FRANK SWETZ

TO KNOW AND TO TEACH: MATHEMATICAL PEDAGOGY FROM A HISTORICAL CONTEXT

ABSTRACT. Historical texts tell us much about how mathematical concepts and techniques were conceived and evolved. But they can also tell us even more. Usually, their contents embody a pedagogy, specific organizational forms and methods used for teaching mathematics. This article isolates and examines several particular pedagogical techniques: the use of an instructional discourse; a logical sequencing of mathematical problems and exercises and employment of visual aids, evident in historical works. It concludes that much of present day mathematical pedagogy evolved from distant historical antecedents.

1. INTRODUCTION

Several years ago while viewing cuneiform tablets at the British Museum, I was deeply moved by what I observed on one clay tablet. In the lower right-hand corner of a small table of multiplication facts dating from the Old Babylonian period (ca 1200 BC) was a finger print, a single, clearly discernible, human finger print. The impression was obviously captured over 3000 years ago when the clay was wet. This human punctuation mark personalized the tablet and its contents for me. No longer did I see just symbols representing mathematical facts and procedures, now I pictured the writer, a person, an individual, associated with that information. My discovery clarified the humanity, the human involvement with the situation, that resulted in this artifact. Implicit in most old mathematical texts are many messages relating to the learning and teaching of mathematics; however, far too often we, as readers, seek only the facts and do not ponder the implicit messages the mathematics, its form and format, tell us.

One such message that, over the years, has been impressed upon me by my research is that much of the extant mathematical writings from the past indicates that their authors were innately teachers, persons who by their privilege of knowing were compelled to teach. They were often concerned with communicating mathematical information to a remote and uninitiated audience and, to assist in this task, they employed a variety of rhetorical and graphical strategies. Their methods and teaching techniques were not haphazard but carefully thought out and organized to serve specific audiences and instructional needs. Quite simply, they evolved a pedagogy for

mathematics teaching. This pedagogy, in several of its primary aspects: the use of an instructional discourse; a logical sequencing of mathematical exercises and problems from the simple to the complex and a reliance on the use of tactile and visual aids, is similar to that employed today. A review and examination of these historical precedents for mathematical pedagogy helps to develop both an appreciation and understanding of the mathematics learning/teaching process.

2. USE OF AN INSTRUCTIONAL DISCOURSE

Despite its critics – "I hear and I forget...", the major venue of mathematics instruction is through "telling", establishing a discourse, either oral or written between the teacher and the learner. This discourse can be teacher dominated with the information flowing in one direction from the instructor to the learner or it can, more desirably, be a two way process, that is, a discussion between the instructor and the student in which the student becomes an active participant in the learning/teaching process. Of course, a secret of good teaching is to make the discourse palatable, that is, both psychologically and intellectually appealing. Robert Recorde (c 1510–1558), author of the first popular, printed mathematics texts in Great Britain,[1] was noted for being highly successful in this endeavour (Howson, 1982). Recorde, in writing primarily for the mathematical practitioners of his time, craftsmen and tradesmen, who where daily users of mathematics but lacked access to further education, produced works intended mainly for self study, i.e. for those "as shall lacke instructors". In doing so he deviated from the standard scholastic genre of the period and initiated a series of pedagogical innovations for the teaching of mathematics (Johnson and Larhey, 1935). He wrote in the vernacular and used two primary modes of exposition: verse and the dialogue. Both practices did not originate with Recorde. The use of verse as an aid for memorization (yes, mathematics learning always involved memorization) was employed in both western and eastern societies at a much earlier date. One of the first European texts on the use of the Hindu-Arabic numeral system, *Carmen de algorismo* by the French monk Alexandre de Villedieu (c 1225), was composed in verse.[2] Medieval Chinese, Hindu and Islamic authors employed a similar technique in many of their technical writings. The dialogue, in Recorde's case between a Master and a Scholar, was a popular instructional device in many didactical works of his period and, no doubt in part, reflected the transition from a purely oral instruction to one employing the printed word. In any case, Robert Recorde considered it the "ryghte teachynge" for his purposes noting in the preface of his *Grounde of Artes* (1542):

> ...because I iudge that to be the easyest way of construction, when the scholar may
> aske eucry doubte orderly, and the mayster may answere to his question playnly.

The use of a dialogue in mathematics instruction is indeed ancient and
pre-dates sixteenth century Elizabethan England. Illustrative conversations
place the learner-reader as a neutral observer in the learning process, an
initially passive observer who is gradually drawn into the train of instruc-
tional thought and hopefully led to the active realizaton or discovery of
the mathematical concepts in question. Vicarious but distant involvement
in the discussion excludes a personal confrontation of objections, conflic-
ing issues are resolved in the dialogue and ignorance is eliminated with a
minimum of embarrassment. Perhaps the classical exemplar of a heuristic
dialogue is found in Plato's *Meno* (4th century BC) where Socrates instructs
a slave boy. He shows the slave a square of side two units and asks him
the measure of the square's side that would result in a doubling of its area.
The boy responds four units upon which Socrates verbally leads him down
a deductive path in which his error is recognized and corrected.

Socrates:	Well how big is the new one? Is it not four times the old one?
Boy:	Surely, it is!
Socrates:	Is four times the old one, double?
Boy:	Why no, upon my word!
Socrates:	How big, then?
Boy:	Four times as big!
Socrates:	Then, my boy, from a double line we get a space four times as big, not double.
Boy:	That's true. (Warmington and Rouse, 1961, p. 44.)

While the passages of the *Meno* were conceived to investigate the philo-
sophical question as to whether virtue can be taught, the dialogue on
geometry stands as an illustrative example of good mathematics teaching
(Freydberg, 1993). But the "Socratic method" of teaching mathematics
was also well known and employed outside of Athens. The earliest sur-
viving Chinese mathematical text, *Zhoubi suanjing*, [Arithmetic Classic
of the Gnomon and the Circular Paths of Heaven] (ca 100 BC) features
a mathematically directed conversation between the legendary Duke of
Zhou and his Grand Prefect Shang Gao. Zhou comments:

> I have heard that the Grand Prefect is versed in the art of numbering. May I venture to
> inquire how Fu Xi anciently established the degrees of the celestial sphere? There are

no steps by which one may ascend the heavens, and the earth is not measurable with a foot-rule. I should like to ask you what was the origin of these numbers?

and Shang Gao replies:

> The art of numbering proceeds from the circle and the square. The circle is derived from the square and the square from the set square... Thus,let us cut a rectangle [diagonally], and make the width 3 [units] wide, and the length 4 [units] long. The diagonal between the [two] corners will then be 5 [units] long. Now after drawing a square on this diagonal, circumscribe it by half-rectangles like that which had been left outside, so to form a [square] plate. Thus the [four] outer half-rectangles of width 3, length 4, and diagonal 5 together make two rectangles [of area 24]; then [when this is substracted from the square plate of area 49] the remainder is of area 25. This process is called 'piling up the rectangles' (ji ju). (Needham, 1954: pp. 22–23.)

Thus the properties of the right triangle are revealed to the reader.

These early authors used the discourse to instruct but they also realized the necessity of motivating and encouraging their student readers. To this end they stressed the importance of mathematics being considered. The Egyptian scribe Ahmes (1650 BC) prefaces his mathematical papyrus with enticing comments that its contents will reveal "a thorough study of all things, insight into all that exists, knowledge of all obscure secrets" (Burton, 1985, p. 36). Writing in the fourth century AD, the Chinese scholar Sun Zi stressed the importance of mathematics for the readers of his text *Sun Zi suanjing*, [Master Sun's Mathematical Classic]:

> Mathematics has prevailed for thousands of years and has been used extensively without limitations. If one neglects its study, one will not be able to achieve excellence and thoroughness. There is indeed a great deal to master when one views mathematics in perspective. When one becomes interested in mathematics, one will be fully enriched; on the other hand, when one keeps away from [the subject], one finds oneself lacking intellectually. (Lam and Ang, 1992: p. 151.)

Similarly, in the sixteenth century, a reader of Robert Recorde's *The Declaration of the Profit of Arithmeticke* (1540) finds the Master and the Scholar discussing the merits of studying mathematics:

Master: Wherefore in all great works are clerks so much desired? Wherefore are Auditors so well fed? What causeth Geometricians so highly enhaunsed? Why are Astronomers so greatly advanced? Because that by number such things they finde, which else would farre excell mans minde.

Scholar: Verily, sir, if it bee so, that these men by numbering, their cunning do attain, at whose great works most men do wonder, then I see well I was much deceived, and numbering is a more cunning thing than I took it to be.

The message that was repeatedly conveyed was that mathematics was good and was important. Its study was a worth activity.

3. COMPOSITION AND SELECTION OF PROBLEMS

While the early authors of mathematics texts wrote about mathematics, they unanimously held to the principle that the learning of mathematics was an active process involving the doing of mathematics, the solving of problems by their readers. In many cases, extensive collections of problems were compiled and incorporated, in various forms, into the learning situation (Neugebauer, 1973; Chace, 1967; Vogel, 1968). Some early mathematical texts have under criticism as being merely collections of problems, for example Babylonian, Egyptian and extant Chinese works; however, it must be remebered that such texts were adjuncts to oral instruction and reinforced the theory explained (van de Waerden, 1983). It should also be remembered as to what audiences such texts were intended to serve: student scribes or apprentices who were preparing to enter into a civil or religious bureaucracy (Ritter, 1989). These scribe trainees may not have had any particular aptitude or interest in mathematics and yet the requirements of their prospective jobs involving taxes, transportation, building construction, etc. involved mathematical problem solving. They had to learn "fast and ready" mathematics – the recognition and typing of specific problems and the application of certain rules, techniques, computational algorithms to obtain a solution. Studying sets of carefully chosen problems supplied an appropriate background.

A collection of mathematical problems does not, in itself, preclude the possibility of pedagogical designs. While some ancient texts may appear to be merely collections of problems, those problems were carefully chosen and through their conception and organization, attest to their author's ability as a teacher.

British Museum cuneiform tablet 15285 from the Old Babylonian period contains a series of geometrical problems given in very brief form. It is believed that the tablet originally contained over 40 systematically arranged exercise problems; however only 30 are wholly or partially preserved (Saggs, 1960).[4] These problems are perhaps reminiscent of modern day geo-board exercises undertaken by school children; however, they are more demanding in their level of conceptual understanding and required a high degree of geometric intuition for their solution. Early assyriologists who studied the tablet and its contents were confused as to its purpose. Initial interpretations described it as a land surveyor's manual designating various field shapes (Gadd, 1922). However, the complexity of the

regions illustrated cast doubt on this theory and it was revised (Caratini, 1957). As any modern geometry teacher could have told the linguists and archaeologists, it is a series of geometrical exercises designed, perhaps, for the purpose of training mathematical scribes. Each problem concerns a square whose side measures 1 US in length (1 US = 60 gar, 1 gar = 12 cubits), these squares are partitioned by straight lines and circular arcs into more complex regions and, for each problem, the area of a specific region is required. The partitioning is accomplished by the use of integral and symmetric divisions of the area; however, it is left to the student to determine the resulting numerical values by a use of intuition and geometric reasoning. As a reader progresses through the sequence of problems he or she encounters an increased degree of conceptual difficulty; solutions to latter problems require the transfer and adaptation of principles learned or used in obtaining solutions for the previous problems. A sample selection of these exercises is given in Figure 1; in each instance, it is required to find the area of the shaded region [my shading]. You may try your skill in obtaining answers to these three thousand year old problems. They provide fascinating exercises in problem solving.

The most comprehensive collection of problems from the ancient world that involve the right triangle is found in the Chinese mathematical classic *Jiuzhang suanshu* [Nine Chapters on the Mathematical Art] (c 100 BC). The ninth chapter of this work is entitled *gougu*, right triangle (Swetz and Kao, 1977). Chinese mathematicians had specific names for the respective sides of a right triangle: *gou*, shorter leg (a); *gu*, longer leg (b) and *xian*, hypotenuse (c). An inspection of the 24 problems of this chapter reveal a sequential ordering based on the mathematical principles being stressed and general conceptual complexity – a pedagogical ordering (Swetz, 1993). See Table I. The initial three problems test the readers's knowledge and comprehension of the "Pythagorean theorem" in its simplest form, i.e. $a^2 + b^2 = c^2$. Problems 4 to 14 require both analytic and synthetic problem-solving skills. These latter problems are posed in rather picturesque and complex mathematical situations that must be unraveled by the reader before the right-triangle problem *per se* emerges. Thinking is required. For example, the fifth and ninth problems are as follows:

A tree of height 20 *chi* has a circumference of 3 *chi*. There is an arrowroot vine that winds seven times around the tree and reaches to the top. What is the length of the vine?

A wooden log is encased in a wall. If we cut part of the wall away, at a depth of 1 *cun* [1 *chi* = 10 *cun*], the width of the [exposed] log measures 1 chi. What is the diameter of the log? [cun = 3.58 cm]

Old Babylonian Geometry Problems

Learning/Solution Sequence
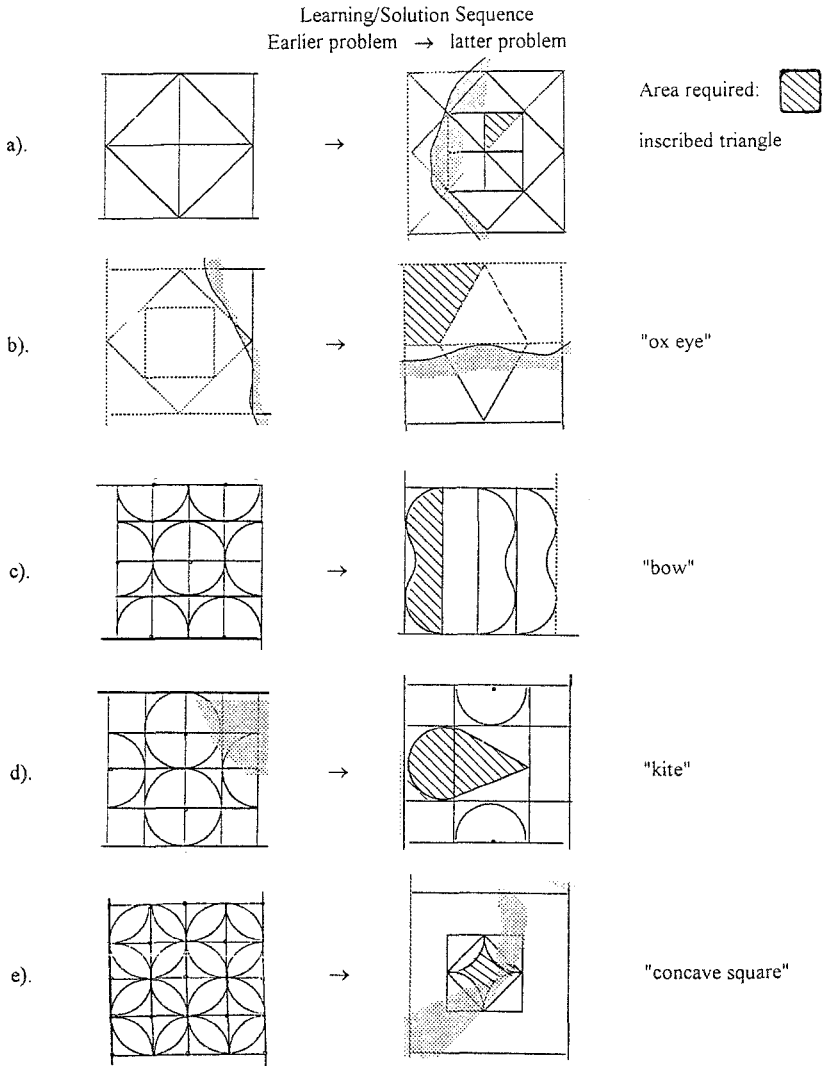Earlier problem → latter problem

Fig. 1. Old Babylonian geometry problems

Problems 15 and 16 broaden solution perspectives and are, in essence, pure mathematics problems; they have remained until today in varous versions as standard secondary school textbook problems:

[A right triangle has] a *gou* of the length 5 *bu* and a *gu* of length 12 *bu*. What is the largest square that can be inscribed in this triangle?[5] [bu = 5 Chi]

[A right triangle has] a *gou* 8 units long and *gu* 15 units long. What is the largest circle that can be inscribed in this triangle?

TABLE I

Classification of *Gouga* problems

Given a right triangle with hypotenuse of length c and legs a and b.

Computational situation		Problem Numbers*
1. Find c, given a and b	Increasing level	1,5
2. Find b, given a and c	of conceptional	2
3. Find a, given c and b	complexity	3,4
4. Find a, given $c - a$ and b	$\mid$	6,7
5. Find c, given $c - b$ and c	$\mid$	8, 9 10
6. Find a and b, given $b - a$ and c	$\mid$	11
7. Find a and b, given $c - b$ and $c - a$	$\mid$	12
8. Find a, given $c + a$ and b	$\downarrow$	13
9. Find a and c, given b and $b + c = 7a/3$		14

* In the text, problems are not numbered. Numbers indicate sequential ordering.

(One can almost hear the mathematics instructor who assigned these prob-lems saying "Draw a picture, draw a picture!".) The remaining exercises of the *gougu* chapter can be described as surveying problems involving similar right triangles and stressing applications of the theory learned.

This format of a mathematics text as a collection of problems, together with detailed solution instructions, has a long history as an educational medium which continues today in the form of study guides and review books, for example, the Schaum's Outline Series in Mathematics. It was not only found in the east. Warren Van Egmond in his scholarly and penetrating study of the *abaci* manuscript and text genre of the early Italian Renaissance notes that these influential works were usually just catalogues of problems. They were written by reckoning masters primarily for the training of merchants who sought ready applications of the "new mathematics" for their trade. Their contents offered a highly specialized, prescriptive approach to problem solving. However, Van Egmond admires the reckoning masters' ability at their choice of appropriate model problems for discussion and is attracted to their personal, conversational style of exposition (Van Egmond, 1976).

The authors of the Babylonian tablet, the Chinese text and the Italian abaci, through their use and organization of mathematical problem solv-ing situations, demonstrate a concern with and pedagogical insights into the human learning process. They were, quite simply, concerned teachers writing for the learning needs of their students.

4. VISUAL AIDS

Contemporary mathematics teachers, especially those involved in elementary education, are urged to incorporate the use of concrete manipulatives and visual aids into their teaching. This practice has been followed by mathematics educators from the earliest times. Babylonian clay tablets from 2000 BC carry geometric diagrams illustrating situations discussed in the associated text. Such diagrams are easily discerned and understood today. It is also interesting to note that the first printed European books to use illustrations were mathematical treatises. Traditional Chinese mathematical texts abounded with illustrations. Often grids were superimposed on the geometric situation to facilitate a solution process called *ji ju* "piling up rectangles". See Figure 2.

Frequently, Chinese authors used colors, i.e. red and yellow, in their geometric illustrations to denote specific regions that would be transformed into a different configuration thus helping to demonstrate the principle that area is preserved under planar transformations (Swetz, 1992). The first author credited with this practice was Zhao Suang (ca 300 AD) who compiled a treatise of 21 theorems involving right triangles. Zhao employed color to clarify the area transformations in his geometrical-based proofs (Gillon, 1977). Color coding was also an integral part of the numerical computing processes of early China. Mathematicians utilized a set of computing rods which were arranged into numerical configurations and then placed in significant positions on a computing board. The rods were manipulated according the established algorithms and a variety of mathematical results obtained (Ang, 1977). These computing rods were also color coded – red for rods representing positive numbers, black for rods representing negative numbers. The numerical dichotomy of positive and negative numbers is thus amplified by a visual dichotomy impacting on the sense of sight. It is interesting to note that today, in a commercial context, positive (profit) and negative (loss) are also associated with the colors black and red although the ordering is reversed.

Liu Hui (ca 263), mathematical commentator and one of the great mathematicians of ancient China, urged his readers to make diagrams on paper, to cut them up and rearrange the pieces. For the Chinese, paper cutting and folding was a readily accepted method of mathematical demonstration (Sui, 1993). Several hundred years later the first English language translation of Euclid's *Elements* was accomplished by Henry Billingsley, Lord Mayor of London, and published in 1570. Billingsley's discussion of solid geometry as contained in Euclid's Book XI was supplemented by "pop-up" diagrams pasted onto the relevant pages thus enabling the reader to interact

Fig. 2. *Xian tu* diagram from Chinese mathematical classic *Zhoubi suanjing*. Note peda-
gogical features: 1. Use of 3-4-5 right triangles; 2. Use of grid network; 3. Use of dissection
proof technique

with the three-dimension figures discussed in the text (Archibald, 1950).
See Figure 3.

With the advent of moveable type in European printing and the accom-
panying popularization of books, mathematics authors devised various
visual schemes to assist in the transmission of knowledge. Positional nota-

EVCLIDIS MEGA
renſis mathematici clariſsimi
Elementorum geometrico
rum libri xv.

Cum expoſitione Theonis in priores x i i i à Bartholomæo
Zamberto Veneto latinitate donata, Campani in
omnes, & Hypſiclis Alexandrini in du-
os poſtremos.

His adiecta ſunt Phænomena, Catoptrica & Optica,
deinde Protheoria Marini, & Data.

Poſtremùm uerò, Opuſculum de Leui & Ponderoſo,
hactenus non uiſum, eiuſdem autoris,
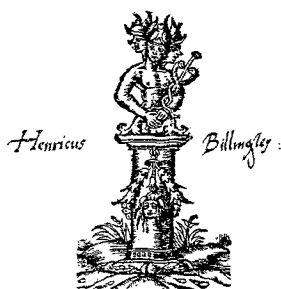
Cum priuilegio Cæſareo.

BASILEAE, PER IOANNEM HER-
uagium, & Bernhardum Brand, Anno
M. D. LVIII.

Fig. 3. (a) Billingsley's 1558 Latin edition title page

tion and associated numerical value as embodied in the use of Hindu-Arabic numerals was a particular important but difficult concept for new learner of arithmetic to grasp. In Europe this problem was solved by employing a graphical scheme whereby number configurations were staggered to associate the number of digits with a designated magnitude. Usually a columnar array of digits was illustrated listing numbers of the smallest magnitude at the top and to the right. In the mind of early medieval Italian scribes, this configuration seemed to silhouette the turrets of a castle and

Fig. 3. (b) Erected pop-up figures Fol 314. Author's student, a distant relation of Henry Billingsley holds the book

they illuminated it as such on manuscripts. The array became known as a *castellucio* or "little castle". See Figure 4.

Multiplication facts were organized into tables, traditionally called *Pythagorean tables*, in which the numerical patterns might be better observed and remembered. These tables could either be square in form or triangular emphasizing the commutative property of multiplication. See Figure 5.

In writing about the Hindu-Arabic numeral system and its computational algorithms, early European textbooks authors often embellished the algorithms' printed form by a diagram usually related to common objects of the time. Although, at first this practice may seem fanciful, it had a pedagogical purpose. Such renderings provided a graphic mnemonic accompanied by a descriptive name both associated particular mathematical operations or a sequence of operations with the resulting visual configuration. A particular operation for which this practice seemed to be most bountiful was the multiplication of two multi-digit numbers where the question and arrangement of partial products was crucial. Thus a computer of the middle ages could choose from a variety of picturesque sounding techniques for multiplication including: (a) *per scachieri*, "by the chessboard" (b); *per gelosia*, "by the grating" (c); *per campana*, "by the bell" (d); or *per coppa*, "by the chalice or cup" (Smith, 1958, pp. 101–128). See Figure 6.
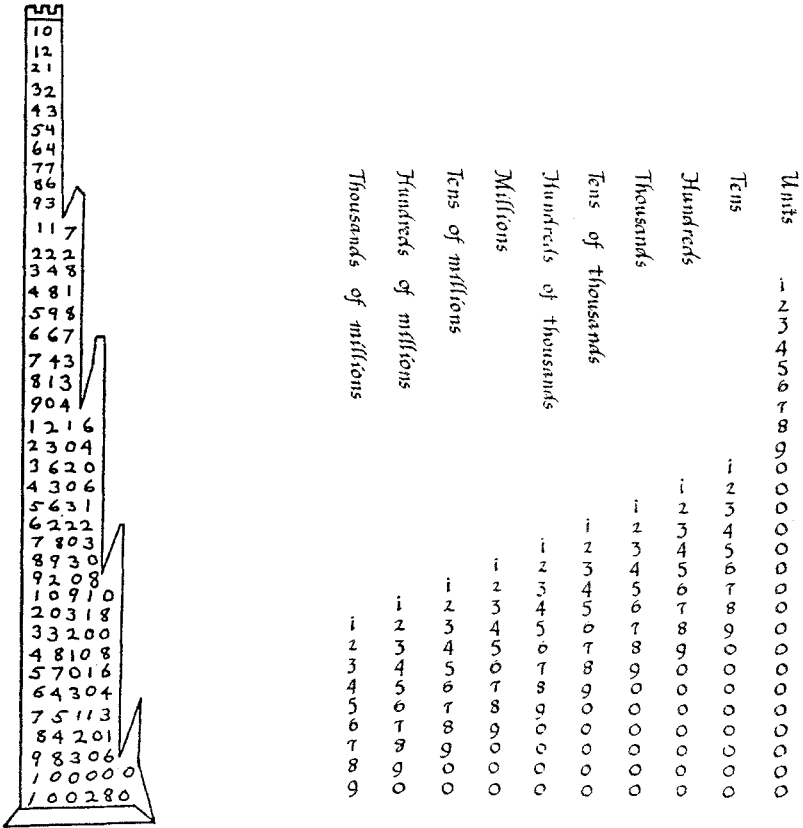
(a) Copy of Castellucio

```
10
12
21
32
43
54
64
77
86
93
117
222
348
481
598
667
743
813
904
1216
2304
3620
4306
5631
6222
7803
8930
9208
10910
20318
33200
48108
57016
64304
75113
84201
98306
100000
100280
```

(b) Numeral array

Thousands of millions	Hundreds of millions	Tens of millions	Millions	Hundreds of thousands	Tens of thousands	Thousands	Hundreds	Tens	Units
									1
								1	2
							1	2	3
						1	2	3	4
					1	2	3	4	5
				1	2	3	4	5	6
			1	2	3	4	5	6	7
		1	2	3	4	5	6	7	8
	1	2	3	4	5	6	7	8	9
1	2	3	4	5	6	7	8	9	0
2	3	4	5	6	7	8	9	0	0
3	4	5	6	7	8	9	0	0	0
4	5	6	7	8	9	0	0	0	0
5	6	7	8	9	0	0	0	0	0
6	7	8	9	0	0	0	0	0	0
7	8	9	0	0	0	0	0	0	0
8	9	0	0	0	0	0	0	0	0
9	0	0	0	0	0	0	0	0	0

(a) Copy of Castellucio from
Ragioni apartenenti all'arismetricha
11th century

(b) Numeral array from
Treviso Arithmetic 1478
*translated from the Italian

Fig. 4. Hindu-Arabic numeral configurations from early arithmetic texts

5. A FINAL THOUGHT

Thus it appears that many of the pedagogical techniques and principles that are favored today have experienced a long and diverse history. That history not only spans time but also cultures and testifies to the fact that while mathematics is universal, to a large extent, so is its pedagogy. The need to communicate mathematical understanding spawned its own devices. Pedagogic techniques were evolved with particular audiences in mind. They were conceived to help popularize mathematics: to more easily reveal its concepts, to facilitate and strengthen a memory of its processes and operations and insure an efficient application of the knowledge learned. For the most part, mathematics and good teaching were bound together

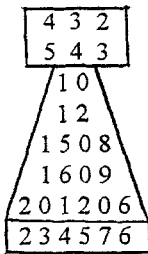a) Square form   (b) Triangular form

Fig. 5. Pythagorean [multiplication] tables from Johann Widman's commercial arithmetic, *Behede und hubsche Rechnung auf allen Kaufmanschafft* 1500 edition
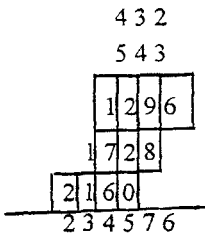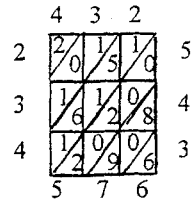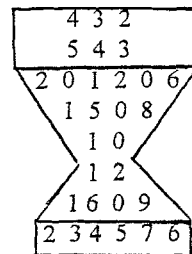

a) by the chessboard   (b) by the grating

(c) by the bell   (d) by the chalice

Fig. 6.   13th–14th Century multiplication algorithms

thousands of years ago establishing a tradition that can yet be enhanced, and in the light of new educational technology, must be enhanced, posing

a challenge to contemporary mathematics educators. Let us hope that challenge, set by our forefathers, will be met.

NOTES

[1] Recorde's major works were: *The Grounde of Artes* (1542), a popular arithmetic that saw at least 45 editions; *the Pathewaie of Knowledge* (1551), a practical geometry based on the first four books of Euclid's *Elements*; *The Castle of Knowledge* (1556), a text on applied geometry; and the *Whetstone of Witte* (1557), an algebra in which the ancestor of the modern equals symbol appears. He also wrote several minor tracts popularizing and promoting the study and use of mathematics.

[2] *Carmen de algorismo*, [Song of the Algorismus], a most influential text in propagating a knowledge of the Hindu-Arabic numerical system on the European scene. The first algorism to appear in England, *The Crafte of Nombrynge* (c 1300) was an interpretation of *Carmen de algorismo*.

[3] Fu Xi was the first of the mythological Three Sovereigns and is credited with inventing, among other things, the drawing compass and the [set] square, the instrument of ancient Chinese geometrical computations. These passages are translated in Joseph Needham, *Science and Civilization in China*, Cambridge, University Press (1954) 3:22–23. For a discussion of the mathematical uses of the set square see Frank Swetz, *The Sea Island Mathematical Manual: Surveying and Mathematics in Ancient China*, University Park, Pa.: Pennsylvania State University Press, (1992).

[4] The author would like to thank Johan Friberg of Goteborg University for his assistance in identifying these articles.

[5] If S is the side of the desired square and a and b are legs of the right triangle then the relationship is found $S = \dfrac{ab}{a + b}$. This fact was "discovered" by a secondary school student and became the subject of an interesting series of "Letters to the Editor" in the *Mathematics Teacher* in 1985, i.e. Morris Wilkenfeld, "Right triangle relationships", (Jan. 1985) 78:12; G. Spenser Lieske, "Right triangles II", (Oct. 1985) 78:498–499.

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FRANK SWETZ
Penn State University,
Mathematical Sciences Department,
Middleton, PA 17057-04898,
U.S.A.