Journal of Mathematical Biology 1, 259—273 (1975)
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Isochrons and Phaseless Sets

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Received May 21, 1974

Summary

Winfree has developed mathematical models for his phase resetting experiments on biological clocks. These models lead him to ask a number of mathematical questions concerning dynamical systems. This paper deals with these mathematical questions. In Winfree's terminology we show the existence of isochrons and establish some of their properties.

This paper deals with questions raised by Winfree [7] regarding the behavior of "isochrons" in mathematical models of biological oscillations. These mathematical questions lie within the domain of dynamical systems [5], the qualitative study of ordinary differential equations. Our discussion will ignore the biological context of these questions, but we shall attempt to present our results in as non-technical a fashion as possible. It is our intention that the exposition be both accessible to non-specialists of dynamical systems and accurate insofar as is possible. Proofs of theorems are contained in the appendices.

We begin with a description of the setting for Winfree's question. The object of ultimate interest is a biological "clock" or oscillation. A model is constructed for the oscillation based on the assumption that its dynamics are determined by the values of a finite number of physical and chemical parameters (temperatures, pressures, free energies, velocities, chemical concentrations, etc.). A multi-dimensional space M is constructed representing the possible values of all these physical and chemical quantities. To say that the dynamics of the system are determined by the values in M at any one time means that there is a *flow* $\Phi: M \times \mathbb{R} \to M$ defined by the condition that $\Phi(x, t) = y$ if the state x becomes the state y after t units of time. The map Φ is to satisfy the usual flow properties

$$\Phi(x, 0) = x \tag{1}$$

and

$$\Phi(x, t_1 + t_2) = \Phi(\Phi(x, t_1), t_2). \tag{2}$$

One makes the additional assumption about the model that M is a smooth manifold (usually a domain in a Euclidean space) and that Φ is a smooth map. The flow Φ then determines a *vector field* X by the equation $X(x) = \frac{\partial \Phi}{\partial t}(x, 0)$.

Conversely, X determines Φ. It is convient to speak sometimes of the flow Φ and sometimes of the vector field X.

Winfree assumes that Φ has a *stable limit cycle*. The *orbit* of the flow Φ through $x \in M$ is the set $\{\Phi(x,t) \mid t \in \mathbb{R}\}$. The orbit through x is *periodic* of period $\tau > 0$ if τ is the smallest positive number with the property that $\Phi(x,\tau) = x$. A periodic orbit γ is a *stable limit cycle* if there is a neighborhood U of γ with the property that if $y \in U$, then $d(\Phi(y,t),\gamma) \to 0$ as $t \to \infty$. The distance d here is the distance function for some metric on M.

A simple example of a vector field whose flow possesses a stable limit cycle is given by the differential equations

$$\dot{x} = -y + x(1 - x^2 - y^2)$$
$$\dot{y} = x + y(1 - x^2 - y^2)$$

in the plane $\mathbb{R}^2$. In polar coordinates these differential equations become

$$\dot{\theta} = 1$$
$$\dot{r} = r - r^3.$$

The circle $r = 1$ is a periodic orbit of period 2π. It is stable because $\dot{r} > 0$ inside the circle (except at the origin) and $\dot{r} < 0$ outside the circle. This implies that r is a monotone function on each orbit and that all orbits except the origin tend to the limit cycle $r = 1$.

If $y \in M$, x is on a stable limit cycle γ, and $d(\Phi(x,t), \Phi(y,t)) \to 0$ as $t \to \infty$, then the eventual behavior of the points x and y looks almost the same. Winfree describes this situation by saying that y is on the *isochron* of x. If some event occurs at one place along γ, then that event will eventually occur at the same times along the orbit starting at y as they will on the orbit starting at x. Winfree asks:

Question A: *Do isochrons exist? Is a neighborhood of a stable limit cycle partitioned into the isochrons of points on the limit cycle.*

The answer to this question is yes if one places a nondegeneracy assumption on the behavior of the flow near the limit cycle. The existence of isochrons in this case is a theorem of dynamical systems which has been known for a few years. We describe now the non-degeneracy condition and state a mathematical theorem answering Question A. A proof of the theorem is discussed in Appendix A.

Let $\phi: M \times \mathbb{R} \to M$ be a flow with a periodic orbit γ of period τ and let $x \in \gamma$. A *cross-section* of γ at x is a submanifold $N \subset M$ with the following properties:

(1) $x \in N$ and $\bar{N} \cap \gamma = \{x\}$. $\bar{N}$ is the closure of N in M.

(2) $T_x N + T_x \gamma = T_x M$.

The second condition says that N is transverse to γ at x. The *Poincaré map* Θ is a map defined on a neighborhood V of x in N with image in N. The map Θ is characterized by the condition that if $y \in N$, then $\Theta(y)$ is the first point of intersection of the forward orbit of y with N when this makes sense. Since the flow Φ is continuous, Θ will be well defined in a neighborhood of x in N. The time of the first intersection will be near τ for points near x. See Fig. 1. One then says that γ is an *elementary* (or *hyperbolic*) limit cycle if the matrix $D\,\Theta_x$ of first partial derivatives of Θ at x has no eigenvalues of absolute value one.

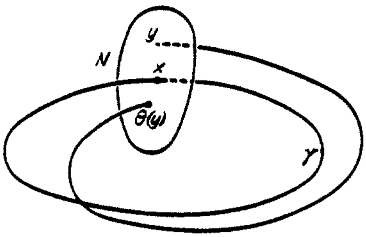

Fig. 1

The eigenvalues of $D\,\Theta_x$ are often called the *characteristic multipliers* of γ. They are independent of the choices of x and N. If γ is an elementary, stable limit cycle, then all of its characteristic multipliers have absolute value smaller than one. Every orbit in a neighborhood of γ tends toward γ exponentially fast.

The existence of isochrons is equivalent to the existence of cross-sections to γ for which the time of first return is identically the period of γ. We seek a cross-section N for which $\Phi(N, \tau) \subset N$. If γ is a stable limit cycle, such a cross-section will be the isochron of its intersection with γ.

The concept of an isochron is closely related to the concept of a stable manifold in dynamical systems. If Φ is a flow on M and S is a subset of M, then the *stable set* of S, denoted $W^s(S)$, is the set of points y for which $d(\Phi(y, t), \Phi(S, t)) \to 0$ as $t \to \infty$. The *unstable set* of S, denoted $W^u(S)$, is $\{y \mid d(\Phi(y, t), \Phi(S, t)) \to 0$ as $t \to -\infty\}$. If an (un)stable set is also a manifold, it is called an *(un)stable manifold*.

The basic theorem we state regarding Question A is the following special case of the Invariant Manifold Theorem [2, 3]:

Theorem A: *Let $\Phi: M \times \mathbb{R} \to M$ be a smooth flow with an elementary, stable limit cycle γ. The stable set $W^s(x)$ of each $x \in \gamma$ is*

(1) *a cross-section of γ,*

(2) *a manifold diffeomorphic to Euclidean space.*

Moreover, the union of the stable manifolds $W^s(x)$, $x \in \gamma$, is an open neighborhood of γ and the stable manifold of γ.

This theorem proves the existence of isochrons for the elementary. stable limit cycle γ of Φ. Note that for any flow Φ, $W^s(\Phi(x, t)) = \Phi(W^s(x), t)$. It follows from this observation that isochrons are permuted by the flow. In dealing with the second and third questions of Winfree, we shall assume that the limit cycle in question is elementary.

The second question Winfree asks is a topological question about the stable manifold of a stable limit cycle.

Question B: *Suppose $\Phi: \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is a flow with an elementary, stable limit cycle γ. Does the frontier of $W^s(\gamma)$ have dimension $\geq n-2$? The* frontier *of a set S is $\bar{S} - \mathrm{int} S$ with $\bar{S}$ the closure of S, and $\mathrm{int} S$ the interior of S.*

M. Hirsch pointed out that the answer to the question is yes for topological reasons. The space $W^s(\gamma)$ is an open set of $\mathbb{R}^n$ homeomorphic to $S^1 \times \mathbb{R}^{n-1}$. Its first homology group is non-zero since γ cannot be deformed to a point in $W^s(\gamma)$. From the Alexander Duality Theorem [6] of algebraic topology, it follows that the complement of $W^s(\gamma)$ has a non-trivial homology class of dimension $n-2$ that links γ. The existence of this homology class implies that the frontier of $W^s(\gamma)$ has dimension at least $n-2$. Details and specific references are given in Appendix B.

We remark that the answer to question B is yes for a flow on a manifold M of dimension n whose homology group of dimension $n-1$ is zero. It is not true on any manifold M as the following example demonstrates. On the three sphere S^3, let X be a vector field which points "down" except at the north and south poles where X has an elementary singular point. Let Y be a vector field on the circle S^1 which is never zero. The sum $X+Y$ defines a vector field on $S^3 \times S^1$ with two periodic orbits γ_1, γ_2 at the {north pole} $\times S^1$ and {south pole} $\times S^1$ respectively. The periodic orbit γ_2 is an elementary stable limit cycle. Moreover, $W^s(\gamma_2) = S^3 \times S^1 - \gamma_1$. Therefore the frontier of $W^s(\gamma_2)$ is γ_1 and the dimension of γ_1 is $1 < 4-2$.

The third question of Winfree concerns the behavior of the isochrons of a stable limit cycle γ near the frontier of $W^s(\gamma)$.

Question C: *For "generic" flows $\Phi: M \times \mathbb{R} \to M$ possessing an elementary stable limit cycle γ, is it true that every neighborhood of every point on the frontier of $W^s(\gamma)$ intersects every isochron of γ?*

Questions A, B, and C motivate a number of experiments performed by Winfree. By adjusting a pair of experimental parameters, he is able to create experiments for which the initial conditions in a model lie on or near the frontier of $W^s(\gamma)$, γ a stable limit cycle. The experimental results for these values of the experimental parameters display one of two phenomena: (1) a destruction of the oscillation entirely, or (2) points arbitrarily close to one another lying on isochrons of every point of γ. The second possibility indicates that all of the isochrons of γ are passing arbitrarily close to a single point on the frontier of $W^s(\gamma)$. Since this situation is the one typically encountered in experiments, is it the one which is also typical of the models in an appropriate sense? This is a natural condition that a reasonable mathematical model should satisfy. The answer to Question C does not seem to exist in the literature of dynamical systems. Here we attempt an answer which appears almost satisfactory.

We illustrate the kinds of phenomena relevant to Question C which may occur for a flow by means of a few examples. The first example is a flow Φ on the plane $\mathbb{R}^2$ whose vector field in polar coordinates is given by the differential equations

$$\dot{\theta} = 1$$
$$\dot{r} = r(r^2 - r_1^2)(r^2 - r_2^2).$$

This vector field has two periodic orbits γ_1, γ_2 given by $r=r_1$ and $r=r_2$ respectively. The periodic orbit γ_1 is an elementary stable limit cycle. The stable manifold $W^s(\gamma_1)$ of γ_1, is the set $\{(r, \theta) \mid 0<r<r_2\}$. If $(r_1, \theta_1)=x \in \gamma$, $W^s(x)$ is easily seen to be $\{(r, \theta) \mid 0<r<r_2 \text{ and } \theta=\theta_1\}$ since the angular velocity of the flow is identically one. Thus Φ is a flow which has an elementary stable limit cycle but does not satisfy the conditions specified in Question C. The frontier of $W^s(\gamma_1)$ contains γ_2. If $x \in \gamma_2$, then a small neighborhood of x does not meet each isochron of γ_1. The isochrons of γ_1 are contained in radial lines of the flow. See Fig. 2.

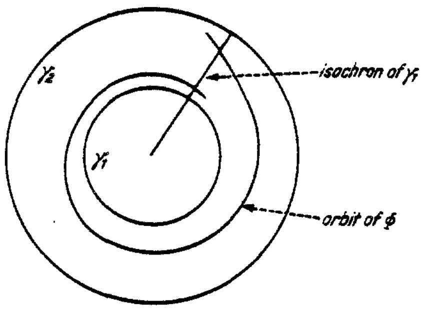

Fig. 2

The second example is obtained from the first by changing the parametrization of the flow in the first example without changing the orbits. Let $f: \mathbb{R} \to \mathbb{R}$ be a smooth function with the properties that

(1) $f(r) \geq 1$,

(2) $f(r)=1$ unless r is near r_1. In particular $f(r_2)=1$,

(3) $f(r_1)>1$.

Let Y be the vector field $f(r)X$ with X the vector field of the first example. The differential equations defining Y are

$$\dot{\theta}=f(r)$$
$$\dot{r}=f(r)\, r\,(r^2-r_1^2)(r^2-r_2^2).$$

The orbits of Y are the same as those of X. There are still periodic orbits of Y at $\gamma_2=\{(r, \theta) \mid r=r_2\}$; $i=1, 2$. Unlike the first example, the periods of γ_1 and γ_2 are different for the flow of Y. The periodic orbit γ_1 has a period $2\pi/f(r_1)$, and the period of γ_2 is still 2π. The effect of this perturbation is to bend the isochrons of Y. See Fig. 3.

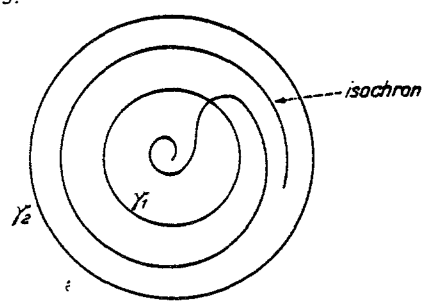

Fig. 3

We are interested in whether the isochrons of Y satisfy the criteria of Question C. The key observation to be made is the following. If we follow the flow for time $2\pi/f(r_1)$ each isochron is mapped into itself, while if we follow the flow for time 2π, each point of γ_2 of γ_2 on the frontier returns to itself. These two requirements are compatible only if the isochrons of γ_1 wind around the annulus $r_1<r<r_2$ infinitely often as they as approach γ_2. In appendix C we give the topological arguments which prove that Y satisfies the criteria of Question C.

The first two examples considered above deal with vector fields in the plane. It is well known that there is a drastic difference in the qualitative behavior vector fields can have in two and higher dimensions. It is reasonable to expect that the discussion of Question C will be considerably more difficult in more than two dimensions. The third example is intended to demonstrated some of the complications which arise in three dimensions.

The third example is described by a pair of perturbations to one flow. Consider the flow Φ on $\mathbb{R}^2\times S^1$ whose vector field X is defined by the differential equations

$$\dot{r}=r(1-r^2)$$
$$\dot{\theta}=0$$
$$\dot{\phi}=1$$

with (r,θ) polar coordinates on $\mathbb{R}^2$ and ϕ the coordinate on S^1. The flow Φ has an elementary stable limit cycle $\gamma=\{0\}\times S^1$. The stable manifold $W^s(\gamma)=\{(r,\theta,\phi)\mid r<1\}$. As in the first example, the isochrons are easily described as the sets $\varphi=$ constant in $W^s(\gamma)$. Denote the frontier of $W^s(\gamma)$ by B.

We now make two perturbations in X. The first perturbation X_1 is obtained by adding a small vector field to X which has a component only in the θ direction and produces a vector field X_1 such that X_1 restricted to B has two periodic orbits γ_1 and γ_2 given by the sets $\{(1,\frac{\pi}{2},\phi)\mid\phi\in S^1\}$ and $\{(1,-\frac{\pi}{2}\phi)\mid\phi\in S^1\}$ respectively. We assume that all other orbits of the flow of X_1 in B flow from γ_1 to γ_2. The second perturbation X_2 of X is obtained by multiplying X_1 by a function $f\colon\mathbb{R}^2\times S^1\to\mathbb{R}$ which is identically 1 outside a small neighborhood U of y_1 and which is identically $1+\varepsilon$ inside a smaller neighborhood of γ_1.

Does X_2 satisfy the criteria of Question C? To answer this question, it is necessary to describe the isochrons of X_2. The isochrons of X_1 are the same as the isochrons of X. If $(r,\theta,\phi)\in W^s(\gamma)$ has a forward orbit for X_2 which does not intersect U, then (r,θ,ϕ) is on the isochron of the point $(0,0,\phi)\in\gamma$ for X_2 since the velocity of X_2 in the ϕ direction is identically one outside U. The period of γ for X_2 is 2π; therefore, the map obtained by following the flow backwards for 2π maps each isochron to itself. Since each orbit inside $W^s(\gamma)$ eventually remains outside U, by iterating the -2π map of the flow, one can determine to which isochron each point belongs.

One finds that the isochrons look like the drawing of Fig. 4. The frontier of each isochron is the closure of a curve in B which approaches γ_1 from both sides. If $p\in B-\gamma_1$, then small neighborhoods of p do not intersect every isochron of γ. If $p\in\gamma_1$, then every isochron contains p in its frontier. On the "half-cylinder"

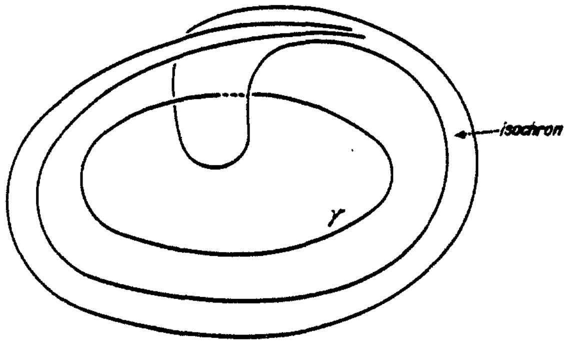

Fig. 4

$\theta=\frac{\pi}{2}$, this example looks like the second example considered above. Elsewhere it has the general features of the first example in relation to the criteria of Question C.

We now state our results regarding Question C. By reparametrization of vector fields along stable limit cycles, it is possible to prove that the set of vector fields meeting the criteria of Question C is dense in the set of all vector fields having an elementary, stable limit cycle (in the space of C^r vector fields with the C^r topology). The question of whether the set is "generic" seems to be difficult. The results we have obtained are contained in the following three theorems.

Theorem C1: *Let M be a compact oriented two dimensional manifold. In the space of C^r vector fields on M having an elementary stable limit cycle γ, there is an open-dense set of vector fields for which every neighborhood of every point in the frontier of $W^s(\gamma)$ intersects each isochron of γ.*

Theorem C2: *Let X be a C^r vector field on a manifold M having an elementary stable limit cycle γ. In the space of vector fields Y such that $Y=X$ on a neighborhood of the complement of $W^s(\gamma)$, there is a neighborhood $\mathscr{U}$ of X and a generic subset $\mathscr{G}$ of $\mathscr{U}$ such that every vector field in $\mathscr{G}$ has a limit cycle β with the property that every neighborhood of every point in the frontier of $W^s(\beta)$ intersects each isochron of β. "Generic" here is used in the sense that a subset of a topological space is generic if it is a countable intersection of open-dense sets.*

Theorem C3: *Let M be a compact manifold and let Σ be the space of vector fields on M satisfying Smale's Axiom A', the strong Transversality Property, and having a stable limit cycle γ. There is a dense open subset of vector fields in Σ with the property that every neighborhood of every point in the frontier of $W^s(\gamma)$ meets each isochron of γ.*

Proofs of these theorems are given in Appendix C. The difficulty in removing the restriction to a class of structurally stable vector fields in Theorem C3 lies in the fact that one has little control on the frontier of the stable manifold of a stable, elementary limit cycle. This frontier may be drastically altered by a perturbation of the vector field. There is no apparent relation between the behavior of two nearby vector fields on the frontier of a common stable limit cycle.

Appendix A:
In this appendix we shall prove the existence of "isochrons" for an elementary, stable limit cycle. This results is not new [2]. Much more extensive theorems have been proved by Hirsch, Pugh, and Shub [3]. Our purpose here is to give a reasonably elementary, self contained proof of the result needed for our applications.

It is useful to reduce the problem somewhat and cast it slightly different terms. The initial assumption is that the flow Φ has a limit cycle γ of period τ. The map $f=\Phi(\cdot,\tau): M\to M$ has γ as a set of fixed points. Moreover, if we consider Df along γ, it has an eigenvector with eigenvalue 1 along γ. All other eigenvalues of Df along γ have absolute value less than one. This implies that we can find a neighborhood U of γ and a metric on M with the properties $f(U)\subset U$ and $d(f(x),\gamma)<d(x,\gamma)$ for all $x\in U$. We can parametrize U as a neighborhood of $\{0\}\times S^1\subset\mathbb{R}^{n-1}\times S^1$ in such a way that at each $x\in\gamma$, Df_x leaves invariant the hyperplane of T_xM tangent to $\mathbb{R}^{n-1}$ and is a contraction on this hyperplane.

Proposition: Let $f:\mathbb{R}^{n-1}\times S^1\to\mathbb{R}^{n-1}\times S^1$ be the time τ map of a flow Φ satisfying

(1) $\{0\}\times S^1$ is a periodic orbit γ of Φ of period τ.
(2) If $x\in\gamma$, then Df_x leaves invariant the subspace tangent to $\mathbb{R}^{n-1}$ and has norm smaller than one on this subspace.
(3) Then there exist invariant manifolds $W(x)$, $x\in\gamma$ such that $W(x)=\{y\in\mathbb{R}^{n-1}\times S^1\mid d(f^n(y),x)\to 0$ as $n\to\infty\}$.
(4) The union of the $W(x)$ is a neighborhood of γ.
(5) The flow Φ permutes the $W(x)$.
(6) $W(x)$ and $\mathbb{R}^{n-1}$ have the same tangent space at x.

The proof of the proposition uses the following lemma which uses the same notation as the proposition.

Lemma: The sequences of functions $\{f^n\}$ and $\{Df^n\}$ are uniformly convergent sequences of functions in a neighborhood of γ. The function $\lim_{n\to\infty} Df^n$ has constant rank 1.

Proof: We need to establish some notation. If $z=(y,\theta)\in\mathbb{R}^{n-1}\times S^1$ then $|z|$, $|y|$, and $|\theta|$ will used for the norms of z, y, θ in suitable coordinate systems on $\mathbb{R}^{n-1}\times S^1$, $\mathbb{R}^{n-1}$, and S^1 respectively. We shall denote by π_1 and π_2 the projections of $\mathbb{R}^{n-1}\times S^1$ onto $\mathbb{R}^{n-1}$ and S^1. If $\varepsilon>0$, there is a neighborhood U of γ such that if $z=(y,\theta)\in U$, then the following estimates hold

(1) $|\pi_2 f(z)-\pi_2(z)|\le\varepsilon|\pi_1(z)|$.
(2) $|\pi_1 f(z)|\le\mu|\pi_1(z)|$ for some fixed $0<\mu<1$ independent of z.
(3) $|Df(z)-D\pi_2(z)|<\varepsilon|\pi_1 z|$.

The first two of these estimates easily imply that the sequence $\{f^n\}$ is uniformly convergent in U. If $z_1, z_2\in U$, then for large enough n we have

$$|f^n(z_1)-f^n(z_2)|<\varepsilon+|\pi_2 f^n(z_1)-\pi_2 f^n(z_2)|$$

since $f^n(z)\to\gamma$ uniformly by estimate (2) above. Now estimates (1) and (2) imply

$$|\pi_2 f^n(z) - \pi_2(z)| \leq \sum_{i=1} |\pi_2 f^i(z) - \pi_2 f^{i-1}(z)|$$

$$\leq \varepsilon \sum_{i=1} \mu^{i-1} |\pi_1(z)| < \frac{\varepsilon}{1-\mu} |\pi_1(z)|.$$

Hence $|\pi_2 f^n(z_1) - \pi_2 f^n(z_2)| < \frac{\varepsilon}{1-\mu} (|\pi_1(z_1)| + |\pi_1(z_2)|) + |\pi_2(z_1) - \pi_2(z_2)|$. This yields an estimate for $|f^n(z_1) - f^n(z_2)|$ independent of n, proving that the sequence f^n is uniformly convergent.

To prove that Df^n is uniformly convergent, it suffices to note that $Df^n(z) = Df(f^{n-1}(z)) \circ \ldots \circ Df(f(z)) \circ Df(z)$. We have an estimate for each term of this composition as $D\pi_2 + E_i$ where $|E_i| < \varepsilon \mu^{i-1} |\pi_1 z|$. Since $D\pi_2$ is a projection ($D\pi_2 \circ D\pi_2 = D\pi_2$), this yields an estimate for $Df^n(z)$ as $D\pi_2 + |\pi_1 z| \cdot$ bounded term. Thus Df^n is uniformly convergent.

It remains to prove that the rank of $\lim Df^m$ is identically one. Since $f^m(z) \to \gamma$ uniformly at an exponential rate, it follows from the mean value theorem that, for sufficiently large m, $n-1$ of the eigenvalues of Df^m will be arbitrarily close to zero. Hence the kernel of $\lim_{m\to\infty} Df^m$ will have dimension $n-1$. This finishes the proof of the lemma.

The lemma implies that the function $g = \lim_{m\to\infty} f^m$ is a submersion in a neighborhood of γ. The implicit function theorem [1] implies that the inverse image of $x \in \gamma$ is a smooth submanifold $W(x)$ transverse to γ. If $z \in W(x)$, then $f^m(z) \in W(x)$ and $f^m(z) \to x$ as $m \to \infty$. Therefore $W(x)$ is the manifold required by the proposition in a neighborhood U of γ. To find the remainder of the isochron of x inside $W^s(\gamma)$, we need merely form $\bigcup_{m \geq 0} f^{-m}(W(x))$. Since g is a submersion, there is a neighvorhood D of x in $W(x)$ which is diffeomorphic to a disk with the property that $f(D) \subset D$. This implies that $\bigcup_{m \geq 0} f^{-m}(D)$ is diffeomorphic to Euclidean space since it is an increasing union of disks.

Appendix B:

This appendix is devoted to the proof of the following theorem:

Theorem B: Let Φ be a flow on R^n having an elementary stable limit cycle γ. If W is the stable manifold of γ, then the dimension of $\bar{W} - W$ is at least $n-2$. ($\bar{W}$ is the closure of W.)

As M. Hirsch pointed out to me, the proof is a corrollary of the Alexander Duality Theorem [6: p. 296, 6. 2. 16] which states that $\tilde{H}_q(\mathbb{R}^n - A) = \bar{H}^{n-q-1}(A)$ if A is a compact subset of $\mathbb{R}^n$. The notation here is that $\tilde{H}_q$ is the reduced singular homology group of dimension q and $\bar{H}^k(A)$ is the direct limit of the cohomology groups $H^k(U)$ for neighborhoods U of A.

If $\bar{W}$ is compact, we apply the theorem with $A = \bar{W} - W$ and $q = 1$. The theorem implies $H_1(\mathbb{R}^n - (\bar{W} - W)) = \bar{H}^{n-2}(\bar{W} - W)$. Now W is a component of $\mathbb{R}^n - (\bar{W} - W)$ and $H_1(W) = H_1(\gamma) = Z$. Hence $\bar{H}^{n-2}(\bar{W} - W) \neq 0$. This implies that the Čech $(n-2)$-cohomology of $\bar{W} - W$ is non-trivial [6: p. 316, 6. 6. 2 and p. 334, 6. 8. 8]. This means that the dimension of $\bar{W} - W$ is at least $n-2$ as was to be proved.

The case in which $\bar{W}$ is not compact is easily reduced to this one. Form the one point compactification S^n of $\mathbb{R}^n$ and apply the above argument on S^n inside of $\mathbb{R}^n$. Note that there is a flow on $\mathbb{R}^3$ with a stable limit cycle γ such that the complement of the stable manifold of γ is a line in $\mathbb{R}^3$. This line is not a homology 1-cycle of $\mathbb{R}^n$. See Fig. 5.

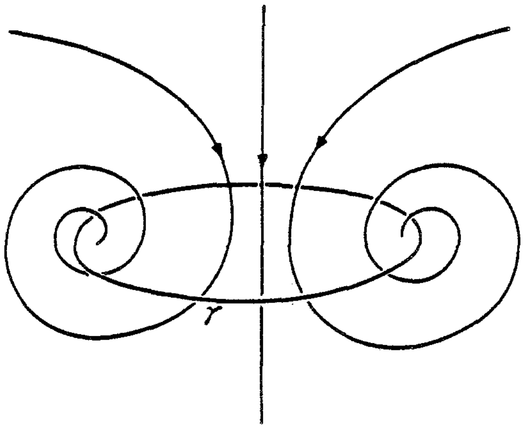

Fig. 5

Appendix C:

This appendix contains proofs of theorems providing partial answers to Question C of Winfree. These theorems are new results on dynamical systems. Throughout this appendix Φ will be a flow with vector field X on a manifold M. The flow Φ is assumed to have a stable elementary limit cycle γ of period τ with stable manifold W. The frontier $\bar{W}-W$ will be denoted B. A point $x \in B$ will be called *phaseless* if every neighborhood of x intersects the stable manifold (isochron) of each point of γ.

We shall first consider the case in which M is two dimensional and $\bar{W} \subset M$ is compact. Peixoto's Theorem [4] implies that there is an open-dense set of vector fields on M having γ as an elementary stable limit cycle which are structurally stable on a neighborhood of $\bar{W}$. If X is structurally stable in a neighborhood of $\bar{W}$, then X has a finite number of singular points on B, all of which are saddles or sources. There are no non-periodic recurrent orbits in B. We shall assume that X has these properties. For each component of B, there are two cases to consider: (1) the component of B contains a singular point of X, or (2) the component of B is a periodic orbit of X.

Case 1: If a component β of B is a single point, it is phaseless. If β is larger than a single point and contains a singular point of X, then β contains a saddle point p of X. The unstable manifold $W^u(p)$ is an orbit of the flow. Hence $W^u(p)$ intersects each isochron transversely. Let σ be a closed curve consisting of a segment of $W^u(p)$ and a segment of an isochron I, both τ flow units apart and chosen so that σ is not contractible in the annulus A between β and γ. Under the $-\tau$ time map of the flow Φ, σ is carried onto another closed curve lying in

$I \cup W^u(p)$. As the map $\Phi_{-\tau}$ is iterated, $\Phi(\sigma, -n\tau)$ tends toward β. Since σ is a cycle in the annulus A, it follows that β will be contained in the closure of $\bigcup \Phi(\sigma, -n t) \subset I \cup W^u(p)$. This implies that β is contained in the closure of I.

We have proved that if β contains a saddle point, then every isochron contains β in its closure.

Case 2: The boundary component β of B is a periodic orbit of X. We assume that A is a two dimensional annulus with boundary $\beta \cup \gamma$ and that every orbit of the flow in A goes from β to γ. Both β and γ are elementary limit cycles of X.

Proposition: If the period τ' of β is different from the period τ of γ, then each point of β is phaseless.

Proof: Let I be the stable manifold (isochron) of $x \in \gamma$. Consider $\bar{I} \cap \beta$. This is a closed set invariant under the time τ map of the flow. We assert that $\bar{I} \cap \beta$ is also connected. If $\bar{I} \cap \beta$ can be written as the union of two disjoint, closed sets K and L and if U and V are disjoint neighborhoods of K and L, then all but a compact portion of $I \cap A$ lies in $U \cup V$. This is impossible since I keeps returning to both U and V as one approaches β on I. Therefore $\bar{I} \cap \beta$ is connected. If $\tau' \neq n t$, then the time τ map of the flow is not the identity on β. A non-trivial rotation of the circle contains no non-trivial closed connected invariant sets. Thus, if τ' is not a multiple of τ, $\bar{I} \cap \beta = \beta$. This proves the proposition unless τ' is a multiple of τ.

If τ' is a multiple of τ but $\tau \neq \tau'$, then we can apply the above argument to the flow of the vector field $-X$, interchanging the roles of β and γ. The argument then proves that the unstable manifold of each point of β for X contains γ in its closure. We conclude that the unstable manifold of each point of β intersects the stable manifold of each point of γ. This implies the proposition.

These arguments for flows on two-dimensional manifolds combine to prove the following theorem.

Theorem C1: On a compact, oriented two dimensional manifold M, there is an open-dense set of vector fields in the space of C^r vector fields $(1 \leq r \leq \infty)$ with the property that if γ is an elementary stable limit cycle, then every point on the frontier of the stable manifold of γ is phaseless.

The theorem follows from the above discussion upon noting that the period of an elementary limit cycle is a continuous function on the set of structurally stable vector fields.

In order to deal with Theorem C2 on higher dimensional manifolds, it is useful to consider rather specific perturbations of a given flow. Let Φ be a flow with an elementary, stable limit cycle γ. We denote the stable manifold of γ by W and the frontier of W by B. There is a differmorphism $\rho: R^{n-1} \times S^1 \to W$ such that the image of $\mu(\mathbb{R}^{n-1} \times \{\theta\})$ is the stable manifold of $\mu(0, \theta)$. The map μ establishes a coordinate system on W which is well suited for our purposes. The vector field of Φ will be denoted by X.

We want to consider flows Φ' which are reparametrizations of Φ. The vector field of Φ is $f\,X=X'$ for some function f. We shall assume that f is a function of a particular sort.

Choose a neighborhood V of γ so that ∂V, the boundary of V, is smooth and transverse to γ and choose $\varepsilon>0$ a small number. We require:

(1) The function f is to be identically $(1+\varepsilon)$ on V.

(2) The function f is to be constant on each translate of ∂V under the flow Φ.

(3) There is a translate U of V so that f is identically 1 outside U.

(4) $X\cdot df>0$ on $U-\bar{V}$.

If f is chosen in this manner and if Φ' is the flow of $f\cdot X=X'$, then the isochrons of Φ' "wrap around" W in the coordinate system described above. Our meaning is described more precisely by the next lemma.

There are two functions we wish to consider. The first $\rho\colon W-V\to\mathbb{R}^+$ is defined by $\rho(z)=t$ if $\Phi'(z,t)\in\partial V$. The second is the S^1 coordinate function $\pi\colon W\to S^1$ defined by identifying S^1 and γ and mapping the isochrons of W for Φ to their intersection with γ.

Lemma: (1) Let J be an isochron of Φ' and R a level surface of ρ. Then π is constant on $J\cap R$. Equivalently $J\cap R$ lies in an isochron of Φ.

(2) Let $\alpha\colon\mathbb{R}\to W-V$ be a curve such that (i) $\alpha(0)\in\partial V$, (ii) the image of α is contained in an isochron J of Φ', and (iii) $d\rho\left(\frac{d\alpha}{ds}\right)\geq\delta$ for some $\delta>0$. Then $\frac{d\pi\circ\alpha}{ds}$ is positive and bounded away from zero in $W-U$.

Proof: The first assertion of the lemma is a direct consequence of the definition of f. There is $t\in\mathbb{R}$ and $\theta\in S'$ such that $\partial V\cap\pi^{-1}(\theta)$ flowed backwards for time t under the flow Φ' is the set $J\cap R$. In the coordinate system determined by the map μ, the component of X' in the S^1 coordinate direction is constant on sets of the form $\Phi'(\partial V\cap\pi^{-1}(\theta),t)$. This implies that π remains constant on each set of the form $\Phi'(\partial V\cap\pi^{-1}(\theta),t)$.

Consider the map $(\rho,\pi)\colon W\to\mathbb{R}\times S^1$ given by $(\rho,\pi)(z)=(\rho(z),\pi(z))$. As a consequence of the first part of the lemma, $(\rho,\pi)\,|\,J$ has a smooth curve as its image. This reduces the proof of the second part of the lemma to a two dimensional problem. Outside $\bar{V}$, we assert that the image of $(\rho,\pi)\,|\,J$ is a curve of positive slope. The reason is that the component of X' is the S^1 direction is a decreasing function of ρ in $U-V$. Moreover, $\rho(z)$ represents the Φ' time it takes z to reach ∂V. Thus, in $\Phi'(V,-\tau')-V$, the larger the value of ρ the longer it takes a point to return to another point with the same S^1 coordinate. This means that the images of isochrons of Φ' by the map (ρ,π) have positive slope outside of $\bar{V}$. The rate at which the slope changes at a point z is determined by $f(z)-(1+\varepsilon)$.

Lemma: Let K be compact neighborhood of γ in W, let I be an isochron of Φ, and let J be an isochron of Φ'. Then $I\cup J$ separates $W-K$ into at least two components. (Indeed, $I \cup J$ separates $W-K$ into a countable number of components.)

Proof: The compact set K is contained in a set $\{z \mid \rho(z)<r\}$ for some $r>0$. Thus we may assume that K is a set of this form. The image of $I \cup J$ by the map (ρ, π) is a one dimensional set which is the union of two curves. The image of I is a set of the form $\{(\rho, \theta) \mid \theta=\theta_0\}$ for some θ_0. The image of J is a curve along which $\rho(\theta)$ is a function with positive derivative bounded away from zero outside the image of U. It follows that the image of J has repeated intersections with I, and that the image of $(I \cup J)-K$ contains curves which separate $\mathbb{R} \times S^1-(\rho, \pi)(K)$. Since $I \cup J=(\rho, \pi)^{-1}((\rho, \pi)(I \cup J))$. $I \cup J$ separates $W-K$.

Let us return now to the consideration of a flow Φ and a point $z \in B$ which is not phaseless. There is an isochron I of Φ such that $z \in \bar{I}$. If Φ' is a perturbation of Φ obtained by reparametrization of Φ in the manner described above, then we assert that z is phaseless for Φ'. If U is a neighborhood of z and β is any curve connecting $B \cap \bar{U}$ to γ in $W-I$, then β must intersect each isochron J of Φ' outside a compact neighborhood of γ. This implies that each isochron J intersects U. This z is phaseless for Φ'.

Let us go farther. Consider the space Γ of vector fields which equal X in a neighborhood of the complement of $W=W^s(\gamma)$. We shall say a flow belongs to Γ if its vector field is in Γ. If z, Φ, and Φ' are as in the last lemma, we assert that z is phaseless for all flows belonging to Γ which are close enough to Φ'. Flows Φ'' near Φ' will satisfy the previous lemma. Isochrons of Φ'' will intersect the isochrons of Φ transversely; consequently, they will "wind around" S^1 in the coordinate system used in the lemma. The same argument which implies that z is phaseless of Φ' establishes that z is phaseless for Φ''. Moreover, note that the set of points in the frontier of W which are not phaseless for Φ form an open set. The perturbation Φ' and further perturbations Φ'' make all these points phaseless. This discussion is summarized by the following proposition:

Proposition: Let Γ be the space of vector fields which equal X in a neighborhood of the complement of $W=W^s(\gamma)$. If z in the frontier of W is not phaseless for the flow Φ of X, then there is a neighborhood U of z in the frontier of W, a perturbation Φ' of Φ belonging to Γ and a neighborhood $\mathscr{U}$ of Φ'' in the space of flows belonging to Γ such that if $z' \in U$ and $\Phi'' \in \mathscr{U}$, then z' is phaseless for Φ''.

With this proposition, we are finally in a position to consider theorem C2.

Proof of theorem C2: Consider a one parameter family of reparametrizations of the flow Φ of the type considered in the above lemmas with the period of γ an increasing function of the parameter. Each point of the frontier of $W^s(\gamma)$ is not phaseless for at most one value of the parameter. Moreover, if z is not phaseless for the parameter value t, then there is a neighborhood of z in the frontier of $W^s(\gamma)$ which is not phaseless for the parameter value t. With the exception of at most a countable number of parameter values, the flows in the one parameter family will have the property that all points in the frontier of $W^s(\gamma)$ will be phaseless. This proves the density assertion of theorem C2. In addition, we have proved that the dense set of flows we have found are interior points of the set of flows for which all points of the frontier of $W^s(\gamma)$ are phaseless. This proves theorem C2.

We now consider theorem C3:

Proof of theorem C3: The density assertion of theorem C3 follows from theorem C2. The openness assertion will be proved by examining the periodic orbits in the frontier of $W^s(\gamma)$. First, we reduce the openness question to one involving periodic orbits. This requires a digression concerning the qualitative structure of vector fields in Σ. Unfortunately, this discussion relies much more heavily on difficult theorems from dynamical systems than the remainder of the paper. The arguments are presented in much less detail than the previous ones.

If X is a vector field in Σ with flow Φ, then there are a finite number of compact sets $\Omega_1, \ldots, \Omega_m$ with the properties:

(1) Each Ω_i is invariant under the flow Φ and contains an orbit of Φ which is dense in Ω_i.

(2) The set of periodic orbits in Ω_i is dense in Ω_i.

(3) The set of ω-limit points of each orbit is contained in one of the sets Ω_i.

(4) Each point in Ω_i has stable and unstable manifolds of complementary dimension. The union of the (un)stable manifolds of points in Ω_i is the (un)stable set of Ω_i [5].

Denote the frontier of $W^s(\gamma)$ by B. From the first property listed above one can conclude that either Ω_i and B are disjoint or $\Omega_i \subset B$. We shall ignore those Ω_i which do not intersect B or γ. (Note that γ is one of the Ω_i.) Let β_i be a periodic orbit in Ω_i if Ω_i is larger than a single point. Then the openness assertion of the theorem reduces to the following two lemmas:

Lemma: If each β_i is phaseless for Φ, then every point of B is phaseless for Φ.

The vector fields of Σ are structurally stable. This implies that the Ω_i and the β_i vary continuously with perturbation. The periods of the β_i also vary continuously with perturbation. Therefore, there is an open-dense of vector fields in Σ for which the period of each β_i is not a multiple of the period of γ.

Lemma: If the period of β_i is not a multiple of the period of γ for the flow Φ, then the points of β_i are phaseless for Φ.

To prove theorem C3, it remains to prove these two lemmas. The proof of the second lemma follows the same argument as the proof of theorem C1. If I is an isochron of γ and τ is the period of γ, then $\bar{I} \cap \beta_i$ is a closed, connected subset of β_i invariant under the time τ map of Φ. If the time τ map of Φ is not the identity on β_i, then $\beta_i \subset \bar{I}$ for every isochron I. This proves the second lemma.

Only the proof the first lemma remains. The key observation is that the intersection of $W^u(\beta_i)$ with any isochron I is always transverse. The reason is that I is transverse to the vector field X and X is tangent to $W^u(\beta_i)$. If $x \in \beta_i$ and $y \in W^s(x)$, it follows that $\overline{W^u(y)} \supset W^u(x)$. Moreover, if $I \cap W^u(x) \neq \emptyset$, then the transversality observation made above implies that $I \cap W^u(y)$. This implies $y \in \bar{I}$. The unstable manifolds of the β_i contain dense subsets of the Ω_i. Consequently, the union of the $W^u(\beta_i)$ contains a dense subset of B. Since the set of phaseless point is closed in B, every point of B is phaseless. This finishes the proof the first lemma and theorem C3.

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