mathematics in their pre-instructional intuitions of negative numbers and in their success in learning and understanding of negative-number concepts and procedures.

Because the rest of the discussion is directed towards teaching fourth graders as specified below, the discussion concentrates on whole numbers rather than on the full range of real numbers.

a. Which mental models for negative or signed numbers are the most appropriate for use at the elementary-school level?

The main pedagogical objectives of a model for whole numbers or signed numbers are, on the one hand, to promote the intuitions of their relative magnitudes and of operating on them while preserving the properties of these operations, and, on the other, to remove the "ambiguity" of the minus sign. A variety of models for these objectives have been examined, such as colored numbers or hot-air balloons (Janvier, 1985), assets and debts (Carraher, 1990; Mukhopadhyay *et al.*, 1990), thermometer with temperatures above and below zero (Murray, 1985; Human and Murray, 1987) and transformations on the number line (Thompson and Dreyfus, 1988).

Janvier (1983) sorts the main models for signed numbers into two categories: equilibrium models and number-line models, as described next.

In the equilibrium model, the numbers are elements of two "opposite" kinds such as black and white marbles, checks and bills, negative and positive charges, etc. Adding is defined as combining and subtracting, as removing or as adding the opposite. Research has shown that although this model is coherent, students do have problems with this model, with regard to subtractions and even with additions.

In the number-line model, numbers are represented either by positions on the number line or by displacements on this line (from the origin or not). Addition is defined as a combination of these positions or displacements. Subtraction may be presented as addition of the opposite or as turning around before moving. This model too may cause some problems because of the ambiguity between states and changes (ibid). Thompson and Dreyfus (1988) demonstrate a successful application of this model, using a computerized microworld which presents integers as transformations of position, integer addition as composition of transformations, and negation as an operator upon integers or integer expressions.

Which of the above models would be the most appropriate for elementary-school students?

Studies by Human and Murray (1987), Peled *et al.* (1989), and Mukhopadhyay *et al.* (1990) suggest that for teaching signed numbers to elementary-school children, the use of models based on embodiment of negative numbers in practical situations is not very beneficial. Thus, it seems plausible to use a number-line model rather than an "equilibrium" model (as categorized by Janvier, 1983 and which applies the embodiment of negative numbers). Fischbein (1987) argues that because there is no model of negative numbers that is obvious and intuitive as well as consistently satisfies all the algebraic properties of the operations on these numbers, pupils should be introduced to negative numbers through a simple model which may be consistently applied only to the additive properties of the whole numbers. He suggests the number line for this aim. The National Council of Teachers of Mathematics (1989) also recommends for the fifth- to eighth-grade curriculum the use of the number line for making comparison between negative numbers.

There is research support for the notion that the number-line model is particularly appropriate for young children because it is very intuitive for them. Human and Murray (1987) found that the strategies children intuitively and correctly use for representing negative integers are those of drawing analogies with positive integers. Consequently, these researchers suggest the mental number-line as the most appropriate intuitive model for negative numbers. Resnick (1983) found that most children establish a mental number line for the positive numbers even before school entry. Initially, children use the number line to compare the relative sizes of numbers and over the first years of school they gradually relate the number-line representation to the operations of addition and subtraction. Once the existence of numbers with minus signs has been noted and thought about, children use symmetry principles to extend the number line to these new numbers too. Peled *et al.* (1989) found that fifth graders who were asked to compare negative numbers or to perform addition and subtraction on them constructed a mental number line that included negative numbers.

To summarize, for gaining initial concepts of negative numbers, those suggested to be taught in elementary school, the number line can serve as a natural mental model.

b. Which of the concepts and procedures of negative numbers are appropriate for teaching elementary-school students?

As presented above, Fischbein (1987) suggests to use the number line as a model for only the additive properties of the whole numbers. Murray (1985) also suggests that an introduction to negative numbers can be given to children as young as fourth graders extending only to the addition of signed integers with no rules and/or algorithms.

On the basis of children's descriptions of how they perceive negative numbers and what operations with signed numbers mean to them, Peled (1991) suggests two four-level dimensions for children's knowledge. A quantitative dimension shows negative numbers as representations of amounts, such as owed money. On a number-line dimension, negative numbers reflect the positive numbers as if arranged on a (mental) number line.

Of the four levels of the number-line dimension as identified by Peled, the two first levels are based on the addition of negative numbers with no rules or algorithms, as suggested also by Fischbein and by Murray. They are described by Peled as follows:

Level 1: This level is represented by knowledge of: (a) the existence of negative numbers to the left of zero on the number line; (b) these numbers' location on the number line (going on the number line from zero to the left, one goes to –1, then to –2, to –3 etc.); (c) notation of negative-numbers through adding the minus sign to positive numbers; and (d) the order relation between these numbers which is an extension of what children already know for positive numbers. Given two numbers, any of which is positive or negative, the one which is further to the right on the number line is the larger number.

Level 2: This level is represented by making for the operations of addition and subtraction simple extensions from the positive number domain to the full domain of whole numbers on the number line. These simple extensions mean going to the right on the number line when adding, and to the left when subtracting. This rule is valid for all cases of the starting number (either positive, zero, or negative), or of crossing the zero (e.g., when a larger number is subtracted from a smaller number). The types of operations possible under Level 2 are: (a, b >0) a+b, b+a, a–b, b–a, –a+b, –b+a, –a–b, –b–a where the first number in each sum is the starting point on the number line from which the movement to the left or right takes place.

Addition and subtraction with signed numbers (e.g., a+(–b), a–(+b) or a–(–b) whereas a, b >0), belong to Levels 3 and 4 and are not considered in our study.

Our choice of concepts and procedures to be taught to fourth graders agrees not only with the suggestions by Fischbein and Murray but also with the recommendations by the National Council of Teachers of Mathematics (1989) for curriculum standards in mathematics for even older students. This publication recommends introduction to negative numbers in fifth to eighth grades as part of the expanding knowledge and understanding of number systems and number theory. The following aspects of negative numbers are suggested to be taught in these grade levels: (a) understanding of negative numbers as an extension of the positive number domain that enables to solve problems of the type $x - y$ when $x < y$ (x, y positives); (b) knowledge of order of magnitude of negative numbers, that is, making comparisons between negative numbers and understanding similarities and differences between negative and positive numbers; and (c) understanding operations on negative numbers as the extension of positive number operations.

c. Which is an appropriate method of instruction for these concepts and procedures?

In a search of related literature we found several studies that examined the success of using different models for teaching of negative or signed numbers, as listed above. However, we did not find any studies that examined alternative class instruction methods for presenting negative numbers. We used an instructional method that largely departs from those mostly prevailing in schools. It employs the computer for promoting autonomous learning processes through solving challenging non-routine numerical problems which neccessitate the investment of mental effort. Autonomous learning promotes students' higher-order thinking skills (Peterson, 1989). The method we used is an extension of the principle of learning from worked examples. Research indicates that students who were assigned to solve examples in material they had not formally learned and who followed for solution the steps of worked examples, exhibited self-learning of new arithmetic concepts and procedures related to these examples (Sweller and Cooper, 1985; VanLehn, 1986; and Zhu and Simon, 1987). Hativa (1992) demonstrated that a computerized practice program which displays full solutions when students fail to solve these examples, led to similar results. Our tasks differ from those used in the other studies of self-learning from examples. Rather than providing students with the actual worked solutions to these examples, the software provides students with tools for thinking about the solutions (e.g., using the number line), for evaluating them, and for promoting number concepts and problem-solving strategies (e.g., looking back for analyzing solutions and improving them).

d. What is the role of students' aptitude in pre-instructional intuitions related to negative numbers and in learning concepts and procedures related to these numbers?

Although effects of students' aptitudes on their learning have been copiously examined, we have not found any studies that investigated aptitude-related differences in learning and understanding of negative numbers. In our study we looked to examine whether there are any differential effects of the treatment on the learning of students with different mathematics aptitudes.

OBJECTIVES OF THE STUDY

On the basis of all the above, the following are the objectives of our study

The study examines the effects on elementary students with different aptitude levels, of working with computer-presented problem-solving tasks in negative numbers. The problems solely involve simple addition and subtraction of whole numbers, as described below, using the number-line model. The effects of this work are examined with regard to:

1. Promoting the following concepts and procedures of negative whole numbers:
   a. Location of numbers on a number line.
   b. Comparison of the magnitude of two numbers (understanding order relations between these numbers).
   c. Estimating the distance (closeness) between two numbers.
   d. Performing simple addition and subtraction, those of Levels 1 & 2 of the number-line model (Peled, 1991). These include only operations of the type: $a+b$, $a-b$, $-a+b$, $-a-b$ ($a > 0$, $b > 0$) and not operations with signed numbers. Addition is perceived as going to the right (on the number line) whereas subtraction as going leftward.
2. Differential promotion of the concepts and procedures listed under Objective 1, and of problem-solving strategies, for students of different aptitudes.

THE METHOD

The Software

The software ("The Arithmetic Challenger", developed by the first author) was designed to achieve the goals of the problem-solving tasks, as presented below. The computer screen presents a goal number to be reached, several constraints on the digits and on the arithmetic operations permitted, and on the maximal number of operations to be used for an optimal solution. Only the permitted digits and operations are allowed (see Appendix). The student starts by typing in consecutively a number, an operation, and a second number. When the student presses the "=" sign, the result of the calculation, performed by the computer, is printed on the screen. The objective at each step of the solution is to approach the goal until it is reached. A number line presents, at each stage, the goal and the current number (which is the result of the last step of computation) so that the student can evaluate, by comparing these two numbers, the size of the difference between them and its direction, that is, whether to add or to subtract from the current number.

When the student reaches the goal, screen feedback provides confirmation and informs the student of the number of steps (arithmetic operations) used to reach the goal. The student is commended if he/she has succeeded in arriving at the solution by the least number of steps possible. The screen presentation of the computations is vertical but upon request the user can get a horizontal presentation. The horizontal presentation enables the student to "look back" over all the steps of his/her solution, analyze the solution, and plan for a more efficient approach if needed. The next section and the Appendix present several illustrations of solution processes.

The software includes several functions of management of students' work. It accommodates rules/algorithms for going through the hierarchical sequence of tasks via several different tracks, and provides each student with the next task to be solved. Decisions made by the software as to moving a student from one track to another are based on an elaborate algorithm which takes into account the student's previous performance.[2] However, the software also enables the user to change the track. The software has the option of recording, saving, and printing all steps in the solution process.

The Problem-Solving Tasks

A sequence of tasks was developed to promote children's concepts of negative numbers. The curriculum was based on the number-line model (which is reinforced visually through representation of all stages of the problem on a number-line on the screen – see Figures 1–3), and of operations on negative numbers (those described above). The problems are numerical rather than verbal (word problems) to avoid difficulties stemming from low verbal ability in young students, and to avoid the embodiment of negative-number concepts which was found to be non-beneficial for children of this age, as described above. In this sequence, a student is permitted to use either one or two given digits that are different from zero and can either use zero or not. Each optimal solution can be arrived at by using no more than four operations. The tasks are organized hierarchically in such a way that each requires some level of generalization from the previous one. The mathematical principles underlying the task solution include: There exists more than one way to arrive at a correct solution; a criterion for quality of solution can be defined; different solutions differ in quality; there is a solution that may be regarded as optimal; and solving problems can meet given constraints.

Table I presents a few portions of this task sequence, along with the optimal solution of each task presented, that is, the solution with the minimal number of steps (of arithmetic operations).

Analysis of the task's cognitive demands

The following are solution processes of three tasks, of similar types, that differ on the number domain of the target number. The processes presented here demonstrate the arithmetic procedures, the strategies, the cognitive demands required for solving our tasks and the significant increase in level of difficulty with going from smaller to larger number domains. The solution strategies presented here are those that we observed and identified from the protocols of students' interviews that took place promptly after the solution session.

Each of the three tasks requires to reach a goal through at most four operations using solely "+" or "–", a digit that is different from "0", and either and the digits "1" and "0".

Task 1, in the tens' number domain [Problem 20, Part II in Table I]

The goal: -28.

First step: I approach as close as possible to -28 using tens: $-10 - 10 = -20$.

Second step: I continue with ones: $-20-1-1-1-1-1-1-1-1 = -28$.

Computer response: The goal has been reached in 9 operations. Try to reach it in 4 operations.

Third step [looking back : I can combine twice $-10 - 1 = -11$.

The solution: $-11 - 11 - 1 - 1 - 1 - 1 - 1 - 1 = -28$.

TABLE I

The sequence of tasks. Using at most four operations

Exercise #	Permitted characters	Goal	Optimal solution (minimal # of operations)
Part I			
8)	1; −;	−3	−1 − 1 − 1
15)	1; 0; −;	−12	−11 − 1
21)	3; 0; −;	−63	−33 − 30
23)	3; 2; 0; −;	−53	−23 − 30 or −33 − 20
28)	1; 0; −;	−32	−11 − 11 − 10
36)	1; 2; 0; −;	−34	−22 − 11 − 1 or −21 − 12 − 1
43)	4; 3; 0; −;	−91	−44 − 44 − 3 or −44 − 43 − 4
50)	3; 0; −;	−75	−33 − 33 − 3 − 3 − 3
55)	3; 0; −;	−306	−303 − −3
63)	1; 0; −;	−132	−111 − 11 − 10 or −110 − 11 − 11
73)	3; 0; −;	−630	−300 − 330
78)	1; 0; −;	−325	−111 − 111 − 101 − 1 − 1
91)	2; 0; −;	−682	−222 − 220 − 220 − 20 or −220 − 220 − 220 − 22
Part II			
6)	3,0; +; −;	−27	−30 + 3
20)	1,0; +; −;	−28	−10 − 10 − 10 + 1 + 1
40)	3,0; +; −;	−327	−330 + 3
50)	3,0; +; −;	−2967	−3000 + 33
60)	2,0; +; −;	−19994	−20000 + 2 + 2 + 2
70)	2,0; +; −;	−177998	−200000 + 22002
80)	3,0; +; −;	−3300267	−3300300 + 33
Part III			
1)	1; +; −;	−9	−11 + 1 + 1
10)	2; +; −;	−198	−222 + 22 + 2
20)	3; +; −;	−597	−333 − 333 + 33 + 33 + 3
25)	3; +; −;	−3600	−3333 − 333 + 33 + 33
30)	2; +; −;	−22438	−22222 − 222 + 2 + 2 + 2

Computer response: The goal has been reached in 7 operations. Try to reach it in 4 operations.

Fourth step: I start anew. Now I try to approach -28 as close as possible from the other side, from "above" [that is, from a number which is larger in absolute-value terms. These terms are often used by students]. I start with tens: $-10 - 10 - 10 = -30$.

Fifth step: Now I need to go "down" [that is, add]: $-30 + 1 + 1 = -28$.

Computer response: Congratulations! The goal has been reached in 4 operations.

Task 2, in the hundreds' number domain [Problem 78, Part I in Table I]
The goal: -325.

First step: I approach as close as possible to -325 using hundreds: $-100 - 100 - 100 = -300$ [Figure 1a].

Second step: I continue, using tens: $-300 - 10 - 10 = -320$ [Figure 1b].

Third step: I continue, using ones: $-320 - 1 - 1 - 1 - 1 - 1 = -325$ [Figure 1b].

Computer response [white window in Figure 1b : The goal has been reached in 9 operations. Try to reach it in 4 operations.

Fourth step [looking back, using a special window – Figure 1c – which presents all steps of the solution, arranged horizontally]: I can combine twice $-100 - 10 - 1 = -111$. I start anew. The solution is: $-111 - 111 - 100 - 1 - 1 - 1$ [typed vertically in 5 steps].

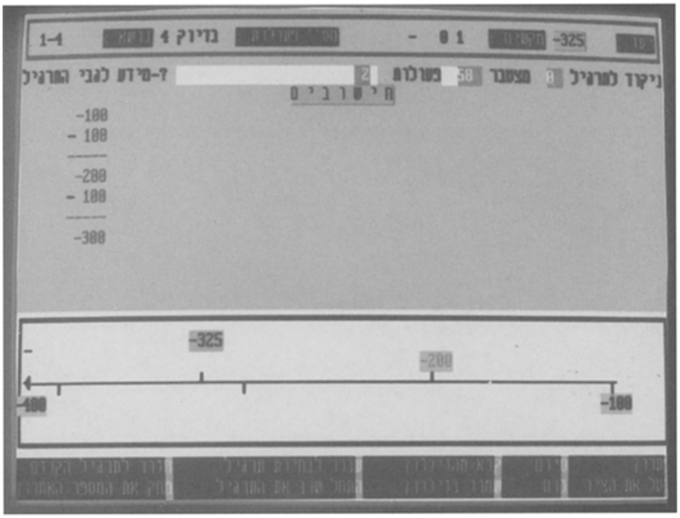
Computer response: The goal has been reached in 5 operations. Try to reach it in 4 operations.

Fifth step [looking back, again : I can combine also $-100 - 1 = -101$. I start anew. The solution is: $-111 - 111 - 101 - 1 - 1$ [Figure 2a].

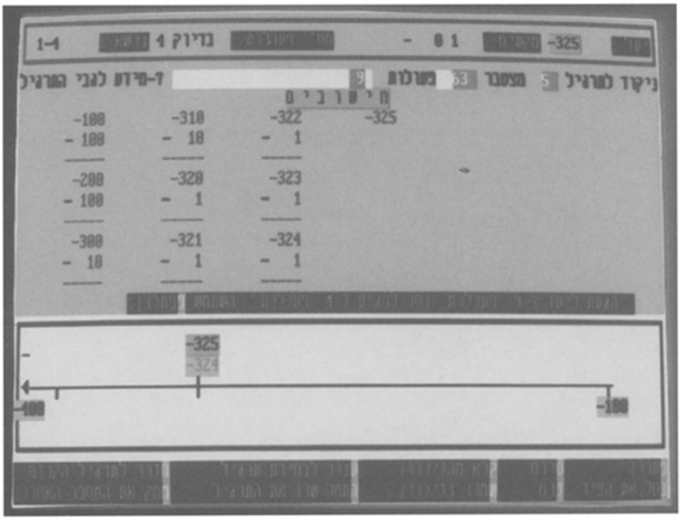
Computer response: Congratulations! The goal has been reached in 4 operations.

After experiencing several exercises of the type demonstrated here, the better students developed a "sense" of when they should go "above" or "to the left of" the target number, when "below it" or "to its right", and of when they could combine several negative numbers into a single number. This was true for number domains that they had already practiced. However, in going to larger numbers, they would again often need several trials (each entails solving the exercises anew) until reaching the optimal solution.

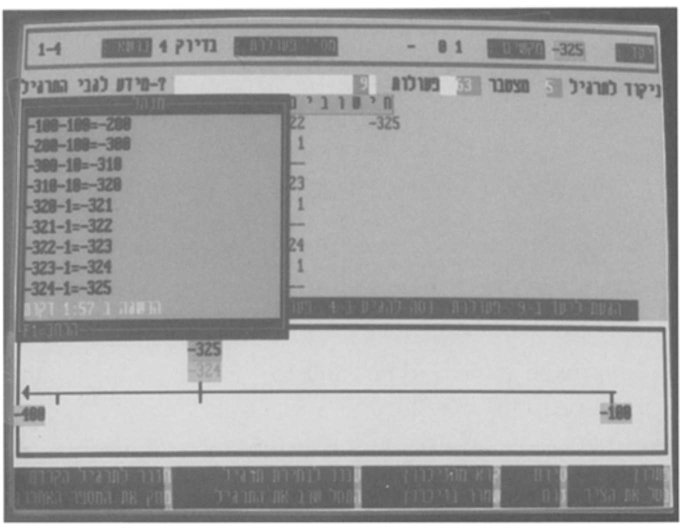
For the next task we present only the optimal solution, as taken from the protocols of one of the best students.



(a)

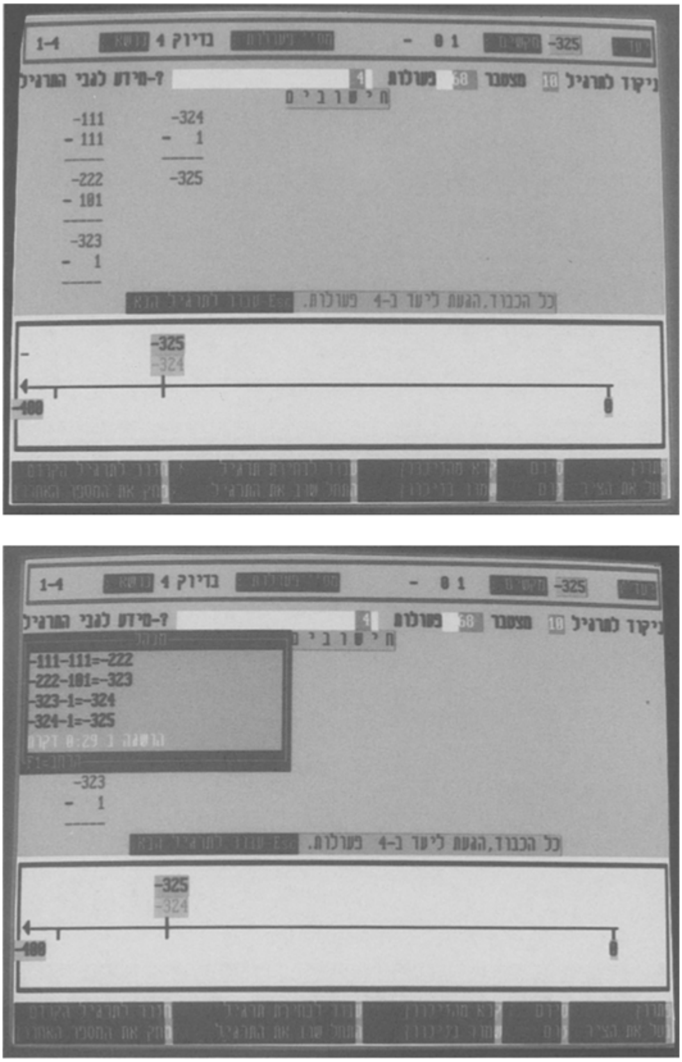


(b)



(c)

Fig. 1. Task #1, 1st Trial.



(a)

(b)

Fig. 2. Task #1, 2nd Trial.

Task 3, in the ten-thousands' number domain [Problem 65, Part II in Table I]
The goal is $-20,793$.

First step: I approach as close as possible to $-20,793$ using ten thousands: $-10,000 - 10,000 = -20,000$.

Second step: If I continue using hundreds, that would mean seven operations which is too many. I should go instead to a "larger negative num-

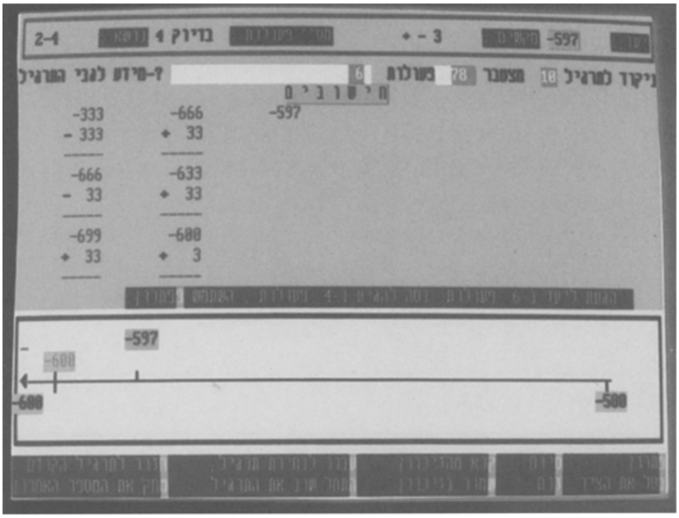


Fig. 3.

ber" that is, I need to "add" another thousand: $-20,000 - 1,000 = -21,000$.

Third step: Now I go "backwards" using hundreds: $-21,000 + 100 + 100 = -20,800$.

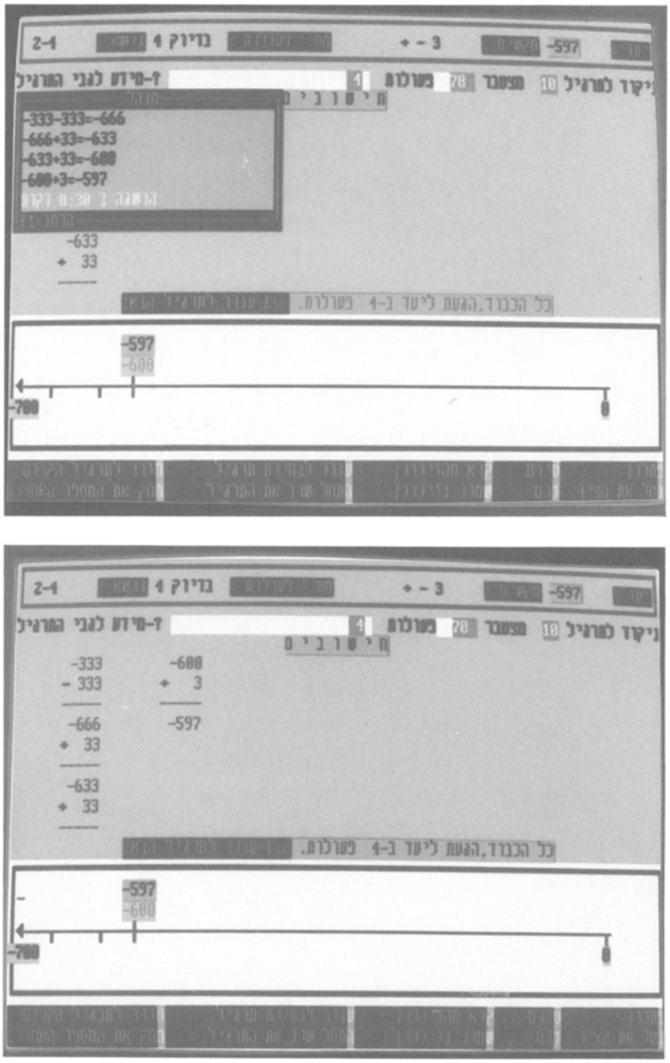
Fourth step: I need to go further "backwards", using tens: $-20,800 + 10 = -20,790$.

Fifth step: Now I need to go "forward" again by ones: $-20,790 - 1 - 1 - 1 = -20,793$.

Computer response: The goal has been reached in 9 operations. Try to reach it in 4 operations.

Sixth step: Now I can combine $-10,000 - 1,000 - 1 = -11,001$, $-10,000 - 1 = -10,001$, and $100 + 10 = 110$. Thus, the solution is: $-11,001 - 10,001 - 1 + 110 + 100$.

Figures 1 and 2 demonstrate the use of the number line to support students' intuition about the size of numbers. The goal and the last result of the computer calculations are marked above the line. The student can request zooming which adjusts the borders to the current situation of the numbers. At the beginning of the solution to Task 3, the number-line borders are set automatically from 0 to -400 (to include the goal of



(a)

(b)

Fig. 4.

−325), as in Figure 2. Figure 1 illustrates using the "zoom" function after the first step.

The main identified strategies students employ in working with all types of tasks in the sequence are: estimation (in considering which number to add or to subtract); generalization from familiar to unfamiliar number domains (of number notation and place-value concepts); looking back;

educated elimination of options; planning ahead; educated guessing; trial and error, and analogy.

The activities and aptitudes needed for successful solution are: generalization of the strategies and of the number patterns from small to large numbers (promoting use of strategies, generalization, "number sense"); considerations of which operation to use rather than performing the computations (promoting "operation sense"); comparisons of a pair of numbers to decide whether to add or to subtract (promoting the understanding of order relations between these numbers); estimating the distance between two numbers, that is, the size of the number to use for approaching closest to the goal (promoting the sense of magnitude of numbers or "number sense"); and taking into account the place value of the digits in the number (promoting the command of place-value concepts). The promotion of "number sense" and of "operation sense" that are excercised through the solution process belongs to the main objectives of the NCTM (1989) standards for school mathematics.

The adaptation of the tasks to students' aptitudes

Rather than adapting *the mathematical curriculum material* to students' aptitudes as almost all studies of adaptive instruction do, this study attempts to adapt to students' aptitude *the investment of mental effort.* This way students at all levels of mathematical aptitude will experience mental challenges at their level.

Adaptation to investment of mental effort is carried out on the basis of the students' ability to skip steps in a hierarchy of the tasks which is related to their ability to generalize mathematical concepts. Resnick and Ford (1981) suggest adapting instruction to learners by decomposing the learning material into hierarchical structures of component skills and by enabling students to skip steps in the hierarchy. "Taking alternative routes through a hierarchy and skipping over certain prerequisites are some of the ways teaching can match specific children's learning characteristics. Some children need direct instruction in all the steps of the learning hierarchy and others seem to profit by skipping steps." (p. 57). The more steps are skipped (or the larger the skip), the larger the generalization that is needed.

Krutetskii (1976) summarizes findings of studies that took place over two decades by Russian cognitive psychologists attempting to identify what comprises "the ability to learn school mathematics". Krutetskii and his colleagues concluded that students with different abilities to study mathematics differed on four dimensions. The one most important dimension is the ability to generalize mathematical material. Pupils able to learn mathematics were characterized by their ability to quickly pass from solving simpler problems to solving *more complex ones of the same type* and to generalize almost without comparing and contrasting, on the basis of analyzing just one (or only a few) examples. In contrast, pupils "incapable" of learning mathematics showed an inability to generalize mathematical material when all the conditions for the generalization were present. They had trouble passing from one level of generalization to another, each level having to be reinforced by a considerable number of examples and solved exercises.

In our study, students with different levels of mathematics achievement move through the hierarchical sequence of tasks along five different tracks. The tracks differ in the degree of generalization between each task and the consecutive one. The first, easiest track covers all tasks in the sequence consecutively. Each additional track skips one step in the sequence of the previous track. Thus, the second track presents each second step of the original sequence, the third track each fourth step, the fourth track each eighth step, and the fifth track each sixteenth step. This is how the level of generalization between two consecutive tasks increases when going from the first to the fifth track.

All experimental students received approximately the same amount of problems, and only the number of domains differed. Lower aptitude students received problems of a small number of domains (e.g., from tens to thousands) whereas higher aptitude students received fewer problems in each domain and a larger number of domains (e.g., from tens to millions).

Table II illustrates the principle of defining the different tracks.

The tasks and the assignment of students into tracks were developed through a long process of composition and examination by observing students working with them and interviewing them on their methods for solution. The final version was examined through observations of four fourth-graders of different abilities working individually with the tasks. These observations served also for evaluating the appropriateness of the tracks to the students' aptitudes, and for identifying both problematic aspects in working with the tasks, and students' strategies and misconceptions.

The procedure

The school in which the study was conducted is located in a suburb of Tel-Aviv, integrating a varied SES student population.[3] Two fourth-grade classes in this school were randomly selected to serve as the experimental and no-treatment groups. The fourth grade was chosen because these students were young enough not to have formally studied the topic of negative numbers, and were still expected (on the basis of the studies discussed

TABLE II

The assignment of consecutive tasks to the five tracks

Exercise #	Number domain	Track I	II	III	IV	V
1	Tens	✓	✓	✓	✓	✓
2		✓				
3		✓	✓			
4		✓				
5		✓	✓	✓		
6		✓				
7		✓	✓			
8		✓				
9		✓	✓	✓	✓	
10		✓				
11		✓	✓			
12		✓	✓	✓	✓	✓
31	Hundreds	✓	✓	✓	✓	✓
32		✓				
33		✓	✓			
34		✓				
35		✓	✓	✓		
51	Thousands		✓	✓	✓	✓
52			✓			
53			✓	✓		
54			✓			
55			✓	✓	✓	
61	Ten thousands			✓	✓	✓
62				✓		
63				✓	✓	
64				✓		
65				✓	✓	✓
71	Hundred thousands				✓	✓
72					✓	
73					✓	✓
81	Millions					✓
82						✓
83						✓

above) to have developed some intuitions and informal knowledge of this topic. Class teachers were asked not to deal with negative numbers in class (which, anyhow, were not in the fourth-grade curriculum).

A test form was developed to serve as both pre- and posttest. The test examined performance on the four objectives listed above.[4] Three experts in mathematics education at the elementary school level judged the extent to which the items represented the objectives of the study. Only those items agreed upon by all experts were included in the test. Each test item consisted of several sub-items that all had the same form, but involved numbers in different number domains – from units to hundred thousands. The scoring assigned a total of 100% to a perfect solution of all items belonging to each objective separately, and to the test in its entirety.

Pre-treatment

Two weeks before the beginning of the treatment, the two classes took the pretest. The scores on these tests along with the students' class grades in arithmetic served for sorting the experimental students into the five tracks. Fourteen students per class were interviewed, and asked to explain their solutions to the pretest items and to perform certain operations on negative numbers.

Two preliminary sessions in the computer lab were designed to acquaint students of the experimental group with the software, the methods, and routines for work. The students worked with positive numbers on tasks similar to those in the negative-number sequence.

The treatment consisted of 12 once-weekly 45-minute sessions, in addition to the regular arithmetic lessons, starting one month after the beginning of the school year. The no-treatment group proceeded with its regular arithmetic lessons without treatment.

Each experimental student worked individually on the tasks in the computer lab. At the end of each session each student received worksheets which included the same problems that he/she solved on the computer, to be performed at home on paper with the use of a calculator. Family members were asked not to provide any help in these assignments.

Because the number of computer stations in the lab (20) was insufficient for all students in the class to work on a one-student-to-one-computer basis, the class was divided into two groups, which visited the computer lab on different days, one hour before the formal beginning of the school day. Work in the lab was supervised by the class teacher and a research assistant. Both assisted with technical problems, supervised students' behavior, and helped students who met an impasse by providing guiding questions and strategic hints, while never offering detailed solutions. When a student felt that the problems received were either too easy or too difficult for him/her and this matter was not handled appropriately by the software management system, the supervisors changed the track for that student.

The supervisors recorded in writing (using a specially prepared key) all questions students asked, their own responses, and their other activities throughout the computer lab sessions. The students' workbooks (in which they documented their worksheets' solutions) were collected after each computer session, and returned before the next one with all solutions checked. All steps of students' solutions done in the lab, recorded by the software, were printed out promptly after each session. These hard-copy computer logs and the workbooks served to introduce changes in students' present tracks or helped instructing students to repeat, in the next session, the solution of a particular problem.

Post-treatment

All classes took a posttest two weeks after the conclusion of the treatment. This test was identical to the pretest taken 16 weeks earlier. The same students who had been interviewed in pretreatment were interviewed again and asked to explain their solutions and solution strategies.

QUANTITATIVE ANALYSIS AND RESULTS

The no-treatment class served here solely to control the effects of maturity, time, and of repeated test-taking. To examine the initial comparability of the two classes on the pretests, differences between the classes were assessed using t-tests. The t-tests were performed for each of the four objectives and for the total score on the test. For the analyses, students were divided into two achievement levels – those assigned at the beginning of the study to use the two lower tracks and those who used the third and fourth tracks. Four students in each class, who were sorted to the fifth, highest track, performed almost perfectly on the pretest. They were excluded from the analysis to avoid ceiling effect. The scores on each of the four objectives and the total score on the test were transformed to a percentage scale. Table III presents means and standard deviations of the students' scores on the four objectives and on the total score, for both the pretest and posttest. Table III also presents the gains from pre- to posttest and the effect size of the posttest performance as compared with the pretest performance. This information is presented for all students and is also broken down on the two levels of achievement. Table IV presents results of an ANOVA for the pretest performance by achievement level of the two classes.

TABLE III

Descriptive statistics of students' performance in pretest and posttest for class by achievement level

Class:	Experimental			No-Treatment		
Ach:	All	Low	High	All	Low	High
n	30	12	18	29	18	11
	M(SD)	M(SD)	M(SD)	M(SD)	M(SD)	M(SD)
Pretest						
a. Comparing numbers mentally	48(32)	23(20)	64(27)	45(29)	27(21)	75(13)
b. Estimating the distance between numbers	60(27)	45(26)	71(24)	58(26)	46(24)	78(17)
c. Operating on numbers	31(34)	12(28)	43(32)	13(27)	2 (7)	32(37)
d. Locating numbers on the number line	70(34)	49(34)	83(27)	54(42)	39(38)	79(35)
Total Score	48(28)	27(20)	62(24)	42(24)	27(16)	68(9)
Posttest						
a. Comparing numbers mentally	72(25)	49(22)	88(12)	54(27)	40(23)	77(12)
b. Estimating the distance between numbers	78(23)	65(24)	87(19)	65(24)	58(23)	77(23)
c. Operating on numbers	56(39)	27(32)	76(29)	22(32)	6 (8)	48(39)
d. Locating numbers on the number line	89(27)	76(37)	97(13)	58(48)	49(49)	73(47)
Total Score	76(24)	53(20)	91(13)	54(25)	40(20)	76(13)
Pre-post differences (Effect Size)	*D(ES)*[1]	*D(ES)*	*D(ES)*	*D(ES)*	*D(ES)*	*D(ES)*
a. Comparing numbers mentally	25(0.8)	26(1.4)	24(0.9)	9(0.3)	13(0.7)	3(0.1)
b. Estimating the distance between numbers	18(0.7)	20(0.8)	16(0.7)	7(0.3)	12(0.5)	−1(−0.0)
c. Operating on numbers	25(0.7)	14(0.5)	33(1.0)	8(0.3)	4(0.4)	16(0.4)
d. Locating numbers on the number line	19(0.6)	27(0.8)	13(0.5)	4(0.1)	10(0.3)	−6(−0.2)
Total Score	28(1.0)	26(1.4)	29(1.2)	12(0.5)	13(0.8)	9(0.9)

[1] D: Difference from pretest to posttest ES: Effect Size

TABLE IV

F-values in ANOVA of students' performance in pretest by class by achievement level

	Class	Ach.	CXA
a. Mental comparison of numbers	1.6	57.3***	0.4
b. Estimation of distance between numbers	0.4	21.7***	0.3
c. Operating on numbers	2.1	17.8***	0.0
d. Location of numbers on the number line	0.6	17.2***	0.1
Total Score	0.3	57.1***	0.3

*** $p < 0.001$

Tables III and IV reveal, for the pretest, no main effect of class for any of the four objectives and for the total score. The F-values are very small except for objectives b and d. The largest pretest difference shows on item d, "operating on numbers", and favors the experimental group. Table III

TABLE V

F-values in repeated measures analysis of students' performance in the tests by class, achievement level, and time of taking the test

	Between Subjects			Within Subjects			
	Class	Ach.	CXA	Time	CXT	AXT	CXAXT
a. Mental comparison of numbers	0.1	77.1***	0.1	37.0***	9.9**	1.4	0.7
b. Estimation of distance between numbers	0.3	23.5***	0.1	13.0***	3.9*	1.7	0.5
c. Operating on numbers	9.2**	43.4***	0.1	13.5***	2.4	2.9	0.1
d. Location of numbers on the number line	3.6	11.9***	0.1	7.0**	4.8*	3.3	0.0
Total Score	1.7	81.2***	0.1	69.6***	12.7***	0.1	0.6

* $p < 0.05$ ** $p < 0.01$ *** $p < 0.001$

indicates, as expected, large differences in the performance of low and high achievers and Table IV supports these findings by showing main effect of achievement for each of the objectives and for the total score.

As for the posttest, Table III indicates that the experimental class performed significantly better and showed much larger pre-post gains and effect sizes than the no-treatment group. This is true for all four objectives and for the total score. Table III suggests also, as expected, large differences on all posttest objectives between the high and low achievers in both classes. However, on examination of the pre-post differences, the low achievers seem to gain more from pre- to post-test than the high achievers on all objectives, except on "operating on numbers" in both classes and on the total score for the no-treatment class. We should note, though, that on items a and b, as well as on the total score, the experimental high achievers do extremely well indeed so that it is possible that a ceiling effect reduced their gains on these items.

Table V presents results of repeated measures using MANOVA of class by achievement level by time of measurement (pretest versus posttest).

Table V indicates that when introducing the effect of time-of-measurement (pre-versus posttest), we see large significant differences in performance related to time, and to class by time. That is, the two classes performed differently on the posttest, as compared with the pretest, to the advantage of the experimental class (as shown in Table III). Because the classes showed no significant differences on the pretest, these findings indicate significant gains in performance favoring the treatment group over the no-treatment group.

A very important finding is that there are neither significant interactions of achievement-level by time-of-measurement nor of achievement-level by time by class. That is, there was no difference between the behaviors of high and low achievers in both classes, and in the posttest as compared with the pretest. We may conclude that the gap between both achievement groups remained as large as it was initially, that is, the treatment had similar effects on the performance of high and low achievers. The overall performance improvement in the experimental class means that all experimental students, of all achievement levels, have gained largely from the treatment.

Students' performance on the objective "Operating on numbers" was different. This is the lowest-rated of the four objectives and in both pre- and posttest it rates lower for the no-treatment group than for the experimental group. Although Table V does not yield interaction of class by achievement-level by time-of-measurement. Table III suggests that the low achievers gained the least on this item whereas the high achievers gained more on it than on any of the others. This result, combined with the non-significant interactions of class by time and of achievement-level by time on this item, may be interpreted as follows: Operations on negative numbers, even in their reduced context as applied in this study, are still too difficult for the lower achievers in the fourth grade. Alternatively, the present method is insufficient for teaching this topic to the weaker students. The high achievers, though, show good potential to learn and understand these operations at the fourth-grade level, by using the methods applied here.

QUALITATIVE ANALYSIS

The next summary of the qualitative analysis is based on pre- and post-treatment interviews with 28 students in both classes. Four of the eight high achievers (two for each class) who were excluded from the quantitative analyses because of possible ceiling effect, were included in this group of interviewees. The interviews aimed to identify students' conceptions of negative numbers, their knowledge and misconceptions related to simple operations on these numbers, the models that they used for operations with negative numbers, and the sources for students' pre-instructional intuitions.

Students' conceptions of negative numbers

The students were asked to explain what is a negative number and were presented by a particular number (e.g., -3) to help them in their explanations. Sixty six percent of the experimental students and 47% of the no-treatment students correctly answered this question in the pre-treatment interviews, and 86% and 55% respectively in the post-treatment interviews.

Answers considered correct were (in students' words): a number below zero; smaller than zero; small than all numbers [positive]; to the left of zero; zero minus something; and subtracting a large number from a small number. In addition, several students answered, rather than in words, either by writing out an example which resulted in a negative number (e.g., $5-8$), or by drawing a number to the left of zero on a number line. Still other students used "embodied" examples based on daily experience, e.g., a thermometer with numbers below zero, or a place that is geographically below sea level. This latter example, presented by several students, is probably unique to Israel, where on the way to the Dead Sea, which is 200 meters below sea level, one crosses several road signs indicating "-50 meter", "-100 meter", etc.

Wrong answers were: a little; there is no such number; I don't know; and zero. The zero answer was given by two students who explained that -3 is a small number and zero is the smallest number that they knew...

Students' performance and types of misconceptions regarding simple addition and subtraction

Students were asked to solve several problems in addition and subtraction with negative numbers. Next is a list of each type of problem and of the related misconceptions that were identified.

a. Subtracting a positive number from 0 [$0 - x, x > 0$].
   Students' misconceptions: $x, 0, 10 - x$ [e.g., $0 - 4 = 4$, or 0, or 6].
b. Subtracting a positive number from a smaller positive one [$x - y$, $y > x > 0$]:
   Students' misconceptions: $y-x, x+y, -(x+y), x, y$ [e.g., $3-8 = 5$ or 11 or -11 or 3 or 8].
c. Adding two negative numbers [$-x - y, x > 0, y > 0$].
   Students' misconceptions: $x-y, -(x-y), x+y, x$ [e.g., $-3-8 = -5$ or 5 or 11 or 3].
d. Adding and subtracting the same number [$-x + x, x > 0$]:
   Students' misconceptions: $2x$ ($x + x$), x [e.g., $-3 + 3 = 6$ or 3].
e. Adding a positive number to a negative number which is either larger or smaller in absolute value [$-x + y, x > y > 0$]:
   Students' misconceptions: $x-y, x+y, -(x+y), x, y$ [e.g., $-8+3 = 5$ or 11 or -11 or 8 or 3].

Table VI quantifies the correct solutions of each type of these operations. The table summarizes the percent of correct answers in pre- and posttests for the two classes.

TABLE VI

Percent of correct solutions to examples with simple operations on negative numbers, in pre- and post-study interviews

Percent Correct:	Experimental ($n = 15$)			No-Treatment ($n = 13$)		
Type of Operation [An example]	Pretest	Posttest	Diff.	Pretest	Posttest	Diff.
Subtracting a positive number from 0 [$0 - 4$]	76	88	12	55	61	6
Subtracting a number from a smaller one [$3 - 5$]	76	88	12	59	64	5
Adding two negative numbers [$-7 - 5$]	40	74	34	31	46	15
Adding and subtracting the same number [$-5 + 5$]	40	73	33	39	53	14
Adding a negative to a positive number [$-12 + 7$]	47	80	33	53	61	8

Table VI supports the previous findings of the quantitative analyses, showing that the percent of correct answers increased (while, necessarily, that of misconceptions decreased) for the experimental group to a larger extent than for the no-treatment group. Table VI supports also claim made in previous studies concerning children's pre-instructional knowledge and their intuitions of simple operations with negative numbers.

Of all types of examples listed in Table VI, those with the highest proportion of pre-instructional knowledge are – subtracting a (positive) number from zero, and a (positive) number from a smaller (positive) one. More than one half of the no-treatment students and more than three quarters of the experimental students correctly performed these operations in the pre-treatment interviews. Pre-instructional knowledge of the other types of operations is less prevalent but is still found in relatively high proportions (approximately 40%).

Models students used for calculations

When asked in their post-experiment interview about their use of the number line, almost all students acknowledged that in the first computer sessions they did use the screen presentation of the number line to compare the given number with the goal and to decide whether to add or to subtract. However, with time there were students who completely stopped using the visual (software-presented) number line and only did mental calculations whereas others used the visual and mental models alternately. All these students developed their own rules and invented their own strategies. Several students proceeded to use a mental number line with zero as a "road sign", separating between positive and negative numbers, and as a "stopping point" to get from one number domain to another. For example, to solve $5 - 7$, these students broke down the 7 into $5 + 2$. They started with going to the left to get to zero: $5 - 5 = 0$ and then continued to the left: $0 - 2 = -2$. That is, they solved: $5 - 7 = 5 - (5 + 2) = 5 - 5 - 2$. However, the majority of students interviewed transferred their knowledge of positive operation to the negative number problems. To illustrate, they would convert a problem with two "successive subtractions" to a problem with two successive additions and then add a "–" sign to the result. Thus, to solve $-5 - 7$, these students first computed $5 + 7 = 12$ and then added the minus sign to get the answer: -12.

The source of pre-instructional intuitions

The students were asked, in the pre-treatment interviews, to explain where they had learned about the existence of negative numbers, and then to perform simple addition and subtraction on such numbers. Of the 28 students, 42% acknowledged that they came across these concepts and procedures via explanations by family members, often prompted by the child's inquiries about temperatures with a minus sign in the TV weather forecast (usually of countries other than Israel), or, as mentioned above, when visiting the Dead Sea, or while watching arithmetic programs on educational TV; 14% referred to games (board or computer) in which you accumulate points but may also lose more points than you have in your credit; and 7% mentioned out-of-school, paid group-enrichment activities that they had attended.

DISCUSSION

The National Council of Teachers of Mathematics recommended in its publication of Curriculum Standards for School Mathematics (1989) to start teaching initial concepts of negative numbers as early as in fifth through eighth grade. Negative numbers would be introduced to this age group as an extension of the [positive] number system. However, in spite of this recommendation and notwithstanding the problems students face in learning negative numbers at a later age, as we showed in our above summary of the research literature, there is very scarce published material on both the teaching and learning of this topic. Recent leading books on research in mathematics education (e.g., Post, 1992 which presents suggestions for teaching mathematics to grades K-8; and Grouws, 1992) completely overlook this topic. An ERIC search produced only very few items, mostly in (PME) conference proceedings (see attached list of references) rather than in research journals.

The present study contributes to the literature in that it examines experimentally, in real classroom conditions, the feasibility of teaching negative number concepts and procedures to students of a relatively young age, indeed even younger than the minimum age recommended by the NCTM (1989). The curriculum for the fifth through eighth grades, as suggested by the NCTM, was fully covered in our study by fourth-grade students. Results show that our students gained significantly on three of the four objectives (locating negative numbers on the number line; mental comparisons of negative numbers; and estimating their size and intervals between two numbers). These gains were achieved through self-learning procedures, by solving challenging numerical tasks. Low achievers gained at least as much as high achievers on these objectives, indicating that the adjustment of the challenge level to students' aptitude, as done in our method, works well.

The objectives that appear to be easiest to grasp for this age group are the comparison of numbers, either mentally or by locating them on the number line. In fact, the experimental high achievers scored almost the maximum on these objectives in the posttest. The only objective that appeared problematic and on which posttest performance is not statistically significant as compared with that of the no-treatment group, is the "operations with numbers", even in the limited form in which it features in this study. To accomplish this objective proved to be particularly difficult for the lower-achieving students, in contrast with the better experimental students. We may conclude either that fourth grade is still too early to teach these procedures to the lower-achieving students, or that these students should receive more direct instruction on this topic than the teaching method used in our study.

Despite low achievers' difficulties with the operations on negative numbers, we would like to suggest that the teaching method used here demonstrates a good potential for providing students of all aptitude levels, starting from a young age, with problem-solving experiences in mathematics. Thus, this method is very appropriate for educating children in accordance with the new NCTM (1989) goals in mathematics education.

We stress, though, that we in no way mean to suggest this method as the sole or best one for teaching negative number concepts. A teacher's explanation in class and practice in word problems is definitely necessary to complement and boost students' understanding of the concepts and procedures involved. We suggest our method as a beneficial and challenging way to enrich students' experiences with negative numbers, which can be integrated into classroom instruction of negative number concepts and procedures.

Our study also reinforces previous research findings of the presence in young students of pre-instructional intuitions and informal knowledge of negative number concepts, and of their capacity to do simple operations with these numbers. This is shown for both classes on the pretest, and for the no-treatment group on the posttest. As expected, high achievers are observed to develop this knowledge to a much larger extent than low achievers. In fact, eight high achievers (four in each class, those sorted to the fifth level) were excluded from the analyses in this study because they performed almost perfectly on the pretest.

To conclude, Murray (1985) suggests that children have a far greater affinity for numbers, number patterns, and simple logic than most current teaching strategies assume, and that, therefore, negative numbers can be introduced to students at younger ages than is done now. Our study supports these claims, demonstrating that certain negative number concepts, e.g., mental comparisons of numbers, locating numbers on the number line, and estimation of distance between numbers, can be taught to all fourth grade students. Simple addition and subtraction can also be taught at this age but their instruction necessitates careful methods, particularly for the less able students.

Limitations of the study and suggestions for further research

Only one class received the treatment. It is desirable to repeat the treatment for a larger population of students. Several issues dealt with here require further studies designed to find out how students structure visual and mental number lines and how this affects the transition to calculations without such support, why "operations with numbers" are more difficult to learn than the other concepts presented here, particularly for lower-achieving students. In addition, though, this study shows that the special instructional method presented here succeeds in gaining its objectives. It does not attempt to compare it with other instructional methods on the method's effectiveness in gaining these objectives.

NOTES

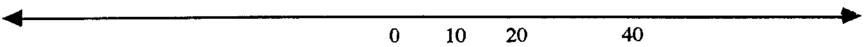
[1] This research was supported by the BASIC RESEARCH FOUNDATION administered by the ISRAEL ACADEMY OF SCIENCES AND HUMANITIES.

[2] These decisions are based on a record of the student's solutions to all previous exercises. The rough description of these decisions is as follows: A correct solution of several exercises which is done in the first trial indicates that the sequence may be too easy for the student and he/she is moved to a more difficult track. On the other hand, failure to correctly solve several consecutive exercises, each in the first three trials, points to difficulties with the rate of advancement of the sequence, and the student is moved to an easier track. There are intermediate situations between these two extremes which also lead to software-initiated changes in the track.

[3] The SES criterion for Israeli students is a weighted measure of three demographic characteristics: father's cultural origin (either Western–North America, Europe; or Eastern–Asia, Africa, Arab), father's level of education, and number of children in the family – each represented by several defined hierarchical categories.

[4] To illustrate, the following are items related to the different objectives.

Objective (a) (location on the number line). "Write on the following number line the numbers: 7, 30, 42, −30, −20, −50". The given number line is:



Objective (b) (comparing numbers). "Mark which of three given numbers is the largest and which is the smallest". In the units domain the three numbers can be 7, −9, −5 and in the thousands domain, −2, 784, −2, 874, −1, 487. Another item: "Is the result of the following computation positive, zero or negative? (answer without making calculations): −5 − 5 (in the units domain); 23 − 32 (in the tens domain). A third type of item: "In order to get from 5 to −1, do you add or subtract? (same for the tens and hundreds domains, the pairs of numbers being respectively −50, −70; 470, −530).

Objective (c) (estimating the distance between numbers). (In the unit domain, "Circle for the following five numbers: −10, −5, 0, 5, 10, the one which is the closest to −7". In the thousands domain students are asked to circle of the five numbers: −2, 500, −2, 400, −2, 000, 2,400, 2500, the one which is closest to −2, 480.

Objective (d) (performing simple addition and subtraction) includes items of addition and subtraction of the types presented above, in different number domains, all for mental calculation. For example, −30 + 40; −70 − 20; −300 + 200. Another type of item: put either + or − in the empty squares □ and put the missing number in the *underlined* space: −4□__ = 2; −25□__ = −30; −400□__ = 0.

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