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IMAGES OF MATHEMATICS HELD BY UNIVERSITY TEACHERS OF MATHEMATICS EDUCATION

ABSTRACT. This article presents data about views of mathematics collected through a national survey of teachers of mathematics education in Canadian universities. The results are compared to those obtained through a similar survey of university teachers of mathematical sciences carried out earlier.

INTRODUCTION

In this article I shall report the results of a study about the views of mathematics held by university teachers of mathematics education (henceforth referred to as mathematics educators) and I shall compare them to the results of a similar study previously carried out among university teachers of mathematical sciences (henceforth referred to as mathematicians) (Mura, 1993).

One of the motivations for studying mathematicians' views of mathematics, apart from their intrinsic interest, was their potential influence on school teachers' views. This motivation, of course, holds as well for studying mathematics educators' views. Furthermore, comparing the two may cast some light on the question of the continuity of the influences that pre-service teachers are exposed to during their training, as they move from mathematics to mathematics education courses. The overall rationale for studying the conceptions of the nature of mathematics held by teachers of any level, school or university, revolves around the likely, though not simple, relationship between such conceptions and the teachers' instructional practices. For a review of the literature on this subject the reader is referred to Dossey (1992), Mura (1993) and Thompson (1992a).

As in the previous study of mathematicians, mathematics educators' views of mathematics will be described here by means of their answers to two questions, the first one asking them to define mathematics and the second one asking them to name up to ten books which have influenced this discipline.

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METHOD

The two questions discussed here, how mathematics educators define mathematics and which books they consider to have had the most influence on the field, were part of a survey of mathematics educators who were regular faculty members of Canadian universities. Other questions included in the survey concerned mathematics educators' social background, education, careers and views about mathematics education. The results of these other questions will be reported elsewhere. The instrument designed to collect the data was a questionnaire comprising 54 questions, 7 of which were open-ended. An English or a French version of the questionnaire was used as appropriate.

Given the small size of the population involved, I attempted to reach all of its members. In order to do so, I sent questionnaires to all those whose name appeared in the mailing list of the Canadian Mathematics Education Study Group or in its directory of current research (Kieran and Dawson, 1992). I also asked each of these persons to name all mathematics educators in their own universities and I then sent questionnaires to the additional individuals identified in this way.

Since the vast majority of universities do not have mathematics education departments, it is up to each individual faculty member to decide whether or not he or she is a "mathematics educator". To take this situation into account, the cover page of the questionnaire contained the following two questions:

(a) Do you hold a tenured or tenure-track position at a Canadian university? and (b) Is mathematics education your primary field of research and teaching?

Those who did not answer positively both questions were not part of the target population and were invited to return the questionnaire without completing it any further.

Altogether 158 questionnaires were sent off by mail. After two reminders, 106 (67%) were returned. Of these, 63 were completed by respondents belonging to the target population and were retained for the present study. The sample consisted of 44 men (70%) and 19 women (30%). The median age of the group was 50 years, with a range from 30 to 64. Forty-one respondents (65%) spoke English at work and 22 (35%) spoke French. Forty-seven (75%) worked in education departments, 13 (21%) in mathematics departments and 3 had joint appointments. Concerning their education, 56 (89%) of the respondents held doctoral degrees: 46 in education (including mathematics education), 8 in mathematics and 2 in psychology. Thirty-five (56%) held university degrees both in education and in mathematics, 16 (25%) did not have any degree in education and 12 (19%) did not have any degree in mathematics.

The first of the two questions discussed in this article concerned the definition of mathematics. It was open-ended and it read: "How do you define mathematics?". It was slightly different from the question that had been asked of the mathematicians which read: "How would you define mathematics?". The change was made in the hope to encourage respondents not to skip the question, like 33% of the mathematicians who returned the questionnaire had done. In both cases a space of eight blank lines was provided for the answer.

Analysis of the data produced by the sample of mathematicians had yielded a set of 12 themes, described in Mura (1993). Analysis of the present data involved first examining each response and noting whether it contained references to any of the previously identified 12 themes as well as to any possible new ones. Indeed, this led me to add two new themes, one concerning inductive thinking[1] and one concerning the cultural nature of mathematics. Thus I arrived at a list of 14 themes, which I used to associate each response with all the themes occurring in it. I found altogether 132 occurrences of the themes, with an average of 2.6 themes per response. I also went back to the responses given by the mathematicians to check whether they contained references to either of the two new themes.

Two judges, who had already played this role in the previous study, were then given the list of 14 themes and were asked independently to associate, as I had done, to each response all the themes that they recognized in it. They were also asked to comment on the themes themselves and to re-examine, as I had done, the responses given by the mathematicians in the previous study, looking for occurrences of the two new themes.

The two judges listed on average respectively 2.4 and 2.6 themes per response. Agreement of the judges' classifications with my original one, that is percentage of the occurrences of the themes identified by the judges that I too had identified, was respectively 83% and 86%. Conversely, of the occurrences of the themes that I had identified, 90% were recognized by at least one judge. As a result of the judges' comments, I modified slightly the definitions of four themes (the two new themes and themes 1 and 4 among the old ones); none of the modifications would have affected the classification of the previous set of data. Taking into account the judges' classifications, I revised my initial one, that is I deleted or added themes associated to each response. Altogether I deleted 8 of the original 132 occurrences of the themes and added 10. I shall present here the results pertaining to the revised list of 14 themes and to the revised classification of the responses. Agreement of the judges' initial (and only) classifications with my revised one is respectively 88% and 91%. Conversely, of the occurrences of the themes identified by myself in the revised classification, 96% had been recognized by at least one judge.

As to the re-examination of the data collected in the previous study, neither I nor the judges found any occurrences of one of the two new themes. We did find a few occurrences of the other new theme and I retained those on which a majority of us (at least two out of three) agreed.

The second question discussed in this article was about the books that have influenced mathematics. It was identical to the one previously asked of mathematicians and it read: "Please identify some of the books which, in your opinion, have had the most influence on the development of mathematics, from ancient to modern times. (Maximum of ten books)". Analysis consisted of straightforward, descriptive statistics.

As noted earlier, some mathematics educators are members of mathematics departments. Hence, before comparing the results from the two studies, it was necessary to verify that the set of "mathematicians" included no mathematics educators. Indeed it turned out that three of the "mathematicians" who had answered the two questions discussed here named mathematics education as their research area. Their responses have been withdrawn before comparing the two sets of data. This is why the results quoted below for mathematicians differ slightly from those reported in Mura (1993).

RESULTS

Definitions of Mathematics by Mathematics Educators

Of the 63 mathematics educators who returned their questionnaires, 51 offered some response to the question of defining mathematics. The statements ranged in length from a single question mark to 125 words.

The following 14 themes have emerged from the content analysis of the material collected in the present study of mathematics educators and in the previous study of mathematicians, as described above. All but two of the themes (number 8 and number 12) had already been identified in the previous study, however the definitions of themes number 1 and number 4 have been slightly expanded. The number in parentheses after each theme is its frequency, i.e. the number of respondents in the present study who made reference to it. Each theme is illustrated by one or more examples. The examples are drawn from the responses given by mathematics educators, except for themes number 5 and number 11 which did not occur in any of the mathematics educators' responses.

1. The creation and study of formal axiomatic systems, of abstract structures and objects, of their properties and relationships ($N = 14$).
   Examples:
   "[...] Mankind creating formal structures."
   "Construction et étude de systèmes formels."
2. Logic, rigour, accuracy, reasoning, especially deductive reasoning; the application of laws and rules ($N = 17$).
   Examples:
   "[...] a way of thinking in reasoned way [...]"
   "Analytical methods [...]"
   "Study of necessary conclusions."
   "[...] jeu intellectuel avec règles précises et annoncées."
3. A language, a set of notations and symbols ($N = 10$).
   Examples:
   "[...] Medium of communication; it is a language. [...]"
   "Mathematics is the study of formal and informal notations of aspects, events or patterns in our world." (Classified also under themes 4 and 7.)
   "[...] Essentially a language. [...]"
   "Moyen d'expression qui facilite la communication d'un raisonnement." (Classified also under theme 2.)
4. Design and analysis of models abstracted from reality; their application. A means of understanding phenomena and making predictions ($N = 16$).
   Examples:
   "Looking at the world through a numerical/spatial/symbolic lens." (Classified also under themes 3 and 13.)
   "[...] A way of thinking about and modelling the world. A means of making predictions based on assumptions."
   "[...] une perception du monde [...]"
   "[...] C'est un outil extrêmement puissant dont le but est de mieux saisir ou comprendre notre univers. [...]"
5. Reduction of complexity to simplicity ($N = 0$).
   Example drawn from the sample of mathematicians:
   "[...] a way of thinking, and in particular a way of looking at complicated unmanageable things and reducing them to simple things."
6. Problem-solving ($N = 6$).
   Examples:
   "[...] It is problem solving involving quantity and space." (Classified also under theme 13.)

"[...] Un ensemble de structures permettant de résoudre des problèmes." (Classified also under theme 1.)

7. The study of patterns ($N = 19$).
   Examples:
   "The organized study of all the patterns there are."
   "The bringing forth of pattern."
   "[...] j'aime considérer les mathématiques comme l'étude des régularités."
8. Inductive thinking, exploration, observation, generalization ($N = 9$).
   Examples:
   "Mathematics is the study of ideas using methods of inductive and deductive reasoning." (Classified also under theme 2.)
   "[...] Medium for exploration and discovery. [...]"
   "[...] This process includes gathering data, observing commonalities of patterns, forming conjectures, and ultimately proving or disproving these conjectures (theorems). [...]" (Classified also under themes 2 and 7.)
   "[...] l'exploration et la rigueur; la particularité et la généralité." (Classified also under theme 2.)
9. An art, a creative activity, a product of the imagination; harmony and beauty ($N = 11$).
   Examples:
   "A creation of the mind; [...] an art."
   "[...] content can be for [...] aesthetic [...] purpose."
   "[...] une science et un art." (Classified also under theme 10.)
   "Un univers de création et de découverte à la fois."
10. A science; the mother, the queen, the core, a tool of the other sciences ($N = 7$).
    Examples:
    "[...] As a science in its own right."
    "C'est "la reine des sciences" comme dirait Bell. [...]"
11. Truth ($N = 0$).
    Example drawn from the sample of mathematicians:
    "[...] The only existing subject in which truth has only one face."
12. Culturally determined content (ethnomathematics) ($N = 4$)
    Example:
    "Mathematics is a socially constructed artifact which reflects cultural mentalities in its content, interpretation and grounding. [...]"
13. Reference to specific mathematical topics (number, quantity, shape, space, algebra, etc.) ($N = 13$).

Example:
"The study of quantity, shape, and related concepts."

14. Other (difficulty, impossibility, futility of defining mathematics, circular delinitions, etc.) ($N = 8$).
Examples:
"I don't."
"Il est impossible de définir les mathématiques, [...]"

Comparison of the Definitions of Mathematics Given by Mathematics Educators and by Mathematicians

Table I presents the frequencies and the percentages of the occurrences of the 14 themes among the responses offered by mathematics educators and by mathematicians. As each respondent may have made reference to more than one theme, percentages need not add up to 100%.

Since the average number of themes per response is 2.63 for mathematics educators and 1.85 for mathematicians, it is to be expected that the relative frequencies of all themes be lower for the latter sample. After correcting for this effect, I carried out a series of chi-square tests, one for each theme where the frequencies were high enough for the test to be reliable. The tests yielded a significant result ($p<0.01$) concerning themes 7, 8 and 14, i.e. there is an interdependence between the professional community to which one belongs and the likelihood of including themes 7, 8 or 14 in one's definition of mathematics. In all other cases there was no significant relationship between these two variables ($p>0.05$). Thus mathematics educators have made proportionally more frequent references to the ideas of mathematics as the study of patterns and mathematics making use of inductive thinking processes. Mathematicians, on the other hand, have produced many more statements classified as "other", in particular statements that avoid giving a definition of mathematics. In this connection, one should remember that mathematicians were also more likely than mathematics educators to skip this question altogether (33% vs. 19%).

Books that Influenced Mathematics According to Mathematics Educators

Among the sample of mathematics educators, 22 respondents out of 63 (35%) declined to name any books having influenced the development of mathematics. Some of them explained their silence. For example, two wrote: "Answering this question would take too much thought and too much time", and "Not my area of expertise". As the mathematicians had done, mathematics educators, too, concentrated on authors rather than

TABLE I

Themes occurring in the definitions of mathematics given by mathematics educators and by mathematicians

	Definitions given by math. educators (N = 51)		Definitions given by mathe-maticians (N = 103)	
Themes	n	%	n	%
1. Formal systems	14	(27.5)	25	(24.3)
2. Logic	17	(33.3)	26	(25.2)
3. Language, symbols	10	(19.6)	10	(9.7)
4. Models of reality	16	(31.4)	30	(29.1)
5. Reduction of complexity	0	(0.0)	3	(2.9)
6. Problem-solving	6	(11.8)	7	(6.8)
7. Patterns	19	(37.3)	5	(4.9)
8. Inductive thinking	9	(17.6)	3	(2.9)
9. Art	11	(21.6)	15	(14.6)
10. Science	7	(13.7)	13	(12.6)
11. Truth	0	(0.0)	4	(3.9)
12. Culturally determined	4	(7.8)	0	(0.0)
13. Specific topics	13	(25.5)	10	(9.7)
14. Other	8	(15.7)	40	(38.8)

books. Here is an example of the kind of justification some of them gave: "I do not think that individual books have an impact as much as the collected works of an individual or possibly a group". Therefore, as I have done for the sample of mathematicians (Mura, 1993), here too I shall report the results by authors rather than by books.

Altogether, the 41 respondents who gave some answers to this question mentioned 81 different authors. The ten authors who received the most citations are Euclid (36), Newton (23), Descartes (16), Whitehead and Russell (12), Bourbaki (11), Al Khwarizmi and Leibniz (9 each), Gödel (7) and Archimedes (6). Cantor, Euler and Hilbert followed with 5 citations each; Diophantus, Pascal, Pythagoras and Viète received 4 citations each; Cardan, Cauchy, Galileo, Klein, Kline, Mandelbrot and Riemann received 3 citations each. The other 57 authors received one or two citations each.

TABLE II

Ten authors more frequently cited by mathematics educators and by mathematicians as having influenced mathematics (with frequencies of citations)

Authors cited by math. educators		Authors cited by mathematicians	
Euclid	(36)	Euclid	(42)
Newton	(23)	Newton	(31)
Descartes	(16)	Gauss	(18)
Russell, Whitehead	(12)	Knuth, Russell, Whitehead	(13)
Bourbaki	(11)	Hilbert	(12)
Al Khwarizmi, Leibniz	(9)	Bourbaki	(11)
Gödel	(7)	Euler	(9)
Archimedes	(6)	Descartes	(8)

Comparison of the Choices made by Mathematics Educators and by Mathematicians Concerning the Books that Influenced Mathematics

As was the case with the question of defining mathematics, proportionally fewer mathematics educators than mathematicians skipped the question of naming books that influenced mathematics (35% vs. 58%). Table II compares the lists of the 10 authors most frequently cited by the respondents in the two samples.

The two lists are very similar: seven out of the ten authors are common to the two lists. Of the ones who occur in the first list only, Al Khwarizmi received one citation by the mathematicians, Leibniz received five and Archimedes received two, while of those who occur in the second list only, Gauss was cited twice by the mathematics educators, Knuth was not cited at all and Hilbert was cited five times.

DISCUSSION

Mathematics educators have proven to be more willing than mathematicians to tackle the two questions discussed here: the percentages of those who skipped them are respectively 19% for the first question and 35% for the second one. The corresponding percentages for mathematicians are 33% and 58%. Furthermore, concerning the first question, a much smaller proportion of mathematics educators than mathematicians gave answers that were classified as "other" and that were more or less equivalent to not answering at all. There are several possible interpretations of this difference between the two samples.

First of all, the wording of the first question was changed from "How would you define mathematics?" to "How do you define mathematics?" precisely in order to discourage skipping the question. Under the conditions of the two studies, it is impossible to know whether the better participation of mathematics educators is indeed due to this change in wording. I should point out that just as the first version of the question elicited the answer "I wouldn't", the second one could and did elicit the answer "I don't". Moreover the wording of the second question was identical for the two samples.

A second interpretation involves the general framework of the two surveys comprising the questions discussed here. The main focus of the survey of mathematicians was the status of women in mathematics, whereas this was not so in the case of the survey of mathematics educators. Those who responded to the first survey may have ignored the questions about views of mathematics because they did not consider them to be relevant to the main subject of the survey.

Finally, it is possible that the different responses of the two samples reflect a real difference in their attitude towards mathematics. Could it be that mathematics educators are more inclined to consider philosophical and historical questions *about* mathematics, or that they are encouraged to do so by the demands of their profession, while mathematicians are too busy *doing* mathematics to pay much attention to these kinds of issues[2]? Or could it be that mathematics educators hold a somewhat more naive view of mathematics that allows them to venture to define it, whereas mathematicians are paralyzed by a greater awareness of the complexity of the question?

Turning now to the views of mathematics expressed by mathematics educators, one can first remark that they exhibit at least as much variety as those expressed by mathematicians. The two images of mathematics as a formal abstract system ruled by logic (themes 1 and 2) and as a model of the real world (theme 4) are both quite widespread. Mathematics is also considered to be both an art (theme 9) and a science (theme 10), both a language, i.e. a form, (theme 3) and a set of specific contents (theme 13). In some cases, the same person mentioned both members of these pairs of complementary ideas: thus five respondents mentioned both themes 1 and 4, three mentioned both themes 9 and 10, and two mentioned both themes 3 and 13.

Compared to mathematicians, mathematics educators were more likely to perceive a kinship between mathematics and the natural sciences, as evidenced by references to inductive thinking processes (theme 8). Theme 7 (the study of patterns) might also be linked to this viewpoint.[3] However, even for mathematics educators, deductive reasoning (theme 2) remains a more saliant feature of mathematics than inductive thinking (theme 8). This is not surprising, since the thesis that mathematics is essentially an empirical science[4] still holds a minority status in the philosophy of mathematics.

Pursuing the comparison of the two groups, one might be tempted to make something of the fact that no mathematics educator touched on the theme of mathematics as truth (theme 11) and that, vice versa, no mathematician depicted this discipline as being dependent on culture (theme 12). However the numbers of respondents who made reference to these two themes are too small to draw any reliable conclusion. In this connection, it may be worth pointing out that 13 mathematics educators (25.5%), compared to 9 mathematicians (8.7%), referred to mathematics as a (human) creation or construction. Only three mathematics educators saw mathematics as discovery, and two of them saw it as both discovery and creation. Among the mathematicians, two mentioned the idea of discovery and both of them mentioned creation or invention as well. The image of mathematics as a human creation does not necessarily contradict the idea that mathematics is akin to the natural sciences, for, within a constructivist framework, the latter are also viewed as human constructions.

In summary, elements of two kinds of conceptions of the nature of mathematics are present among mathematics educators, as among mathematicians: "formalism" and "constructivism" (the quotes are meant to signal that these are not necessarily coherent, fully formed, consciously held philosophies, but rather attitudes and tendencies)[5]. Instrumentalist (traditionalist) and Platonist views are practically absent, with the possible exception of the following response which does have a metaphysical ring: "Une science éblouissante en ce qu'elle sollicite tant l'esprit humain et lui ouvre les portes de l'au-delà".

On the whole, the ideas expressed by mathematics educators and by mathematicians show more similarities than differences. The same conclusion holds even more strongly concerning the second question examined here: the sets of authors cited by mathematics educators and by mathematicians as having had the most influence on mathematics are remarkably similar. The small difference observed can be attributed to a greater importance attached by mathematics educators to authors from earlier epochs. Together with their relative willingness to answer the question, this greater attention to the more distant past might reflect a greater familiarity with the history of mathematics among mathematics educators. As to the striking resemblance between the answers of the two groups, it may indicate the status of ancient, well-established discipline attained by mathematics.

CONCLUSION

First of all, the limitations of the kind of research that I have conducted must be underlined. By writing a few words about mathematics in response to a questionnaire, one cannot display the richness of one's vision of this subject, as one might in the course of a personal interview. Moreover, and perhaps more seriously, mental images are often diffuse, incoherent and partly unconscious, hence difficult to articulate. No doubt, what each participant in the present research has produced offers but a small portion of his or her ideas about mathematics. However, the drawbacks of the method that I have used are compensated by the possibility of gathering data from a large population. Indeed, my aim was to obtain a broad description of the most salient features of the images of mathematics prevailing among a professional community, and not to give a detailed account of the views held by its individual members.

The main conclusions reached in the study about mathematicians (Mura, 1993) hold here too. Within each of the two communities, the images of mathematics vary considerably. Given this variety, it can be stated that, globally, the images of mathematics that university students encounter among their teachers do not change dramatically, as they move from mathematics to mathematics education courses. In particular, formalist views do not disappear from the students' learning environment, and their presence among university teachers of both mathematical sciences and mathematics education contributes to explain and justify their prevalence among school teachers. Individually, of course, as chance will have it, each student may be exposed to a series of teachers holding different views, but the direction of the change can scarcely be predicted, except for an increase in the likelihood of encountering, during mathematics education courses, the ideas that mathematics makes use of inductive thinking and that the study of patterns is central to this discipline.

The present study, like the previous one, also indicates that mathematics educators stand on shaky ground when we make value judgements about school teachers' beliefs. We may count on most of our colleagues' support when we criticize the instrumentalist or Platonist conceptions of mathematics, but this is no longer true when it comes to formalism and constructivism. And if there is no consensus among either mathematicians or our own professional community, how can we maintain that one belief is more desirable than another?

In discussing research into students' conceptualization of advanced mathematical concepts, Tall (1992) urges researchers to consider the nature of their own perceptions of those same mathematical concepts. He borrows from Lavoisier the metaphor of "creases in the mind", creases that cause us, individually or collectively, to see things in a certain way. He warns researchers about the danger of not acknowledging that their own conceptions contain idiosyncrasies dependent on personal and cultural experience, just as the students' do. I think that the same considerations hold, if anything even more strongly, for research into the conceptions of mathematics as a whole.

Consciously or unconsciously, researchers might place views similar to their own on top of a hierarchy of conceptions of mathematics. Such a standpoint should be either avoided or explicitly acknowledged, so that its consequences on the research process and products may be taken into account. Moreover, favouring one's own views carries a special difficulty for those who embrace a constructivist philosophy, like many researchers in the field of the images of mathematics do, for, as von Glasersfeld (1990, p. 19) has pointed out, by its very nature, constructivism cannot prove itself to be the "true" epistemology.

Turning to more practical issues, the present study brings to light a possible obstacle in the path of the current reform movement in mathematics education that has received little attention until now. Thompson (1992b) describes the gap between the image of mathematics underlying current mathematics education reform in North America and that which shapes much of what goes on in schools under mathematics instruction. She finds the two images to be in sharp contrast and suggests that this "may be the single greatest obstacle to achieving mathematics instruction as envisioned in many reform documents." Thompson then goes on to examine the experience gathered through many intervention programs designed to bring teachers' prevailing images of mathematics and mathematics teaching in line with the ideals of the current reform movement and concludes that it is very difficult to trigger the cognitive restructuring necessary to transform in depth the teachers' conceptions of mathematics. The results obtained in the present study, and in my previous one, show that there may well be a second gap to bridge: namely the one between the view of mathematics advocated by those engaged in the mathematics education reform movement and the images held by other segments of the mathematics and mathematics education communities within universities. The coexistence of these diverse images at the university level tends of course to justify school teachers' resistance to changing their own conceptions.

The above conclusion may have a subduing effect on the enthusiasm of those who strive to bring about change in mathematics education. However, a realistic assessment of the difficulties involved in the task may foster the patience and persistence needed to accomplish it. Respect for the teachers' personal rhythms of change and a measure of detachment are undoubtedly further assets. In this connection, Pimm (1993) offers provocative and refreshing thoughts, concluding as follows:

> I think we should examine [...] critically our *need* (lust?) for the teachers we work with to change. [...] Their change is not our business; how when and if they change is surely their concern alone. [...] I believe it is dangerous to lose sight of how difficult personal change can be – and we should not talk lightly or glibly about it, let alone expect or demand it. (p. 31)

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NOTES

[1] By "inductive thinking" I mean here the way of reasoning which uses individual ideas or facts to reach a general rule or conclusion, and not the technique known as mathematical induction.

[2] According to Borel (1983, p. 9), "there is a rather natural reluctance for a practicing mathematician to philosophize about mathematics instead of just giving a mathematical talk. As an illustration, the English mathematician G. Hardy called it a 'melancholy experience' to write *about* mathematics rather than just prove theorems!"

[3] Concerning the idea of mathematics as the study of patterns, with one single exception, all of the mathematicians and mathematics educators who expressed it were English-speaking. In fact, the word "pattern" has no exact equivalent in French; one frequent translation of it, "régularité", covers only a part of the range of meanings of the English "pattern".

[4] For a thorough exposition of this thesis, see, for instance, Kitcher (1983). The idea that mathematics does not stand apart from the natural sciences has gained more support since the advent of computers and their use in mathematical research. A new international journal, first published in 1992, even carries, boldly, the title *Experimental Mathematics*.

[5] By formalism and constructivism, I do not mean here the schools of thought on the foundations of mathematics founded respectively by Hilbert and by Brouwer, but rather

two types of perceptions of mathematics that have been described within an educational framework (see for instance, Dionne, 1988, p. 130–133). In this sense, formalism presents mathematics as a finished product; it emphasizes unifying concepts, rigour and precision in language and symbolism. By contrast, constructivism portrays mathematics as an activity. It places in the foreground the thinking processes of those who "do" mathematics (be they students or professional researchers), such as the finding of relations and the building of theories from real experiences.

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