JUST A MATTER OF DEFINITION

Raffaella Borasi, *Learning Mathematics Through Inquiry*, 1991, (pbk) $23.50 (US)
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'Girl number twenty unable to define a horse!' said Mr Gradgrind, for the general behoof of all the little pitchers. 'Girl number twenty possessed of no facts, in reference to one of the commonest of animals.'
(Charles Dickens, *Hard Times*)

There is a widespread confusion with regard to definitions in mathematics, and some seem to act according to the rule:

If you can't prove a proposition,
Then treat it as a definition. [...]

I think it is about time that we came to an understanding about what a definition is supposed to be and do, and accordingly about the principles to be followed in defining a term. [...] it seems to me that complete anarchy and subjective caprice now prevail.
(from a letter by Gottlob Frege to David Hilbert, 27th December, 1899)

Never take a description of the origin of an idea for a definition.
(Gottlob Frege, *The Foundations of Arithmetic*)

Something is an axiom, *not* because we accept it as extremely probable, nay certain, but because we assign it a particular function, and one that conflicts with that of an empirical proposition. ... An axiom, I should like to say, is a different part of speech.
(Ludwig Wittgenstein, *Remarks on the Foundation of Mathematics*)

Much of this essay review is about words: about words themselves, about meanings, contexts, uses, and connotations evoked by the use of particular words chosen for particular ends. Novelist Fay Weldon (1991, p. 17) writes of the importance of words:

> You will just have to take my word for it, that the words a writer uses, even now, go back and back into a written history. Words are not simple things: they take unto themselves, as they have through time, power and meaning: they did so then, they do so now.

From any culture, embedded and embodied in a language, there is a mathematical history and tradition. Consequently, I shall illustrate some of the points and questions raised for me by reading Raffaella Borasi's book *Learning Mathematics Through Inquiry* with historical examples. Although hers is a book about the potentialities (and one particular actuality) of mathematics teaching in schools, some of the questions it raises can be illuminated by previous traditions and past changes of mathematical focus. These traditions weigh heavily at times – and teachers are, in part, representatives of tradition in the classroom. One key question for me with regard to teaching and learning mathematics is how to encounter such traditions without being obliterated by them.

A central term that will be under scrutiny is that of *definition* itself. The term *definition* is one of a handful of meta-mathematical marker terms (others

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include *axiom, theorem, proof, lemma, proposition, corollary*), terms which serve to indicate the purported status and function of various elements of written mathematics. How are we to come to learn about the functions these terms label, and the discriminatory power that can result from their use?

As ever, it can be both interesting and important to look at the use of any such term in more everyday settings, in order to appreciate its range and colour when used in mathematical settings. Definitions, by definition, place limits around what is defined. They are often seen as stripped down, curt, exclusive rather than inclusive. To define is to omit, to exclude, to focus attention on the part at the expense of the whole.

Definitions can have a sense of finality, of hard boundaries to what something may or may not mean. And both senses of the modal verb 'may' are present: the sense of tentative possibility and the sense of being allowed, of being granted permission. And to demand that someone 'define their terms' is a hostile act, a declaration of argumentative intent.

One common context arises in connection with dictionaries: 'Do you want a dictionary definition?'. What we get in dictionaries is usually a gloss of the various senses of a word and sometimes some historical exemplars of its original meanings and uses.[1] Yet we also ask about the *meanings* of words and look for them in a dictionary too. And a mathematical dictionary frequently does offer us *mathematical* definitions. So is a mathematical definition anything other than a definition of a mathematical term?

A second contributory source is provided by the context of photography. Stroebel and Todd (1974, p. 48), in their dictionary of photographic terms, offer "the clarity of detail of an image as perceived by the viewer" as a gloss for (photographic) *definition*; and later, for *field*, propose "the entire subject area imaged within the circle of good definition of a lens". A camera registers more detail than the viewer can, but it is specific and fixed – the wholistic and variable aspects of human sight are gone. Increased *definition* involves bringing an object or image into greater (or sharper) focus. But increased focus can also sometimes exclude – by narrowing the field of view.

In this sense, focusing a camera both stresses and ignores, to use Caleb Gattegno's telling phrase. And in the context of problem solving, Gattegno (1981, p. 42) has commented:

> Problem solving is a little up in the air and requires a little more definition. And when I say 'definition' I use the word in its optical sense: where there is better definition, where you can see more, can see more clearly, not that you have the words for it. That's an important aspect of how the mind works in terms of definitions. I want a better definition of the problem, not a better phraseology for it.

What is Borasi's Book About?

Telegraphically, the first two, short chapters set the scene for how she (a University lecturer in mathematics education) came to be teaching mathematics for a semester to two teenage (grade 11) girls Katya (K) and Mary (M), in an alternate school

School Without Walls in Rochester, New York State, and why she opted for a unit on mathematical definitions. The next six chapters chronologically detail the ten classes that were held, enriched by extensive, transcripted excerpts of their conversations and reproduced examples of student written work, together with Borasi's reflections on both her mathematical intents and educational purposes.

Chapters 9 and 10 offer reflections on the experience, both from her own and the students' perspectives, as well as her evaluation of the teaching experience, including a discussion of what Borasi herself learnt about definitions. The final two chapters (a quarter of the book) offer more general, philosophical thoughts on 'humanistic inquiry' as an approach to the enterprise of teaching mathematics in school. Lastly, there is a short appendix on methodological considerations and, surprisingly, no index. In this review, I shall stay with Borasi's broader concerns for the first part, before focusing in quite sharply and specifically on the topic of mathematical definitions.

The book is well produced, with virtually no typographical errors (although there are, for instance, a couple of places where set-theoretical symbols have been omitted on pp. 94 and 97), and the extensive student work was reproduced well. The classroom transcriptions are interesting and extensive. The choice of bold, italic and ordinary type for the words of Borasi, Mary and Katya respectively visually gives a (probably unintended) uneven feel and, in particular, a heavy weighting to the teacher's voice. The widespread use of recorded speech gives us access to some aspects of the personalities of the participants. The difference in level of the two students' written work provides a sense of the teaching situation Borasi was contending with.

Throughout the book we have a description of Borasi's practice, followed by after-the-event reflections on it. She is in places disarmingly honest about her own learning and things she would have done differently. On a number of occasions we hear her kicking herself mentally. There are also many insightful asides and discussions about the possibilities she sees. For instance, when reflecting on her not accepting an opportunity, she suggests:

> Perhaps it is really only when the teacher herself has no prior knowledge of the problem under study that she can genuinely *participate with* the students in a problem-solving activity. (p. 37)

My response here is were that to be the case, namely genuine participation with students, then the teacher would have ceased to act as a teacher. There would be no meta-processing or meta-commenting and the entire group focus could be with the problem alone. Learning might still take place, but no teaching would. (It is also unclear from the discussion whether she actually saw this possibility *at the time* and deliberately steered away from it.) This contrasts interestingly with, for example, the account on p. 97 where she definitely is in the role of teacher, pushing and challenging and offering putative counter-examples.

Borasi makes many interpretations of what her students were gaining from their work together. Because of the detailed transcripts, we are placed in a better position than with most published work for checking out for ourselves many of her claims. Nonetheless, I quite often found myself thinking: 'that is too strong a claim', or 'this reflects your agenda not that of the students'. For instance, the opening sentence of Chapter 6, on the role of context, reads:

> As a result of their work with the extension of exponentiation beyond the whole numbers, Katya and Mary had encountered the unexpected fact that mathematical concepts and definitions, far from being absolute, actually depend on the context in which they are interpreted. (p. 75)

Yet Borasi herself comments on the previous page: "I was a bit surprised by the student's easy acceptance of the fact that the original meaning of exponentiation as 'repeated multiplication', employed in its initial definition, had to be relinquished in order to extend this operation". I feel that the earlier quotation reflects the significance for Borasi of what they were doing, but I see no evidence to justify the claim that this was the students' sense too. Another instance for me comes on page 90. "In addition, through their thinking and their actions they revealed their appreciation of the criteria established by the mathematics community for mathematical definitions." I shall make more comments about this later.

Nonetheless, the epistemological sophistication of some of her students' remarks is striking, and provides one of the most telling parts of the book. For instance, in response to the written question:

> *Are you satisfied with this definition [of polygon]? How could we ever know if it is correct? Explain your answer.*

Katya wrote:

> I am satisfied with this for the time being but I think possibly as I start to use them more and more, I may desire something more exact. (p. 55)

Borasi on a number of occasions refers to such expressions as 'tentative' (e.g. p. 74, p. 143) which for me can have overtones of hesitancy. What I see here is an expression of a possibly temporary, but stable, resting place, somewhere I can put my weight with some confidence. But *stable* does not necessarily mean permanent. I recall being at a conference with Borasi and her making a comment that has stayed with me. "Why", she asked, "do we [as lecturers] need to fool our [university mathematics] students into thinking they can be safe?" Which brings us up against the notions of trust and safety, both of which crop up time and again in their class discussions.

> R: So, instead here we have three [pointing to the result $3 \times 180°$]. So do you trust my theorem more, or do you trust this?
>
> M: [without hesitation] Trust ourselves. (p. 52)

And later on (in a discussion about possible assessment questions):

> M: My idea was [...] that we would be given something we would not be familiar with, [...] which we had to find a definition for, [...] a good definition that we were ... safe with.

And again, talking algebra:

> M: ... the equations ... I'm not really steady with that yet and I get sort of scared.

The overlapping terminology of the moral and the mathematical I find fascinating. *Correct* (or *proper* or *good*) definitions, the *right* way to proceed or behave,[2] can also be heard to speak of bourgeois concerns of moral respectability.[3] It is also possible to view many discussions about proof, certainty and rigour in a similar light.

What is the Mathematical Content of the Course?

The mathematical problems and tasks chosen, such as defining the familiar notion of a circle, creating a definition for a polygon, extending exponentiation beyond the whole numbers, revision of the definition/notion of 'circle' when using the 'taxicab' metric, or defining the notion of variable, allow quite specific philosophical themes to be discussed, as well as mathematics to be done. Borasi comments:

> I believe that mathematical definitions become abstract and meaningless unless we analyze them 'in action', as an integral part of regular mathematical activity. Furthermore, I think this episode in particular provides a good illustration of how reflection and inquiry about mathematical definitions could easily become a theme cutting across the whole mathematics curriculum rather than just a topic in an isolated unit. (p. 41)

One instance of her receptiveness to novelty comes where Raffaella Borasi looks at the possibilities of disjunctive definitions, after Katya proposes "an isosceles triangle is a triangle with two equal sides or two equal angles" (p. 33). Borasi makes some remarks about definitional form and its relation to function and lists this as one of the items that she learnt about definitions as a result of teaching this course.

> It manages to capture the two most fundamental and useful properties of isosceles triangles. At the same time, it does not present the problem most common to redundant definitions: that is, requiring more than what is strictly necessary to identify the concept and consequently imposing unnecessary constraints whenever we need to verify whether a given object is an instance of the concept or not. (p. 33)

On the contrary, such definitions allow us *more* opportunities for verification as in certain circumstances one or other of these properties may be easier to see or prove. It seems an instance where the passion for minimalism in mathematical definitions may on occasions prove a false economy[4] – and may also *force* us to choose which of these two properties is more fundamental – or more useful – in itself. It also raises questions about how to contend with equivalent definitions, a topic I return to later.

Definition versus Property

One question which Borasi's book raises loudly and clearly is the problematic distinction between 'property' and 'defining property'.[5] In an earlier article, Borasi (1987, p. 40) quotes a teacher with whom she had been working on the topic of definition:

> I am still thinking about definitions. It seems that in our class discussions about definition, we often used the word 'definition' when we really meant 'description'. It is not necessarily true that a good mathematical definition must give a good image or understanding of a concept. It is most often the case that the understanding of the concept comes before the technical definition can be understood. ... Possibly, we would have much more success at stating the attributes of a good definition if we would differentiate carefully between what we mean by description and what we mean by definition.

Borasi asked the same question explicitly of the two students (p. 40).

Can you try to distinguish between a definition and a property?

K: I see them as mostly the same thing. Definitions can be a group of properties or one exact property, but a circle has properties that are also common to other geometric figures, so one property alone would not usually be an accurate definition, but most definitions are made up of properties. I guess I don't understand the difference.

In this same chapter, Borasi comments that 'in most cases, we cannot expect a definition to provide all the necessary instructions for producing an instance of the concept in question' (p. 41).

Historically, there is a nice distinction (see Molland, 1976) between definitions of curves *by genesis* and definitions *by property*. Definitions by genesis involve telling you what you have to do to produce the curve, whereas definitions by property involve specifying a property that the curve has. Thus, a parabola may be defined as a particular way of sectioning a cone (what you have to do to get one), whereas a circle may be defined by means of the property that all straight lines from the centre to the curve have the same length.[6] This reflects the original use of curves, not as objects of study themselves initially, but as construction devices to help solve other problems. (There is a similar distinction arising in computing between *procedural* and *structural* definitions: see e.g. Leron, 1988.)

Again we have a tension between the more intuitive instructions for generation and the more abstract provision of a property. But the properties may well be more useful for proving results. Fauvel writes (1987, p. 21)

> It was thus by the end of the fourth century, at the latest, that the conic sections were defined (by genesis) as sections of cones [...] Of course, such a definition is of little use in itself unless further mathematical properties can be deduced. In fact from these definitions it is not a difficult application of plane geometry to find their *symptom*: the condition which points of the curve satisfy. The symptom of the parabola is $(\text{ordinate})^2 = (\text{abscissa})\cdot(\text{constant parameter})$. [...] By the time we meet conic sections in the work of Archimedes, they generally make their appearance directly through the symptom. That is, Archimedes worked from this condition, taking it to be something evidently known and not needing to be derived afresh, in such a way that the 'definition by genesis' involving cones had now in effect become a 'definition by property'. It is now the symptom of the curve that serves as its defining property.[7]

But even for curves such generating descriptions or properties are seldom unique. (For instance, a circle can be defined by genesis as the locus of all points obtained by rotating a right-angled triangle about the hypotenuse.) So such singling out for our own purposes can be the only distinction (recall the Wittgenstein comment about axioms being a different part of speech), not one of kind between definitions and properties. And if we cannot for whatever reason 'lower the ladder', let our students in on those purposes, then *la mystification mathématique* is with us to stay. It seems to me plausible that one's purposes may only perhaps be appreciated (if then) at the *end* of a teaching period – so how are we to keep our students with us?

Humanistic Inquiry

In some sense, Borasi's work here is an exploration of the possibility for what she calls 'humanistic inquiry' and she casts her work within the general framework of school mathematics reform in the United States. This idea is characterised under four headings:

- a view of *mathematics* as a *humanistic* discipline;
- a view of *knowledge* more generally not as a stable body of established results but as a dynamic process of inquiry:
- a view of *learning* as a generative process of meaning-making that is personally constructed;
- a view of *teaching* as providing necessary support to students' own search for understanding.

Borasi offers the mini-course on definitions as an exemplification of this notion. There is a fascinating discussion on pp. 90-2 concerning the relation between properties and a definition when looking at a new context, but it also raises for her pedagogical uncertainties when faced with a student resolution which is epistemologically quite different from her own.

> [This episode] made me see how difficult it is in practice to be true to a style of teaching that lets students pursue genuine inquiry, in which they have a say in decisions about the directions worth following and the criteria to be used in evaluating results. The lack of control over 'what's going to happen next' that the teacher might experience in these circumstances is not easy to live with and radically undermines traditional expectations about teacher and student roles in a mathematics class.

This section in particular (and her book in general) puts the lie to glib comments about how easy it is to be open and student-orientated in a mathematics classroom. Borasi concludes (p. 211) by commenting:

> I was often surprised in the mini-course at my own resistance to deviating from the lesson plan or engaging with the students in unexpected explorations whose potential value I could not immediately evaluate. Indeed, only when we become involved in instructional innovation do we appreciate what an alternative pedagogical model really means in practice and realize the nature and strength of our own pedagogical beliefs.

Borasi has certainly been able to immerse the students in a mathematical culture. She also writes:

> I have used the term humanistic to try to convey the complexity of this view of mathematics – that is, mathematics as a fallible, socially constructed, contextualized, and culture-dependent discipline driven by the human desire to reduce uncertainty but without the expectation of ever totally eliminating it. (p. 163)

I dislike the common sliding from non-absolute to 'as fallible as anything else' in recent writing about the nature of mathematics. Borasi claims:

> Once we realize that mathematical results are neither predetermined nor absolute, we also have to accept the fact that mathematics is as fallible as any other product of human activity. (p. 162)

Really? Why? Differences in degree are still differences, surely. Why does the fact that purposes and intentions change imply that mathematics is *fallible*, rather than merely mutable? And I think it important to remember that as well as students who recoil from the apparent indifference of mathematics, there are also students for whom mathematics provides a refuge, a relatively safe haven from the sometimes intense uncertainties of adolescence.

There is also something incongruous in her choice of definitions, surely one of the most technical, internal aspects of pure mathematics, as an exemplification of humanistic inquiry which by Chapter 11 apparently has much more to do with applications of mathematics in society and general concerns of mathematics education. I found the last two chapters rather diffuse, having lost much of the focus that had been so exciting in the earlier ones.

In the final chapter, Borasi assembles a number of strategies to initiate and support students' mathematical inquiry. There is much here of interest (and of specific benefit to teachers examining the range of their ways of working) and some are well illustrated by instances from the mini-course. Three in particular caught my interest: focusing on non-traditional mathematical topics where uncertainty and limitations are most evident, using errors as 'springboards for inquiry', and exploiting the surprises elicited by working in new domains. This chapter also allows Borasi to draw on the extensive investigative work she has undertaken over the past ten years.

However, I felt the book fell into two parts – the first ten chapters and the last two. While there were some illustrations drawn from the mini-course in this latter part, I felt that the last two chapters were setting out a far more general and broader agenda for teaching and learning in school mathematics, one that only had certain connections with what had gone before. Borasi herself tells us: 'Readers who prefer to examine the theoretical framework up front should refer to Chapter 11 for a discussion of the pedagogical assumptions and to the Appendix for a discussion of research methodology' (p. 5).

My difficulty was that I had expected a summary *generated* by the mini-course. Instead, I got a far more general agenda for classroom change only occasionally *illustrated* by the mini-course. These last two chapters do not follow from the rest of the book, but instead come from her life experience of mathematics education in general. It would indeed have been possible to start with these and then to conduct a case study to see if any of them were borne out. But I did feel somewhat miscued. One reason for this is explored in the next section.

Styles of Writing

In general, the book is written in an engaging and open style. However, it exhibited traces of three discourses which struck me as noteworthy.

The first was that of conventional 'research': for instance, 'evaluating the results', 'typical events', 'collect the data', 'critique of the instructional design', ... I found this somewhat incongruous, given the introspective, reflective nature of the account offered. I wouldn't classify what she has done as research in a traditional sense, but I don't think it is any the less for that. On the contrary, her mode of exploration of the issues I have outlied above I felt was both innovative and insightful. But the intermittent use of 'research jargon' in what seemed to me an inappropriate setting resulted in dissonances that I found interrupted my reading. It also triggered uncertainties about the intended audience.

The second was the relatively frequent references made to 'the mathematics community' or 'the mathematicians': 'Mathematicians have chosen, ... had to conclude, ... have decided' (p. 70). It struck me that the students do not have access to this group – and invoking them is a discursive move that puts the justification for something outside of the here-and-now (and is not necessarily any the worse for that). But who are *they* and who has the right to speak on *their* behalf? Do they never disagree? There is something monolithic, in part resulting from the use of the definite article, in the same way that it is common to speak of '*the* definition' of something. There is a predominance of pronouns in this book, arising partly from the wealth of reported speech, partly the authorial 'I' and contrastive 'we' (in various senses), as well as 'they'. This notion of community is reflected in Borasi's use of 'we' and 'us', as contrasted with the personal 'I'.[8]

In her account, we read statements like "So, we want a definition to be able to identify only circles" (p. 22), where the 'we' here is plausibly the three of them present. But on p. 44, we read:

> Reflecting on this kind of situation could help students appreciate that definitions are really *created by us*, even in mathematics, where everything may seem rigid and pre-determined (at least to most students).

Here, the *us* may be 'human beings'. On page 70, when attempting to make "students aware of the alternative chosen by the mathematics community", Borasi says to the two young women, "We say, ... So we may say ..."; here, she seems to be talking *for* that community. Whereas in the next sentence she has nicely personalised her suggestion for them to look at the question of evaluating nought to the nought: "I'll show you another thing that gave me a problem. What do you think is nought to the nought?" The personal in mathematics discourse is still a topic awaiting systematic exploration.

The third discourse was that of absolutist mathematics. And I want to point out from the outset that I am not intending to pick holes or waggle fingers here. On the contrary, I want to point out how pervasive such metaphors and ways of speaking must be if someone as sophisticated and thoughtful as Borasi writing on this topic nonetheless still uses expressions like 'the true nature of mathematics' (p. 7), 'a correct definition' (p. 27), 'What a polygon really is' (p. 45), 'mathematicians had to conclude' (p. 70) 'unavoidable problems' (p. 68, 72), 'the concepts ... required a precise definition' (p. 93), 'the in-depth analysis of a list of incorrect definitions of a given concept' (p. 155). What is the subversive appeal of this way of speaking?

What Are Mathematical Definitions and What Are Such Definitions For?

Borasi offers us 'an inquiry into the nature of mathematical definition' (p. 27), which seems to imply there is an object called 'a mathematical definition'. Or is it the uses of definition in mathematics? Certainly, she contrasts mathematical and non-mathematical definitions as one of her class activities, though the contrast was in terms of form rather than function. Is definition a mathematical *topic* in the same sense as equations or functions or circles? The book's title is *Learning Mathematics Through Inquiry* and by the end I wondered whether it could be more accurately rendered 'learning *about* mathematics through enquiry', a more meta-activity, which is not to say that the two students did not also learn mathematics in the process.

Central to her work is the (for me) somewhat *a priori* categorisation of attributes of definition (is it a *definition* of definition or merely a listing of properties?) which can be found on pages 17-8:

> In an earlier inquiry on definitions ... the following points emerged as commonly accepted requirements for mathematical definitions:
>
> – *Precision in terminology.* All the terms employed in the definition should have been previously defined, unless they are one of the few *undefined terms* assumed as a starting point in the axiomatic system one is working with.
> – *Isolation of the concept.* All instances of the concept must meet all the requirements of its definition, while a noninstance will not satisfy at least one of them.
> – *Essentiality.* Only terms and properties that are strictly necessary to distinguish the concept in question from others should be explicitly mentioned in the definition.
> – *Noncontradiction.* All the properties stated in a definition should be able to coexist.
> – *Noncircularity.* The definition should not use the term it is trying to define.

Appreciation of this list to some considerable extent formed Borasi's content agenda for her students and guided her choice of mathematical tasks and her orientation and focus within them. This list also immediately raised my hackles, partly due to its undiscussed arrival ready-made. My initial reaction turned into the slightly more productive activity of looking for counter-examples.

Firstly, it seems to assume 'being an example of' is an unproblematic notion. And where have all these *shoulds* and *musts* come from? What is the difference between a definition and a property? Is it merely a change in status accorded to the statement, a shift in emphasis? Can we exclude a purported definition on the basis of this characterisation? Do I have to claim something to be a definition for it to be one? If we reject a definition, what is it then, what does it become? The above seemed to be more a philosopher's than a mathematician's or an educator's list of desirable properties.

Consider my first unease. Definitions can be used as advance organisers by allowing us to focus (our, our students') attention on certain salient features of situations which often *will become* (the tense is crucial here) examples of a concept. In the English language, an example has to be an example *of* something. Hawkins (1980, p. 44) writes:

> This drawing was not in her mind an *example* of anything geometrical. It was not a consequence but a starting point. But after we have developed some theorems, the drawings *become* examples.

Definitions can therefore act as filters through which to view mathematical situations.

They can also allow us to accord something the status of example or not. To take a simple example: is 1 a prime number or not? Well, it depends. If our definition is "Any whole number that is divisible only by one and itself', then, yes. If we use 'Any number which has exactly two factors', then it is not. These two definitions are equivalent for every other whole number. In part, our choice boils down to whether or not we *want* 1 to be prime?

There are many statements that require modification depending whether 1 is taken to be prime or not. Prime factorisation (if yes), because of non-uniqueness; prime decomposition (if no), because in that case 1 is a whole number with no prime decomposition. But these are arguments that come from the desire to have clean, sophisticated results. It illustrates the fact that we must have some understanding of a concept and crucially what we want that concept to do for us *before* we are able to judge whether or not a proposed definition is a good one (or one adequate for our purposes).

The linguist John Searle claims (1969, p. 9):

> Any extensional criterion for a concept would have to be checked to make sure it gave the right results, otherwise the choice of the criterion would be arbitrary and unjustified.

The length of a parametrised plane curve $(x(t), y(t))$, where $a \leq t \leq b$, is defined to be

$$\int_a^b \sqrt{x'(t)^2 + y'(t)^2}\, \mathrm{dt}\,.$$

We *check* that the definition agrees with our *known* examples (e.g. length of a straight line, perimeter of a circle), which encourages us to accept it as a valid characterisation of length. It also acts as a device for extension in that we can now *calculate* the lengths of curves we were previously unable to. It also suggests the possibility of non-rectifiable curves (since integrals sometimes fail to exist) and so acceptance of a definition may sometimes entail a revision of our intuitions about a notion.

There is frequently a proposition concealed behind a definition, verifying that the definition does indeed do what we want it to do. And so, as ever, behind it all are human *intentions*. Euclid Book IX, definition 9 asserts: "Equal solid figures are those contained by similar planes equal in magnitude and multitude". From its use, the required sense of *equal* seems to be 'same volume'.[9]

David Fowler has suggested a planar, pseudo-Euclidean definition of equal polygons: "Equal polygons are those contained by lines equal in magnitude and multitude". If the intent is to capture 'same perimeter', then this definition is fine. However, the parallel intent to the actual Euclidean definition (IX,9) would be 'same area'. From our plane geometric knowledge (and although Heron's formula can be read as a verifying proposition sanctioning the use of this definition for triangles), quadrilaterals and polygons with a greater number of sides flex, thereby changing the area (though many school students apparently believe the contrary when a rectangle is flexed into a parallelogram).

It was only in 1978, that the mathematician Connelly (1981) exhibited a flexible polyhedron, a Euclidean-constructible object which invalidates the apparent *intent* of Euclid IX,9 and hence offers, in some sense, a counter-example to a definition.

Residual Questions about Definitions

We say that, in order to communicate, people must agree with one another about the meanings of words. But the criterion for this agreement is not just agreement with reference to definitions, e.g. ostensive definitions – but *also* an agreement in judgments. It is essential for communication that we agree in a large number of judgments.
(Ludwig Wittgenstein, *Remarks on the Foundation of Mathematics*)

I merely list here (with a brief discussion) some questions that arose for me from reading Borasi's book.

(a) *When and why might I (we? you? they?) change a definition?*
Lakatos' (1976) often-quoted work *Proofs and Refutations* explores the tangled web that can exist between the triad:

theorem statement,
theorem proof, *and*
concept definition.

In Chapter 4, Borasi uses his central example of polyhedron:

> This is a good example of how new developments in the discipline occasionally invite mathematicians to modify an existing *concept* and, consequently, its *definition*, in order to make it more interesting and useful mathematically – quite a blow for anyone believing in the absolute truth of mathematical results! (pp. 44-5)

But was it the concept of polyhedron that was modified first, or merely 'its' definition? Concepts and their definitions sometimes feel like quantities and their measures – on occasions, as with value and its supposed measure 'money', the measure has more existence than the quantity, and the measure may in some sense bring the quantity into existence.

In particular, Lakatos raises the topic of proof-generated definitions. This seems particularly problematic in terms of teaching mathematics, because of needing to perceive the definition as a tool custom-made to do a particular job that cannot be known by those trying to learn it, certainly not with an order of presentation that seems to require definitions to come first.

But his work also raises the prospect of disagreement both about definitions but also about judgements about whether or not the criteria of a definition are met (e.g. the technique of monster-barring putative counter-examples). This brings us up against varied ways of seeing that a definition can bring about in order that purported examples are seen as such. (Borasi fails to mention the extensive work of Nicolas Balacheff on polyhedra and definitions within that.)

> 'Consider: 'Our mathematics turns experiments into definitions'.
> (Wittgenstein, 1965, p. 383).

The history of Stokes' theorem provides a good example of this claim. Michael Spivak's book *Calculus on Manifolds* (1965) has at its core a proof of Stokes' Theorem for a class of manifolds. The proof of this theorem takes only a handful of lines – in some sense, it is a computational triviality. All the power is in the definitions. Perhaps this is a general process in mathematics. Namely, that definitions accrete intention and purpose with regard to a mathematical theory, and in necessary consequence, move away from intuitive, neophyte senses. If this is the case, it is easy to see why coming to grips with an organised, codified mathematical theory is such a challenging task for a relative novice.

(b) *Are definitions used for anything else in mathematics besides proving theorems?*
Recall the oft-quoted remark by Yu Manin that 'a good proof is one that makes us wiser'. I suggest a good definition is one that makes us think. The tag 'By definition' seems to be an invitation *not* to think – resulting in something that is automatic or perhaps 'for free', and consequently often not valued highly. Yet, as instances that Borasi has described show, the application of a definition leaves scope for defence as well as mere acquiescence.

In addition, it can involve an enormous amount of work. There is a sense of 'simply checking' the definition, yet this can be far from a simple matter. For instance, can you *check* whether zeta(3) is irrational? Is part of the difficulty predominantly that irrationality is usually defined negatively?

(c) *In the same spirit as Borasi's comments on disjunctive definitions, what about negative definitions?*
'A set is *connected* if it can't be decomposed into two non-empty, open disjoint subsets.' Why can we not easily get past a negative characterisation of irrational numbers (those whose decimal expansions do not terminate – even their name is in reference to what they are not)? What about symbolic definitions ('A differential equation is an expression of the form ...')? Borasi writes (p. 94) about equations formally in terms of an equivalence relation between expressions. Yet transitivity would imply that if $x + 3 = 5$ and $x - 2 = 5$ then $x + 3 = x - 2$? But do we want this last entity to be an equation? When does an equation become an object (the first algebraic object)?

(d) *Do I have to claim something to be a definition for it to be one?*
Is it actually a status claim (recall the protest by Frege that I cited at the outset)? Can I turn a theorem into a definition (Newton *defined* the integral as the anti-derivative)? Sometimes, the changes or shifts are less evident. How could it be that the one-to-one correspondence between the whole numbers and the squares be a paradox to be resolved to Galileo and to be seen as a property (on occasion *the* defining property) of infinite sets by the time of Dedekind.[10] What has changed to allow that altered perspective? Fisher's study (1966) of the demise of invariant theory describes a ghost town of the Platonic world where no one ever goes these days. We have collectively almost forgotten about it. What of those definitions and distinctions? Do they cease to exist because no one attends to them any more?

(e) *Are definitions ever equivalent?*
Just because two definitions are logically equivalent may not mean we have no preferences for other reasons. Larry Copes (1979, p. 384) has offered the guiding suggestion:

> Given the chance to define something in a less-intuitive but more generalisable way, have the definition make sense, then lay the 'groundwork' for later generalisation of the definition by proving the logical equivalence.

But logical equivalence may not be the same as functional equivalence. *Equivalent* can mean 'equivalent for our purposes'. And how often are mathematicians' purposes and intents revealed or discussed? Definitions help when you want to prove something. But the help they offer will vary. Why is it so hard to find out about mathematicians' purposes and intentions – even to the point that they have them? Purposelessness is so often the complaint to which students grappling with mathematics at a variety of levels give voice – and is it any wonder when it is so rare to find a discussion of purpose?

Under what circumstances would I say 'That is not a definition'? Would I ever say that a definition (or concept) where the *true* one? Omar Khayyam (in his *Discussion of Difficulties in Euclid* – see Amir-Moez, 1959) dealt with both the parallel postulate and the definition of ratio in Euclid's *Elements*. He refers to two definitions of ratio, *Famous ratio* (the Eudoxan one, the celebrated, intricate Book 5, definition 5) and *True ratio*, where the latter is based on the process of *anthyphairesis* (continued subtraction in turn) – what we think of today as the Euclidean algorithm giving rise to continued fractions, but directly applied to geometric entities. The sequence of multiple-numbers, e.g. once, twice, five-times, thrice, ... *is* the ratio.[11] Khayyam also makes the definition: '*Like* magnitudes are those whose difference has a meaning': in other words, pairs of magnitudes to which the anthyphairetic procedure can be applied – a completely functional and process-related definition.

One reason for studying the history of mathematics is to discover instances of definitions being superseded by others and trying to examine the reasons for this. The work of mathematician David Fowler (1979, 1981, 1987) on pre-Euclidean mathematics is an attempt to reconstruct an older conception of ratio which he believes was eradicated by the more powerful, abstract proportion theory methods to be found in Book V of Euclid. He feels this reconstruction is necessary in order to make sense of much of the Elements, particularly Books II and X (the latter of which, in terms of the number of lines, comprises one third of the entire work!). What were those Greek mathematicians trying to do that resulted in this work? Questions of intention and purpose arise, but where do we go to find out about them? Because it is surely only in relation to purpose and intention that judgments about the fittingness, success or rightness of different definitions can be made.

(f) *Can definitions bring objects into existence?*

The way definitions were employed in Borasi's book (not surprisingly, given the audience and setting) always seemed to be *after* the fact – formulating or rendering more precise a characterisation. There is an important switch of field and ground when a definition produces the concept rather than the other way round. For example, a new definition might be proposed by analogy with an existing definition.

(g) *Defining versus specifying?*
Finally, there is also the confusion between defining and specifying. Although it is common to say we 'define' $f(x)$ to be ..., in fact we are merely specifying which function we are referring to, selecting the focus of our attention and then assigning it a chosen name, one rich with past associations. The term 'definition' might fruitfully be reserved for concepts.

Conclusion

> What we call the beginning is often the end. And to make an end is to make a beginning. The end is where we start from.
> (T. S. Eliot, *Burnt Norton*)

An English friend of mine travelled to the north of Finland recently to stay with someone she hardly knew: but, as she liked travelling, had nonetheless accepted his casual invitation to visit. The plane was delayed and she arrived after midnight in a remote city near the Arctic circle. The official at the airport, instead of asking the more customary question 'What is the purpose of you visit?' asked instead, 'What is the *meaning* of your visit?'. The effect on her state of mind was devastating. While purpose and meaning[12] are sometimes treated as interchangeable, purpose and intention are at times importantly different from meaning and significance – in mathematics, as elsewhere. And none of these are commonly well expressed (or even to be found at all) in mathematical writing.

Borasi has given us a 'warts-and all', particular view of being in a classroom with her, as well as letting us in on some of her intentions, purposes and reflections. The life and enthusiasm in her writing is manifest, as are the subtlety and insightfulness of many of her reflections on teaching practice. Its avowed particularily (and unusual setting) should not concern us in the least. As her sometime doctoral supervisor and now her geographically and epistemologically proximate colleague Stephen Brown has written: "One incident with one child, seen in all its richness, frequently has more to convey to us than a thousand replications of an experiment conducted with hundreds of children" (1981, p. 11). And in Oscar Wilde's words, Borasi's is a book deserving of Lady Bracknell's observation on John Worthing's life, that it it one 'crammed with incident'.

There is a principle attributed to the psychologist Abraham Maslow of not looking at what is currently happening, but looking for the extremes of possibility, in order to test what might be achievable. To that extent, Borasi's work is somewhat of a Maslow study. As ever, one perennially interesting question for any teacher who reads this is: 'What can I take away from her actuality to add to my possibilities as a teacher?' It is one of the most subtle but important questions to be worked at in teacher education.

I don't often find the question 'What is generalisable from this work?' offers a useful cutting edge. A far more pertinent – and necessarily personal – question I feel is 'What is appropriable?'. For me, important categories include: being reminded of things I may have once known, but have lost sight of; being encouraged to explore something I thought inaccessible to certain groups at certain stages;

being stimulated to think hard about what I believe about something. Judged by all three of these criteria, and often *because* I disagreed with her detailed analyses, Borasi's book has been successful for me. And in consequence I warmly invite you to explore what it has to offer you.

NOTES

[1] The new Collins COBILD English language dictionary (1990) worked from a huge computerised corpus of extant texts. It aims to offer contemporary usages, rather than the more conventional accretions of sense. It offers the most frequent usages first, rather than working etymologically and from the most literal meaning and earliest recorded usage. Its definitional style is also quite unusual, frequently involving whole sentences (e.g. '*Feline* means belonging or relating to the cat family') and addressing the reader (e.g. "If you *ferret something out,* you ..."). What is the (unstated) conventional structure of mathematical definitions?

[2] In the US, the prison department is known as the Department of Corrections.

[3] Carl Jung, in his autobiographical work *Memories, Dreams, Reflections* (1965, pp. 27-8) writes of his bewilderment and fear at school with mathematics in general, and with algebra in particular:

> The teacher pretended that algebra was a perfectly natural affair, to be taken for granted, whereas I didn't even know what numbers were. They were not flowers, not animals, not fossils; they were nothing that could be imagined, mere quantities that resulted from counting. [...] No one could tell me what numbers were, and I was unable even to formulate the question. To my horror, I found that no one understood my difficulty. [...] Whenever it was a question of an equivalence, then it was said that a=a, b=b, and so on. This I could accept, whereas a=b seemed to me a downright lie or fraud. I was equally outraged when the teacher stated in the teeth of his own definition of parallel lines that they meet at infinity; [...] My intellectual morality fought against these whimsical inconsistencies, which have forever debarred me from understanding mathematics. [...] All my life it remained a puzzle to me why it was that I never managed to get my bearings in mathematics when there was no doubt whatever that I could calculate properly. Least of all did I understand my own *moral* doubts concerning mathematics.

[4] Borasi calls this *essentiality* and observes on p. 41 that this may suggest modifications to her earlier stated requirement – see later for her full list of requirements of a definition.

[5] And what are we to make of the capitalist overtones and the claim that 'property is theft'?

[6] I am grateful to John Fauvel's (1987) lucid account of these matters.

[7] And the way one or other 'property' gets associated as the defining criterion is a clear instance of the process of metonymy at work (see Pimm, 1988). With the conic sections, the symptoms become algebraically expressed as second-order equations in two unknowns, and then it seems as if conic sections, the name implying a definition by genesis, actually are algebraic objects!

[8] Elsewhere (Pimm, 1987), I have explored some of the wide range of uses of the pronoun 'we'.

[9] Recall this example is taken from a geometric culture, where geometry is not only the particular content, it is also the very *language* of mathematics. For us, 'same volume' is a numerical equivalence not worthy of mention, in part because the numerical measure *is* the definition of volume. I have long been worried by *Elements* common notion 4: "All right angles are equal to one another".

[10] For Cantor, a set A is *finite* if it is equipollent to 1, 2, 3, ..., n, otherwise it is infinite. For Dedekind, A is *infinite* if it is equipollent to a proper subset of itself, otherwise it is finite. Thus, what for Dedekind was a definition, for Cantor was a property, a switch of field and ground.

[11] The ratio of the side to the diagonal of any square is characterisable as once, twice, twice, twice, twice, for ever. Interestingly, the simplest repeating, non-terminating 'ratio': once, once, once, once, ... (*ad infinitum*), comes from the anthyphairesis of side and diagonal of the regular pentagon, ostensibly the icon of the Pythagoreans. See John Fauvel's review (1989) of David Fowler's treatise *The Mathematics of Plato's Academy* for further details on anthyphairesis.

[12] I recently read an airline ticket where the following words appeared: 'For the purposes of ..., "ticket" means ...'. Thus, the context is specified in advance by the purposes, and *then* a 'meaning' is assigned.

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