RUHAMA EVEN AND DINA TIROSH

SUBJECT-MATTER KNOWLEDGE AND KNOWLEDGE ABOUT STUDENTS AS SOURCES OF TEACHER PRESENTATIONS OF THE SUBJECT-MATTER

ABSTRACT. Pedagogical content knowledge is made up of several components. In this paper we concentrate on one of these: teachers' planned presentations of the subject-matter. We deal with two main sources of this component of pedagogical content knowledge: knowledge about the subject-matter and knowledge about students. Illustrations are given in two mathematical domains: functions and undefined mathematical operations. The paper concludes with a discussion of the nature of teachers' knowledge and the interconnections between the three constructs: subject-matter knowledge, knowledge about students, and knowledge about ways of presenting the subject-matter.

INTRODUCTION

The recognition that pedagogical content knowledge is an important characteristic of teacher knowledge is growing fast. One major issue related to this kind of knowledge is its sources. Obviously, the teacher's own experiences, both as a learner and as a teacher, influence pedagogical content knowledge. Exposure to relevant developmental and cognitive research, including learning theories, and interactions with students, are other factors. Another source of pedagogical content knowledge is the nature and depth of teachers' own subject-matter knowledge of the material they teach.

Not only does pedagogical content knowledge derive form different sources, it is made up of several components. In this paper we concentrate on one of these: teachers' planned presentations of the subject-matter. Special attention is paid to teachers' planned reactions to students' questions and hypotheses. We deal with two main sources of this component of pedagogical content knowledge: knowledge about the subject-matter and knowledge about students.

Subject-matter

Even though it is usually assumed that teachers' subject-matter knowledge and pedagogical content knowledge are interrelated there is little evidence

Educational Studies in Mathematics **29:** 1–20, 1995.
© 1995 *Kluwer Academic Publishers. Printed in Belgium.*

to support and illustrate the relationships. A simplistic explanation for this situation draws attention to the fact that "pedagogical content knowledge" is a relatively new notion (Shulman, 1986). However, there is more to this issue than initially meets the eye. Different conceptions of teachers' subject-matter knowledge have evolved throughout the years. Not many years ago, teachers' subject-matter knowledge was defined in quantitative terms – by the number of courses taken in college or teachers' scores on superficial standardized tests (Ball, 1991; Begle, 1979; Wilson *et al.*, 1987). But these "measures" are problematic. In recent years, teachers' subject-matter knowledge has been analyzed and approached more qualitatively, emphasizing cognitive processes and understanding of facts, concepts and principles, and ways in which they are connected and organized. Epistemological knowledge about the nature of the discipline has also received more attention (Ball, 1988, 1991; Even, 1990; Leinhardt and Smith, 1985; Shulman, 1986; Tamir, 1987).

Another possible explanation for the lack of research on the interrelations between teachers' subject-matter knowledge and pedagogical content knowledge has to do with different conceptions of the role of the teacher in the process of learning. Curriculum development during the 1960's and 1970's viewed the teacher's role as that of implementing an expert made curriculum. Teacher-proof curricula, the extreme outcomes of this process, assumed that children could learn directly from ready-made curriculum materials while the teacher, instead of teaching, would adopt a role of manager and facilitator. This was most apparent during the period of individualized instruction. Accordingly, most studies of teachers that were conducted in those days adopted a similar approach. Process-product research and the later research on effective teaching were basically "content free" and tried to identify generic teacher behaviors that seemed to be effective (Brophy and Good, 1986; Gage, 1978). The identified "effective" instructional behaviors tended to be connected with the management of classrooms rather than with content pedagogy. The mid 1980's marked a change in conceptions of the teacher's role in promoting learning; which now came to include setting mathematical goals and creating classroom environments to pursue them; helping students understand the subject-matter by representing it in appropriate ways; asking questions, suggesting activities and conducting discussions. Subject-matter knowledge is much more critical for this "new" role of the teacher.

Shulman (1986) distinguishes between two kinds of understanding of the subject-matter that teachers need to have – knowing "that" and knowing "why":

> We expect that the subject-matter content understanding of the teacher be at least equal to that of his or her lay colleague, the mere subject-matter major. The teacher need not only understand *that* something is so; the teacher must further understand *why* it is so (p. 9).

Students

Shulman (1986) also argues that, in addition to knowledge of the subject-matter per se, teachers' knowledge should include: "the ways of representing and formulating the subject that make it comprehensible to others" (p. 9). He further emphasizes the need to consider students' ways of thinking, stating that teachers need to be familiar with the conceptions and preconceptions that students bring with them to the learning.

This last component of teachers' knowledge necessitates a body of knowledge of common students' conceptions. Such knowledge has been gathered mainly in the last two decades of extensive cognitive research on student learning, which has yielded much useful data on student conceptions and thinking in mathematics. Many studies have shown that students often make sense of the subject-matter in their own way which is not always isomorphic or parallel to the structure of the subject-matter or the instruction (e.g., Even, 1993; Hershkowitz *et al.*, 1987; Kieran, 1992; Schoenfeld *et al.*, 1993; Tirosh and Graeber, 1990).

Knowledge about students is one aspect of teachers' pedagogical content knowledge. Another aspect is that of teachers' choices of presentations of the subject-matter to students. To make appropriate decisions for helping and guiding students in their knowledge construction certainly requires an understanding of student ways of thinking. A teacher who pays attention to where the students are conceptually can challenge and extend student thinking and modify or develop appropriate activities for students. Starting from students' limited conceptions, the teacher can help build more sophisticated ones.

Research on teaching has only recently started to include investigations of teacher knowledge and understanding of students' ways of thinking related to specific topics, as well as the issue of the nature and quality of teachers' responses to students' questions, remarks and ideas (Ball, 1988; Even, 1989, 1993; Even and Markovits, 1993; Leinhardt *et al.*, 1991; Maher and Davis, 1990; Peterson *et al.*, 1991; Strauss and Shilony, in press; Tirosh, 1993). One explanation for the lack of research in this direction has to do with the fact that research on learning and learners, and research on teaching and teachers have been following separate tracks for a long time.

In this paper we concentrate on two main sources of teachers' choices of ways of presenting the subject-matter: one is related to subject-matter knowledge, the other to understanding students' ways of thinking. Following is a discussion of these sources in the context of teaching mathematics. Illustrations of the sources and their effect on teacher presentations of the subject-matter are taken from our studies in two mathematical domains: functions and undefined mathematical operations. Let us start with a short background description of these studies.

FUNCTIONS AND UNDEFINED MATHEMATICAL OPERATIONS

Function is one of the most important and fundamental concepts in mathematics. A mathematical function is defined as any correspondence between two sets which assigns to every element in the domain exactly one element in the range. Functions do not have to be described by any specific expression, follow some regularity, or be described by a graph with any particular shape. However, explicit requirements of functions are: (1) they should be defined on *every* element in the domain, and (2) for each element in the domain there should be only *one* element (image) in the range; this condition is also known as univalence.

A quick way to tell if a given graph is the graph of a function is to use the "vertical line test": A graph is the graph of a function if and only if each vertical line cuts the graph in no more than one point.

As the history of the development of the concept of function shows, univalence was not required at the beginning. Freudenthal (1983) attributes this requirement to the desire of mathematicians to keep things manageable. The development of advanced analysis created the need to deal with differentials of orders higher than one, and, therefore, to distinguish independent from dependent variables. Therefore, it became too difficult to work with multivalued symbols and the univalence requirement was added to the definition of a function.

All basic mathematical operations on the real numbers (e.g., addition, subtraction, multiplication, division, exponentiations and roots) are functions. Thus, the definitions of mathematical operations should fulfill the requirements of functions, namely the univalence requirement and the requirement that the operation should be defined for every element in the domain. Apart from that, a definition of a mathematical operation should fulfill the general requirements for any mathematical definitions, that is, it should be non-contradictory, non-circular and well defined (i.e., the definition of the operation should not depend on the representatives of the numbers involved in the operations). Problems with defining a mathematical operation on the real numbers arise when any possible definition of the operation cannot fulfill either one of the requirements for mathematical definitions, or one of the requirements for functions. This happens when:

(1) any possible definition of the operation contradicts other acceptable definitions or theorems of the mathematical system. For instance, any definition of 4/0 violates the definition of multiplication as the inverse of division – if 4/0 is the number c, then $c \times 0$ should be 4, but $c \times 0 = 0$, therefore 4/0 is undefined;

(2) the operation cannot be defined for all elements in the domain. In this case, it is possible to restrict the domain for which the operation is defined. For instance, the operation of division is defined for all real numbers, except for a zero divisor;

(3) the definition depends on the representatives of the numbers involved in the operation. One example is related to the attempt to extend the definition of exponentiation as $b^{m/n} = \sqrt[n]{b^m}$ when b is a negative number and m/n is a rational number. Let us refer, for instance, to $(-8)^{1/3}$ and $(-8)^{2/6}$. Extending the definition of exponentiation results in assigning different values to the different representatives of 1/3: $(-8)^{1/3} = \sqrt[3]{(-8)} = -2$; $(-8)^{2/6} = \sqrt[6]{(-8)^2} = 2$;

(4) the definition violates the univalence requirement, as there are at least two possible definitions for the operation. In these cases one of the two following possibilities is selected – either only one of these numbers is chosen as the definition of the operation, as in the case of $\sqrt{4}$, which could have been defined as either 2 or -2, but is defined as 2; or a choice is made not to define this mathematical operation, as in the case of 0/0 which could have been defined as any number. Such undefined operations are sometimes called undetermined.

Functions and undefined mathematical operations are part of the school curriculum. Function is a major topic at the secondary school level, and undefined mathematical operations are taught within the framework of specific topics.

In what follows we shall use data from our studies on teachers' conceptions of functions and undefined operations as sources of illustrations. All the items to which we refer are included in the high-school curriculum in Israel. Even though some are not taught explicitly, teachers who are expected to teach this material ought to understand them thoroughly.

Participants in the study of teachers' knowledge about functions were 162 prospective secondary mathematics teachers in the last stage of their formal preservice preparation at eight midwestern universities in the USA. Data were gathered in two phases. During the first phase, 152 prospective teachers completed an open-ended questionnaire. This questionnaire included non-standard mathematics problems addressing several interrelated aspects of function knowledge (Even, 1990), and "students" mistaken solutions or misunderstandings to be analyzed or responded to. After responding to the questionnaire, in the second phase of data collection, an additional ten prospective teachers were interviewed. The probing focussed on subjects' explanations of what they had answered on the questionnaire and why, on their reactions as teachers to students' conceptions and on questions related to the questionnaire but requiring more general, longer or more thoughtful responses related to the teaching and learning of the concept of function.

The study on teachers' conceptions of undefined mathematical operations explores 33 Israeli secondary mathematics teachers' conceptions of four undefined mathematical operations ($4/0$, $0/0$, 0^0, $(-8)^{1/3}$). The participant teachers were first asked to answer a questionnaire which included defined and undefined mathematical expressions by providing numerical solutions if possible, and if not, by explaining why not. All the subjects were then individually interviewed and were asked to describe their in-class reactions to a list of suggested definitions of 4/0, 0/0, 0^0 and to $(-8)^{1/3}$ which were presented as if they were made by students.

SUBJECT-MATTER KNOWLEDGE

During the last years research on teachers' mathematics knowledge has been going through a new blooming. This research focuses mainly on studying teachers' understanding of specific mathematical topics which are included in the school curricula (Ball, 1990; Even, 1989, 1993; Tirosh and Graeber, 1990). By and large, it was reported that many teachers do not have a solid understanding of the subject-matter they teach. In fact, serious misunderstandings were found at the level of mere knowledge of rules, procedures and concepts of almost every topic investigated (i.e., the concept of zero, division, proof, function). Thus, insufficient subject-matter knowledge, on the part of teachers, does not seem to be a sporadic, infrequent phenomenon, but rather a widespread one whose consequences for the actual teaching should be investigated.

Although "disaster studies" on teachers' mathematical content knowledge have been known for quite sometime, only recently attempts are being made to study how such problems affect teachers' reactions to students' questions and ideas related to specific mathematical topics.

Knowing that

The most basic level of subject-matter knowledge is "knowing that". This includes declarative knowledge of rules, algorithms, procedures and concepts related to specific mathematical topics in the school curriculum. "Knowing that" is certainly important as a basis for adequate pedagogical content knowledge; for questions teachers ask and activities they suggest. This claim seems trivial but "knowing that" is not always straightforwardly and easily identified, as we can learn from the following.

When asked to define a function most prospective teachers in the study correctly mentioned the univalence requirement. They also correctly classified given mathematical objects into functions and non-functions, basing their judgement on this requirement. Yet, further probing often revealed that their knowledge of this requirement was rather superficial as is illustrated in Brian's case.

At the beginning of his interview, Brian seemed to know that functions have to be univalent. He used the univalence requirement in his definition of a function (the emphases in the following quotations were added): "A thing which maps every element in a domain set onto another *unique* element in the range set." When asked to give an alternative version of this definition for a student who does not understand it, he also emphasized the univalence requirement: "For every number you put into a function you get *only one* number back out."

In addition to memorizing the univalence requirement, Brian correctly based his decisions as to whether given objects were functions by using this requirement. For example, he decided that the following:

$$g(x) = \begin{cases} x, & \text{if } x \text{ is a rational number} \\ 0, & \text{if } x \text{ is an irrational number} \end{cases}$$

is a function because: "There is an assignment of a *single* value to each number." He also correctly used the "vertical line test" to support his decision to accept a given graph as a function.

As we can see, Brian seemed to know that functions must be univalent and he understood how to use that requirement in the process of deciding whether a mathematical object was a function. However, at the same time, Brian thought that familiar graphs such as circles and ellipses are functions (even though they do not satisfy the univalence requirement and therefore are not functions). Having these two contradictory pieces of knowledge about functions simultaneously did not cause any conflict for Brian until he was asked to explain the "vertical line test" which he referred to as important to teach to students. He drew the graph of a circle. At this moment he faced a conflict: "Uh, a circle is a function, but a circle doesn't pass the line test." And the confusion continued: "I have a problem trying to make an ellipse and call it a function based on my definition."

Brian did not have any problem in using the univalence property until he was confronted with a contradiction: "A circle is a function, but a circle doesn't pass the line test." So, even though it seemed at first that Brian did know whether something is (or is not) a function, apparently he had conflicting schemes in his cognitive structure. On the one hand, he (correctly) "knew" that functions have to be univalent. On the other hand, he (wrongly) "knew" that a circle and an ellipse are functions.

This contradicting knowledge left its marks on Brian's pedagogical content knowledge. As we saw earlier, before becoming aware of the conflict, his presentations of functions to students followed the ones that he encountered as a student, and correctly included the univalence requirement. However, after becoming aware of the conflict, Brian wrongly decided that the vertical line test (which is a quick way to check univalence) "is an over-generalized tool" and said that he would use it with his students only for linear functions – an unnecessary, pedagogically unwise restriction.

In the above case most participant teachers had some adequate knowledge about the task. This did not happen when teachers in the undefined operations study were presented with $(-8)^{1/3}$. In fact, the vast majority of the teachers incorrectly argued that $(-8)^{1/3} = -2$ because $(-8)^{1/3} = \sqrt[3]{-8} = -2$.

The teachers were then asked to assume that one of their students suggested that $(-8)^{1/3} = -2$ because $(-8)^{1/3} = \sqrt[3]{-8} = -2$, while another student argued that $(-8)^{1/3} = 2$, because $(-8)^{1/3} = (-8)^{2/6} = \sqrt[6]{(-8)^2} = \sqrt[6]{64} = 2$. The reactions of those who knew that $(-8)^{1/3}$ is undefined largely differed from those whose own responses were that $(-8)^{1/3} = -2$. Teachers who correctly argued that $(-8)^{1/3}$ is undefined struggled with the suggested definitions, each of which, on the one hand, seemed reasonable, but on the other hand, contradicted their knowledge that $(-8)^{1/3}$ is undefined. They were quite sure the students' definitions were inadequate, and attempted to assess what could be wrong with them. The majority of the teachers who argued that $(-8)^{1/3} = -2$ also remained convinced their own definition was correct. A common reaction of these teachers consisted of ruling out the second student's definition by using a wrong mathematical argument as an explanation. Gal exhibited such a reaction:

Gal: I'm sure that $(-8)^{1/3} = -2$. I am sure that the second student is wrong.
Interviewer: Can you explain...?

Gal: [interrupting] This is what I am trying to find out. There is a mistake here which I have not yet found. I know both arguments cannot be true. If both were true $(-8)^{1/3}$ would have been undefined, but it is defined. The second student needs to understand that *all* undefined mathematical operations include 0 [sic]; $(-8)^{1/3}$ does not include 0, and thus $(-8)^{1/3}$ is defined and is -2.

Another common reaction consisted of partially accepting the student's definition but, at the same time, remaining convinced that $(-8)^{1/3} = -2$. The following snapshot, taken from an interview with Einat, is one example.

Einat: I'll explain to him that 1/3 and 2/6 are the same, but in this case, if you use 2/6 instead of 1/3, you get a different, incorrect answer.
Interviewer: So, do you think that 1/3 and 2/6 are the same number?
Einat: It is the same quantity, but it is not exactly the same.
Interviewer: Is it the same point on the number line?
Einat: Yes, but the appearance of 1/3 is different from that of the 2/6. This difference must have some meaning and it really has. In this case, $(-8)^{1/3} = -2$, but $(-8)^{2/6} = 2$.

Both these reactions reflected the teachers' limited understanding of undefined mathematical operations. Gal's reaction reflected her under-extended notion of undefined mathematical operations. Einat's fragile conception of equivalent fractions, according to which 1/3 and 2/6 are not always the same, allowed her to accept that $(-8)^{1/3}$ is one number while $(-8)^{2/6}$ is another, a conclusion that violates the requirement that a mathematical definition should not depend on its representatives.

Teachers, especially when they let their students explore and raise questions, may find themselves in unplanned situations such as has happened to Brian, Gal and Einat in the interviews. These cases illustrate how teachers' pedagogical decisions are based, in part, on their most basic subject-matter knowledge – "knowing that". In these cases, inadequate knowledge on the part of the teachers led them to provide the students with responses that were mathematically inadequate.

Knowing why

"Knowing that" though certainly important, is not enough. Knowledge which pertains to the underlying meaning and understanding of why things are the way they are, enables better pedagogical decisions. The following sections illustrate how "knowing why" affects teachers' decisions about the presentation of the subject-matter.

Most of the subjects who participated in the undefined operations study knew that 4 divided by 0 is undefined. However, when asked to explain "why", most could not supply any appropriate explanation. Some provided a rule-based argument – both to themselves and to students. Vered, for example, argued that "in mathematics there is such a rule that one cannot divide by 0." She also advocated this rule-based approach as adequate to students' inquiries. The following snapshot describes her reaction to a student's suggestion that 4 divided by 0 is 0.

Vered: I'll tell them that it is forbidden to divide by zero.

Interviewer: And if they ask why?

Vered: I will tell them that that's how it is. My students will never ask such a question because I will tell them and they will know, and I won't invite them to suggest their own definitions. I'll tell them that there are certain axioms in mathematics that they should memorize.

Interviewer: And if you get a very stubborn student and he still asks why?

Vered: I'll tell him that it is not allowed to divide by 0. I'll explain that mathematics and physics are different. In physics I can explain everything in terms of nature, of reality. Mathematics is not like that. In mathematics we have rules, and we operate according to them. These rules often do not seem reasonable. When studying mathematics, one has to adopt these rules and to operate accordingly. There is no reason and there is no point at all in looking for explanations. One just has to accept them.

Evidently, this short illustration reveals Vered's own limited conceptions of the reasoning behind the decision not to define 4/0. It seems that she had memorized the statement that $a/0$ is undefined, and was willing neither to attempt to question the logic behind this decision nor to challenge it.

Apparently, Vered viewed mathematics as a bag of unexplainable rules that students should accept, memorize and use. She considered the teaching of mathematics as a process in which students absorb what they are told. As a result she could not seriously consider students' responses. She expected her students to unquestioningly memorize the rules, much like she had done.

A similar view of mathematics was observed when teachers who participated in the function study were asked about the issue of univalence. Most were familiar with the univalence requirement and included it in their definition of a function: "A function is a relation such that a number in the domain can only be matched to *one* number in the range" (emphasis added). They also used the requirement as a criterion for checking whether given mathematical objects were functions.

When asked to explain the importance of univalence, the subjects gave two kinds of immediate responses which showed that they did not really know. They either simply stated they did not know or claimed that the role of univalence was to distinguish between functions and non-functions. Even after prompting, most did not know why it should be important to distinguish between functions and non-functions. Like in Vered's case, some described the origin of the requirement as arbitrary:

It seems like that whoever decided to call that a function just made it one of the requirements: if it looks like a graph, like this, and has only one, and I'm going to call that a function.

Not knowing why univalence is needed and accepting it as an arbitrary requirement influenced the prospective teachers' pedagogical content-specific decisions. Obviously, they could not explain to a student why univalence was needed. More than that, when asked to explain what a function is to a student with difficulties, many did not do so. Instead, they tended to provide the student with the "vertical line test" as a rule for getting the right answers without needing to understand: "By graphing the function and doing the vertical line test, a line never crosses the graph more than once." These prospective teachers chose to present students with easy rules that overemphasize procedural knowledge at the expense of and without concern for meaning. One of the subjects claimed:

If they're told to figure out whether it's a function or not, using the definition, they probably wouldn't be able to do it. If they know the vertical line test works, even if they don't know why it works, they can see right away why this is a function, because they can go through with a ruler or a straight edge and vertically go across the function, looking for places where there are two points.

Another task that asked teachers to explain "why" referred to quadratic functions. In this case, in contrast with the previous ones, some teachers attempted to go beyond simply providing the students with rules to follow.

The general form of a quadratic function is $f(x) = ax^2 + bx + c$. "a" in the expression is positive (negative) if and only if the graph opens upward (downward). When presented with the following question (see Figure 1) almost all the prospective teachers correctly stated that when the graph of a quadratic function looks like ∩, "a" (in the equation) should be negative. A vast majority of the subjects stated a rule as an explanation: "$a < 0$ since the graph opens downward." When asked to explain *why* the rule works, most did not explain the "why" but rather admitted that they just memorized it.

Again, not knowing why the rule works influenced the prospective teachers' pedagogical content-specific choices. The subjects were presented with a situation where a student asked them *why* "a" in the quadratic equation had to be negative if the parabola looked like ∩. Those who treated the relationship between the graph and "a" as a rule to memorize, suggested a nice exploration of quadratic functions with positive and negative "a"s so that the student could find the pattern. The only problem with

This is the graph of the function $f(x) = ax^2 + bx + c$.
State whether a, b and c are positive, negative or zero.
Explain your decision.

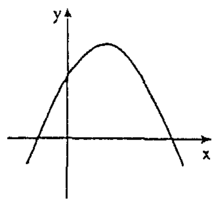


Fig. 1. The relationship between the coefficients and the graph of a quadratic function

this approach was that it did not help the student understand why the rule works:

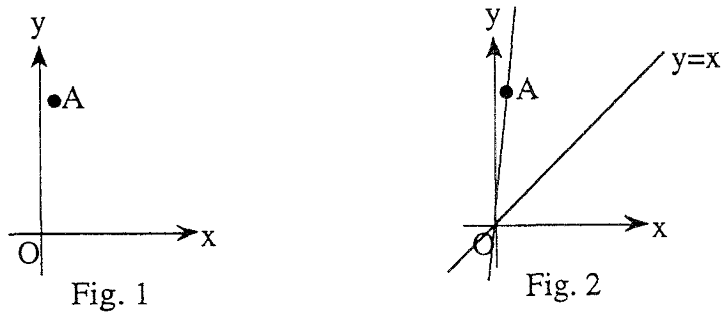
I think that the best way to teach this is when you're having students graph them. Give them a whole set of these, interchanging negatives and positives... . And then have them see if they can find the pattern. ...Have them make several different graphs. And then cut them out and try to put them in groups. ...Some would, probably, group all the ones that are down the same and all the ones that are up... . And by that way, kids can say, "What was common in this to make it a group?" And then, as a class, I think that they could come up with it. And then they would remember it. By reading it in a book I don't think they will.

By asking students to try several examples and find the pattern, these people ignored the fact that the student has already found a rule, and they did not relate to the essence of the student's question: Why does this rule hold?

In contrast with the above approach the following excerpt illustrates an approach based on better subject-matter understanding – understanding why the rule holds. Both the first described approach and the following one start with the sketching of graphs. However, instead of suggesting the sketching of several graphs in order to find a pattern, the following approach suggests to sketch the basic graph of $y = x^2$ and then to follow the change in the graph when the y-values are multiplied by a negative number:

Start from graphing several parabolas, using basic $y = x^2$ so $f(x) = x^2$. And showing how that changes (going back to the translations) and having them realize they're essentially multiplying the y values by a negative so they are rotating the whole thing about the x-axis. And that what forces it to point down, the opening down.

A student was asked to find the equation of a line that goes through A and the origin O (see Figure 1). She said: “Well, I can use the line $y = x$ as a reference line. The slope of line



AO should be about twice the slope of the line $y = x$, which is 1 (see Figure 2). So the slope of line AO is about 2, and the equation is about $y = 2x$, let’s say $y = 1.9x$.”
What do you think the student had in mind? Is she right? Explain.

Fig. 2. The slope of a line

KNOWLEDGE ABOUT STUDENTS

A teacher’s decision about whether a certain student’s response is correct is based on that teacher’s content knowledge. But this, by itself, is not enough for developing a reaction that can help the student construct his/her knowledge. Such reaction should take account of common students’ conceptions and ways of thinking related to specific mathematical topics (“knowing that”). S/he should be able to understand the reasoning behind students’ conceptions and anticipate sources for common mistakes (“knowing why”).

Understanding students’ ideas and the reasoning behind them often constitutes a real challenge, as we can learn from the following. Teachers were presented with a situation (see Figure 2) in which they had to respond to a student’s explanation which resembled a mistake that students often make when they learn about linear functions: That the slope of a linear function varies in straight proportion with the angle between the line and the x-axis (e.g., twice the angle means twice the slope).

A large group of the subjects who did not understand the reasoning the student used described what the student did as estimation:

“She had the right idea but she was off in her gross approximation of $y = 2x$.”

“She was thinking that the slope needed to be steeper which is good. She thought using a decimal would make the graph tighter.”

"No, the slope of AO is not necessarily twice that of $y = x$. She must be careful in her estimation."

While estimation can serve as a description of what the student did, these teachers completely ignored the (wrong) connection between angles and slopes which the student had made. Telling the student that she must be careful in making estimations would not help the student realize that her assumption about the linear relationship between a slope and an angle was wrong.

The above kind of explanations of the student's way of thinking is very different in its nature from the type of explanation that about half of the participating teachers gave. These teachers did identify the student's mistaken way of thinking. This is illustrated in the following excerpt:

The girl thinks that because the $y = x$ line, which is a 45 degree angle (whether she knows that's a 45 degree angle or not) and the y axis is twice that, 90 degrees. So she thinks that this has slope 1 [points to the graph of $y = x$], so this one has slope of 2 [points to the y-axis]. I can see where she made a mistake, so she says this [the slope of line AO] is just a little bit less than this [the "slope" of the y-axis], so that's going to be 1.9. That's what she did.

The above was said by a teacher who knew that the slope of a linear function does not vary directly with the angle between the line and the x-axis. On the basis of this knowledge the teacher not only realized that the student was wrong, but also pinpointed the source for the mistaken way of thinking. However, as in the case of Brian's knowledge of univalence, "knowing that" the slope of a linear function does not vary directly with the angle between the line and the x-axis, is not a matter of "knowing" or "not knowing". It is more a matter of "situated knowledge", as became clear when the teachers were asked to estimate the slope of the line AO.

Assuming that one unit on the x-axis equals one unit on the y-axis, one can estimate the slope of the given line AO by comparing Δy's with the corresponding Δx's. The slope approximated this way is 9. Interestingly, even teachers who realized that the student was wrongly using angles to estimate a slope, based their estimation on a similar hidden assumption. This is illustrated in the following excerpt:

It's going to be just a big number over a little number. That's all I can say about it. The slope is going to be a real big number. I'd say 100. I don't know. It's just going to be a really big number.

These people knew that slope is "rise over run". But they did not use this knowledge. They looked at the graph and assumed that the result of dividing vertical change by horizontal change would be "very big". More than that, those who tried to find the ratio of vertical to horizontal change, counted and found a slope of 10–15. This number was counter-intuitive to their feeling that a line "close" to vertical (a large angle) should have a huge number for slope:

Um, the slope would probably be (pause), may be about $y = 15x$. I might be wrong. I am probably wrong. Because I know the vertical line has an infinite slope. It's a pretty steep slope.

These teachers recognized the wrong use of angles for estimating slopes when dealing with the familiar case of slope 2, but used exactly that mistaken way of thinking when the situation involved non-familiar linear graphs. It is probable that these teachers would not be able to identify certain adequate responses as such.

In the cases described above, the teachers were presented with a common students' response and explicitly asked to explain its possible sources. A student's response, by itself, usually does not provide enough information for detecting his/her way of thinking. In many cases it is necessary for the teacher to pose assumptions about the student's ways of thinking, to test them, and to construct a reaction accordingly.

In the case of 4/0, teachers were provided with two common students' responses. One suggested that "$4 : 0 = 4$. When you divide by zero, you cannot actually perform the division, and thus you're left with the entire quantity." The other said that "$4 : 0 = 0$, dividing by 0 is impossible and thus the answer is 0." The teachers were asked to describe their in-class reactions to each of these students. They were not explicitly asked to provide explanations for the sources of the students' mistakes. Obviously, these mistakes could evolve from conceptualizing zero as nothing, from viewing division only as sharing, from both these conceptions, or from others yet to be explored. Therefore, the first response to these students should have consisted of further probing in an attempt to better understand their line of reasoning, and then to react accordingly.

The analysis of the teachers' reactions to students' definitions however revealed that most of them did not attempt to make this initial inquiry to better understand their reasoning. In fact, the vast majority of the teachers judged the students' answers only in terms of being right or wrong, and provided them with their own explanations for the right answer. Avi was one of these teachers.

Avi: The second student says that it is impossible to perform division by zero. I'll explain to him why division by zero is undefined. I'll start with $8 : 2 = 4$, and I'll ask them why this is so. The answer should be that $4 \times 2 = 8$, and I think they know it. Then I'll explain that in order to see what $4 : 0$ could be, we shall divide 4 by various numbers that get closer and closer to 0. We shall then calculate $4 : 2$, $4 : 1$, $4 : 1/2$, $4 : 1/4$, and we shall see that when the divisor gets smaller the quotient gets bigger. We shall then do the same with $6 : 2$, $6 : 1$, $6 : 1/2$ and so on. Later we shall try with

$4:(-2), 4:(-1), 4:(-1/2)$ and they will see that in this case, when the divisor gets bigger the quotient gets bigger. They will then get the feeling that $4:0$ cannot be defined.

Interviewer: Does this also hold for the first student?

Avi: Yes. I can use this explanation for him too.

Avi constructed his reaction upon an intuitive sense of limit. This reaction was based on the problematic notion of division by positive and negative fractions. When such an approach is used, there is a need to carefully assess students' understanding of these notions. In fact, the appropriateness of using this approach is partially determined by the students' understanding of these notions. Yet, it seems that Avi was unaware of the need to assess the students' conceptions of these divisions. Further, his immediate conclusion that this same reaction could work for both students may indicate that he made only a superficial attempt to understand the reasoning behind the students' suggested definitions.

Very few teachers tried to carefully examine the students' ways of thinking. A rather unique reaction is that of Maya who made careful assumptions about the students' reasoning, tried to find procedures for validating her assumptions, and to define means by which she could discuss these cases not only with one specific student, but with the entire class. She tried to identify both strengths as well as weaknesses that should be addressed in the discussion.

Maya: I'll first refer to the second student. It seems to me [pause]. It is possible that for him zero exemplifies nothing, and then the divisor is nothing, and then there is no division, which is, for him, nothing, which is zero. First, I need to better understand his thinking. What meaning does he assign to 'if we are not dividing?' If his understanding is in line with what I developed here, then it is important to free the operation from the specific content of division [pause] I mean, for instance, we shall have to check and see how he operates with negative numbers. We can start with division by fraction, for instance, $4:1/2$, and try to build on the notion of division as the inverse of multiplication. [She goes on...]

Interviewer: Does this also hold for the first student?

Maya: [Thinking] With the first one [reading his suggestion] ... this is even more demanding. He says that it is 0, and it is not clear to me what he thinks about zero. It takes a lot to understand that $a * 0 = 0$ for a non zero a. It means that the class shall have to deal with the idea that division by zero means not doing division... .

CONCLUSION

Pedagogical content knowledge includes several interrelated aspects. In this paper we dealt with teacher presentations of the subject-matter, especially when faced with students' questions, ideas or hypotheses. Clearly, teacher responses to students may have different aims, such as encouraging cooperative work among students, making students feel good, etc. We analyzed teachers' responses in light of the potential development of meaningful learning. Based on research in the field, we focused on two important sources of this aspect of pedagogical content knowledge: subject-matter knowledge and knowledge about students.

In respect to the first source, we discriminate between "knowing that" and "knowing why". Within the mathematical education literature these two kinds of knowledge are frequently discussed and there is a general agreement that understanding of the subject-matter requires both (e.g., Hiebert, 1986; Nesher, 1986; Skemp, 1976), and that teachers should therefore have both kinds of knowledge (e.g., Ball, 1990; Leinhardt, 1988; Skemp, 1976). While in the literature "knowing that" and "knowing why" are rather sharply defined; when it comes to specific subject-matters and specific contexts, their respective scopes become vague. As we saw in this paper, it is not always clear what it means to "know that" about a function; or to "know why" about undefined mathematical operations. Our studies indicate that sometimes participant teachers did not "know that": they did not know the definitions or incorrectly solved problems presented to them. However, in many cases it was impossible to precisely determine if a certain teacher "knew that". For example, some could correctly quote definitions and at first seemed to know how to use them, but when faced with "problematic" cases (which, of course, are subjective) the teachers became unsure about their own original definitions and occasionally even changed them. When it comes to "knowing why", things are not less complicated. Some subjects knew why a specific case was set in a certain way, but could neither explain what lay behind the general structure nor correctly solve problems related to "extreme" cases. Therefore, even though we provided illustrations of the influence of teachers' subject-matter knowledge on their pedagogical content specific choices, the issue of the nature of teacher subject-matter knowledge needs further investigation.

We have suggested that the terms "knowing that" and "knowing why" are also useful when dealing with teachers' knowledge about students. "Knowing that" in this context refers to research-based and experienced-based knowledge about students' common conceptions and ways of thinking in the subject-matter. "Knowing why" refers to general knowledge about possible sources of these conceptions, and also to the understanding of the sources of a specific student's reaction in a specific case. In this paper we mainly refer to the first, i.e., teachers' understanding of possible sources of a certain student's response. Our data suggest that many of the teachers made no attempt at understanding the sources of students' responses. When asked directly, they found it difficult to explain why students reacted the way they did. Sensitivity to students' thinking becomes even more difficult under the pressure of real teaching instead of an interview setting. Therefore, we suggest that teachers' awareness of sources of students' responses be developed. This can be based on existing research literature.

A main conclusion that can be drawn from this study is that teacher education should explicitly refer to topics included in the high-school curriculum, such as functions and undefined mathematical operations. These topics have already been studied by the teachers during their own high-school years. However, as we saw in this paper, one cannot assume that teachers' subject-matter knowledge with respect to the two aspects ("knowing that" and "knowing why") are sufficiently comprehensive and articulated for teaching.

Clearly, teachers do not study explicitly about students' conceptions and ways of thinking in mathematics during their own studies in high-school. Therefore, teacher education should emphasize the two aspects of knowledge about students mentioned in this paper.

REFERENCES

Ball, D.L.: 1988, *Knowledge and reasoning in mathematical pedagogy: Examining what prospective teachers bring with them to teacher education*, unpublished doctoral dissertation, Michigan State University, East Lansing, MI.

Ball, D.L.: 1990, 'Examining the subject-matter knowledge of prospective mathematics teachers', *Journal for Research in Mathematics Education* **21**(2), 132–143.

Ball, D.L.: 1991, 'Research on teaching mathematics: Making subject matter knowledge part of the equation', in J. Brophy (ed.), *Advances in research on teaching*, Vol. 2, JAI Press Inc., Greenwich, CT, pp. 1–48.

Begle, E.G.: 1979, *Critical Variables in Mathematics Education*, Mathematics Association of America and the National Council of Teachers of Mathematics, WA.

Brophy, J. and Good, T.: 1986, 'Teacher behavior and student achievement', in M.C. Wittrock (ed.), *Handbook of Research on Teaching* (3rd ed.), Macmillan, NY, pp. 328–375.

Even, R.: 1989, 'Prospective secondary teachers' knowledge and understanding about mathematical functions (Doctoral dissertation, Michigan State University, 1989)', *Dissertation Abstracts International* **50**, 642A.

Even, R.: 1990, 'Subject matter knowledge for teaching and the case of functions', *Educational Studies in Mathematics* **21**, 521–544.

Even, R.: 1993, 'Subject-matter knowledge and pedagogical content knowledge: prospective secondary teachers and the function concept', *Journal for Research in Mathematics Education* **24**(2), 94–116.

Even. R. and Markovits, Z.: 1993, 'Teachers' pedagogical content knowledge of functions: characterization and applications', *Journal of Structural Learning* **12**(1), 35–51.

Freudenthal, H.: 1983, *Didactical Phenomenology of Mathematical Structures*, Dordrecht: D. Reidel Publishing Company.

Gage, N.: 1978, *The Scientific Basis of the Art of Teaching*, Teachers College Press, Columbia University, NY.

Hershkowitz, R., Bruckheimer, M. and Vinner, S.: 1987, 'Activities for teachers based on cognitive research', in M.M. Lindquist and A.P. Shulte (eds.), *Learning and Teaching Geometry, K-12, 1987 Yearbook*, National Council of Teachers of Mathematics, Reston, VA, pp. 222–235.

Hiebert, J. (Ed.): 1986, *Conceptual and Procedural Knowledge: The Case of Mathematics*, Lawrence Erlbaum Associates, Inc., NJ.

Kieran, C.: 1992, 'The learning and teaching of school algebra', in D.A. Grouws (ed.), *Handbook of Research in the Teaching and Learning of Mathematics*, Macmillan, NY, pp. 390–419.

Leinhardt, G.: 1988, 'Expertise in instructional lessons: An example from fractions', in D.A. Grouws, T.J. Cooney, and D. Jones (eds.), *Effective Mathematics Teaching*, National Council of Teachers of Mathematics, Reston, VA, pp. 47–66.

Leinhardt, G., Putnam, R.T., Stein, M.K., and Baxter, J.: 1991, 'Where subject knowledge matters', in J.E. Brophy (ed.), *Advances in Research on Teaching*, Vol. 2, JAI Press, Greenwich, CT, pp. 87–113.

Leinhardt, G. and Smith, D.A.: 1985, 'Expertise in mathematics instruction: Subject matter knowledge', *Journal of Educational Psychology* **77**, 247–271.

Maher, C.A. and Davis, R.B.: 1990, 'Teachers' learning: Building representations of children's meanings', in R.B. Davis, C.A. Maher, and N. Noddings (eds.), *Constructivist Views on the Teaching and Learning of Mathematics* (Journal for Research in Mathematics Education: Monograph Number 4, pp. 7–18), National Council of Teachers of Mathematics, Reston, VA.

Nesher, P.: 1986, 'Are mathematical understanding and algorithmic performance related?, *For the Learning of Mathematics* **6**(3), 2–9.

Peterson, P.L., Fennema, E., and Carpenter, T.P.: 1991, 'Teachers' knowledge of students' mathematics problem solving knowledge', in J.E. Brophy (ed.), *Advances in Research on Teaching: Vol. 2. Teachers' Subject Matter Knowledge*, JAI Press, Greenwich, CT, pp. 87–113.

Schoenfeld, A., Smith, J., and Arcavi, A.: 1993, 'Learning – The microgenetic analysis of one student's evolving understanding of a complex subject-matter domain', in R. Glaser (ed.), *Advances in Instructional Psychology* (Vol. 4), Erlbaum, Hillsdale, NJ, pp. 55–175.

Shulman, L.S.: 1986, 'Those who understand: Knowledge growth in teaching', *Educational Researcher* **15**(2), 4–14.

Skemp, R.R.: 1976, 'Relational understanding and instrumental understanding', *Mathematics Teaching* **77**.

Strauss, S. and Shilony, T.: (in press), 'Teachers' models of children's minds and learning', in I. Hirschfeld and S.A. Gelman (eds.), *Mapping in Mind: Domain Specificity in Cognition and Culture*, Cambridge University Press, Cambridge.

Tamir, P.: 1987, *Subject Matter and Related Pedagogical Knowledge in Teacher Education*. Paper presented at the annual meeting of the American Association for Educational Research, Washington, DC.

Tirosh, D.: 1993, 'Teachers' understanding of undefined mathematical expressions', *Substratum: Temas Fundamentales en Psicologia Education* **1**, 61–86.

Tirosh, D. and Graeber, A.: 1990, 'Inconsistencies in preservice teachers' beliefs about multiplication and division', *Focus on Learning Problems in Mathematics* **12**, 65–74.

Wilson, S.M., Shulman, L.S., and Richert, A.: 1987, "'150 ways of knowing": Representations of knowledge in teaching', in J. Calderhead (ed.), *Exploring Teacher Thinking*, Holt, Rinehart, and Winston, Sussex, pp. 104–124.

RUHAMA EVEN
Department of Science Teaching,
Weizmann Institute of Science,
Israel – 76100 Rehovot.

DINA TIROSH
School of Education,
Tel Aviv University,
Israel - 69978 Tel Aviv.