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MAKING THE TRANSITION TO FORMAL PROOF *

ABSTRACT. This study examined the cognitive difficulties that university students experience in learning to do formal mathematical proofs. Two preliminary studies and the main study were conducted in undergraduate mathematics courses at the University of Georgia in 1989. The students in these courses were majoring in mathematics or mathematics education. The data were collected primarily through daily nonparticipant observation of class, tutorial sessions with the students, and interviews with the professor and the students. An inductive analysis of the data revealed three major sources of the students' difficulties: (a) concept understanding, (b) mathematical language and notation, and (c) getting started on a proof. Also, the students' perceptions of mathematics and proof influenced their proof writing. Their difficulties with concept understanding are discussed in terms of a concept-understanding scheme involving concept definitions, concept images, and concept usage. The other major sources of difficulty are discussed in relation to this scheme.

1. INTRODUCTION

In the United States the transition to proof is abrupt. The only substantial treatment of proof in the secondary mathematics curriculum occurs in a one-year geometry course. Lower-level university mathematics consists primarily of calculus, where few, if any, written proofs are required of students. In many calculus textbooks, precise definitions of limits, ε-δ proofs, and other forms of rigor have been relegated to optional sections or removed altogether. Thus, many students begin their upper-level mathematics courses having written proofs only in high school geometry and having seen no general perspective of proof or methods of proof. Furthermore, at many colleges and universities students are expected to write proofs in real analysis, abstract algebra, and other advanced courses with no explicit instruction in how to write proofs.

This abrupt transition to proof is a source of difficulty for many students, even for those who have done superior work with ease in their lower-level mathematics courses. In order to address this problem, a growing number of universities and colleges offer transition, or bridge, courses to teach students how to effectively communicate in the language of mathematics

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and, in particular, how to write formal proofs like those required in upper-level courses. A variety of textbooks are available for these courses (for example, Bittinger, 1982; Fletcher and Patty, 1988; Morash, 1987; Smith, Eggen and St. Andre, 1990; Solow, 1990). The present study was conducted in one such course to determine the nature of students' cognitive difficulties in making this transition to proof.

Although a number of empirical studies have addressed students' difficulties with proof, many have dealt with high school geometry, and relatively few have dealt with university students. Nevertheless, the literature suggests the following areas of potential difficulty that students encounter in learning to do proofs: (a) perceptions of the nature of proof (Balacheff, 1988; Bell, 1976; Galbraith, 1981; Lewis, 1987; Schoenfeld, 1985), (b) logic and methods of proof (Bittinger, 1969; Duval, 1991; Morgan, 1972; Solow, 1990), (c) problem-solving skills (Goldberg, 1975; Schoenfeld, 1985), (d) mathematical language (Laborde, 1990; Leron, 1985; Rin, 1983) and (e) concept understanding (Dubinsky and Lewin, 1986; Hart, 1987; Tall and Vinner, 1981; Vinner and Dreyfus, 1989).

These studies suggest that the ability to read abstract mathematics and do proofs depends on a complex constellation of beliefs, knowledge, and cognitive skills. It is not at all clear, however, which of these factors are the most salient for capable undergraduate mathematics students taking a first course that emphasizes proof nor how these factors interact with one another.

2. METHODOLOGY

The purpose of the present study was not to verify an existing theory or to test a priori hypotheses. Rather, the intent was to develop a grounded theory (Glaser and Strauss, 1967) of the students' difficulties with proofs by observing the students in the context of a regular mathematics course and by attending to the professor's and the students' perspectives on the difficulties involved in learning to do proofs.

I conducted two preliminary studies and the main study in undergraduate mathematics courses at the University of Georgia during the winter, summer, and fall quarters of 1989. I conducted the first preliminary study in an introductory group theory course and the second preliminary study and the main study in a new transition course entitled An Introduction to Higher Mathematics. (After a one-year trial period, the course became a requirement for a degree in mathematics.) According to the instructor, Dr. Pierce, the major purposes of the course were to teach the students to read and do proofs and to introduce them to certain mathematical ideas that pervade advanced mathematics. The topics covered during the course included mathematical logic and methods of proof, the principle of mathematical induction, elementary set theory, relations and functions, and the real number system. The required proofs were short deductive proofs in which inferences were based largely on definitions and axioms. The professor and the textbook (Fletcher and Patty, 1988) provided all the definitions, axioms, and theorems.

Of the 16 students in the class, 8 were undergraduate mathematics majors, 6 were undergraduate mathematics education majors, and 2 were graduate mathematics majors. I selected two mathematics majors and three mathematics education majors as key participants. They represented a variety of mathematics backgrounds and abilities and were willing to meet with me for interviews and tutorial sessions outside of class.

The data were collected primarily through nonparticipant observation of class each day, interviews with the professor and the students, and tutorial sessions with the students outside of class. The inductive data analysis procedures were influenced by the constant comparative method (Glaser and Strauss, 1967).

3. FINDINGS

Dr. Pierce relied largely on direct instruction and was very conscientious in explaining the concepts and proofs in detail. He explicitly taught standard methods of proof and involved the students in class by asking questions, encouraging them to ask questions, and soliciting examples. He attempted to maintain a pace that allowed the students to understand the material, and he was readily available for help outside of class. The students' comments about the professor and the course were invariably positive.

Although some of the students' difficulties in learning to do proofs could be attributed to a lack of diligence, it appeared to me that many of their difficulties were cognitive and they would have encountered these difficulties despite diligent studying. Specifically, I found the following seven major sources of the students' difficulties in doing proofs:

D1. The students did not know the definitions, that is, they were unable to state the definitions.
D2. The students had little intuitive understanding of the concepts.
D3. The students' concept images were inadequate for doing the proofs.
D4. The students were unable, or unwilling, to generate and use their own examples.
D5. The students did not know how to use definitions to obtain the overall structure of proofs.

D6. The students were unable to understand and use mathematical language and notation.

D7. The students did not know how to begin proofs.

In addition, the students' perceptions of mathematics and proof influenced their proof-writing performance and were sometimes a hindrance to their success.

Figure 1 presents a model of the major sources of the students' difficulties in doing proofs. The boxes indicate the major areas of difficulty, and the arrows indicate that a difficulty or lack of understanding in one area led to further difficulties in another area.

The seven difficulties and the students' perceptions of mathematics and proof are discussed in terms of the concept-understanding scheme illustrated in Figure 1. Because D1-D5 are directly related to the three parts of this scheme, the discussion will focus on these five difficulties and the students' perceptions of proof. Further discussion of D6 and D7 and other findings can be found in Moore (1990).

3.1. *The Concept-Understanding Scheme*

Vinner and others (Dreyfus, 1990; Tall and Vinner, 1981; Vinner, 1983; Vinner and Dreyfus, 1989) have distinguished between the definition of a mathematical concept, the *concept definition*, and the cognitive structure in an individual's mind associated with the concept, the *concept image.* The former refers to a formal verbal definition that accurately explains the concept in a noncircular way, as might be found in a mathematics textbook, whereas the term *concept image* refers to the set of all mental pictures that one associates with the concept, together with the all the properties characterizing them. The concept image is derived from the examples, diagrams, graphs, symbols, and other experiences one has with the concept.

Whereas Vinner and others have been interested in the differences between the set of objects determined by the concept definition and the set of objects determined by one's concept image, I found the distinction useful in clarifying the different ways in which one must understand mathematical concepts to use them in proofs. Furthermore, the data revealed a third aspect of concept understanding, *concept usage*, which refers to the ways one operates with the concept in generating or using examples or in doing proofs. The term *concept-understanding scheme* refers to these three aspects of a concept: definition, image, and usage. This scheme was evident in the data from the professor, from the textbook, and from the

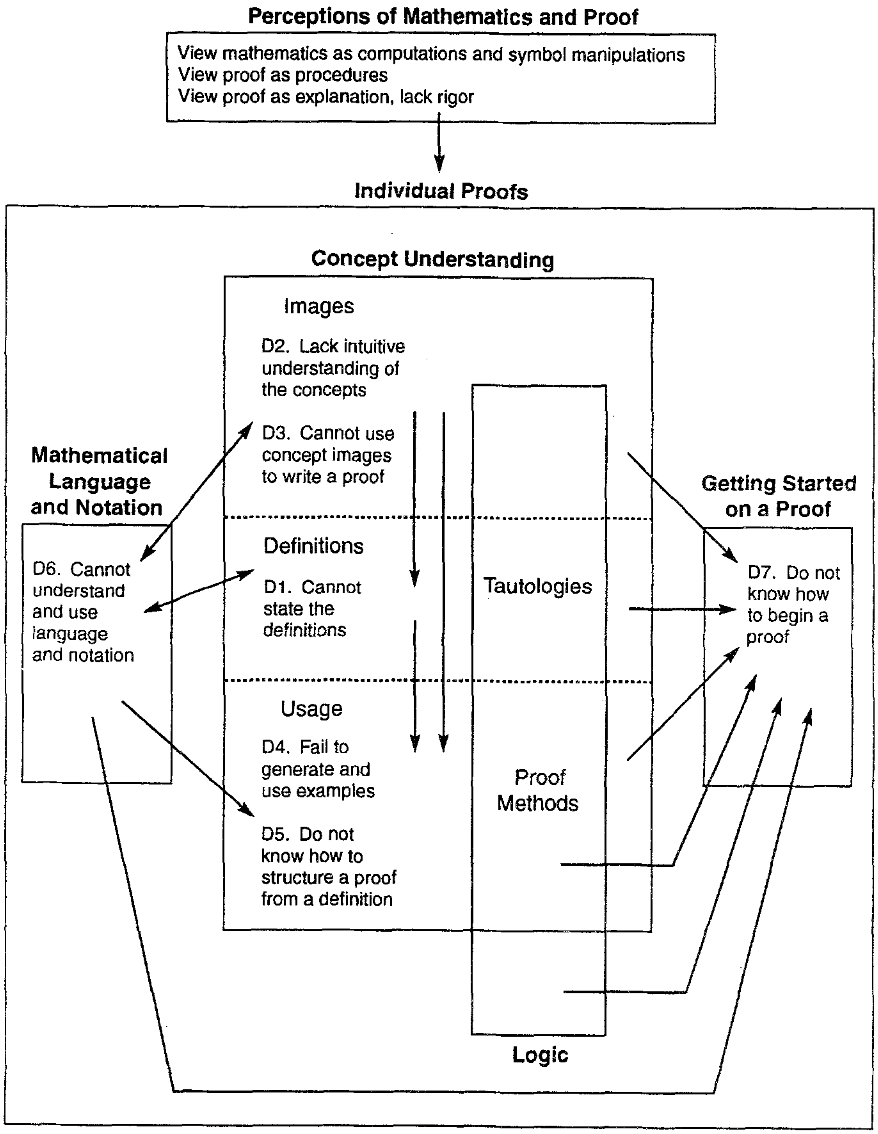

Fig. 1. Model of the major sources of the students' difficulties in doing proofs.

students, and it was useful in explaining many of the students' difficulties with proofs in the course.

As an illustration of the concept-understanding scheme, consider the *equivalence class* concept.

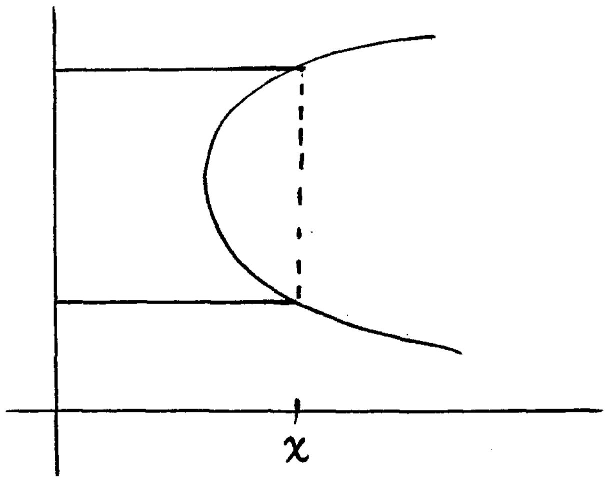

Fig. 2. Cherie's diagram for [x].

Definition. Both the textbook and the professor gave the following definition:

Let R be an equivalence relation on a set S, and let $x \in S$. The *equivalence class of* x, denoted $R[x]$ (or simply $[x]$ when R is understood), is the set $\{y \in S : (x,y) \in R\}$. (Fletcher and Patty, 1988)

The professor also defined $R[x]$ in words without using set notation: $R[x]$ is the set of all second coordinates of ordered pairs belonging to R having first coordinate x.

Image. The professor used a "family" metaphor to lead up to the equivalence class concept. Before introducing equivalence relations, he defined $R[x]$ for a relation R (not necessarily an equivalence relation) that is a subset of $A \times B$ as the set $\{y \in B : (x,y) \in R\}$ and called it the *family of* x. Thus $R[x]$ contains all the "relatives" of x; it is all things related to x.

The textbook authors suggested that an equivalence class be thought of as a box:

The set $[x]$ is called the *equivalence class of* x (but "box-x" is shorter and suggests the right way to think about $[x]$; it is a set after all, a box so to speak filled with x and all its relatives). (Fletcher and Patty, 1988, p. 94)

When I asked Cherie, one of the five key students, what $[x]$ is, she replied, "That's wherever x is, you're looking for the y values." She drew a picture like Figure 2 to illustrate how she thought about equivalence class.

Usage. There are several aspects of working with equivalence classes in proofs that students must learn. First, the definition says an equivalence class is a set, and so they can work with equivalence classes in proofs the way they work with sets. Second, in order to use the defining property, students must know how to use the notation. They must know that the expressions $z \in R[x]$, $(x,z) \in R$, $x\ R\ z$, and $x \simeq z$ all mean the same thing and be able to choose the best symbols for the particular task at hand. Third, one definition may be easier to use than another, and the students must learn how to use the various mathematically equivalent definitions of a concept and choose among them for different tasks.

Although the professor did not explicitly think in terms of this concept-understanding scheme, on many occasions throughout the course his instruction revealed the three aspects of the scheme. His comments to the class following the third test provide another example of the scheme. He began by telling the class that most of them had had trouble showing the function $f{:}\mathbb{R} \rightarrow \mathbb{R}$ given by $f(x) = 2x + 3$ maps $\mathbb{R}$ onto $\mathbb{R}$.

> You want to show that f maps $\mathbb{R}$ onto $\mathbb{R}$. What does that mean? Informally it means that everything in $\mathbb{R}$ gets hit. How do we say that formally? It means this: $\forall y\ \exists x\ (f(x) = y)$. Every y comes from some x – that's what being *onto* means. So for any y you have to find an x. So the way to say it is: Let $y \in \mathbb{R}$ be given. And then your task is either to show that an x exists, by some fancy method, or just exhibit the x that works.

Notice that he began with an informal explanation of *onto*, then gave the definition, then gave a second informal explanation, and finally explained how to use the definition in a proof.

The scheme applies to certain aspects of mathematical logic as well as to definitions. Tautologies play a role similar to that of definitions. The methods of direct proof, contrapositive proof, and contradiction proof are the usage aspect of logic because these methods show how to use the tautologies to write proofs.

3.2. *Concept Definitions*

While tutoring the students outside of class, I observed a number of instances in which they were unable to state important definitions that had been emphasized in class (D1). For example, just before the second test Cherie and Alan could not define the terms *relation* and *equivalence class of x*, even though these topics had been studied in class each day for more than a week. Because most of the proofs throughout the course depended largely on the definitions, not knowing the definitions was often a reason for the students' failure to produce a proof.

A common reason for the students' difficulties with the definitions was that the concepts were seen as abstract by the students. They had difficulty finding or creating mental pictures of the concepts, and without an informal understanding of a concept they could not learn the written form of the definition.

Several of the terms that the students seemed to have trouble learning were defined using set-builder notation. The notation used in those definitions involves both symbols and language, including quantifiers, that the students had trouble understanding. For example, Cherie could not understand the set-builder definitions of *image* and *inverse image* of a set under a function without first gaining an informal understanding of those concepts through examples and diagrams. Furthermore, a student's concept image may be sufficiently different from the definition that he or she cannot translate the image into written symbols.

The students' beliefs about mathematics and proof offer another explanation for why they neglected to learn definitions as they should have. Some students did not sense the important distinction between precise definitions and informal explanations based on concept images. The clearest example of this lack of sensitivity to the precision required in definitions occurred in an interview with Vicki, a senior mathematics major, following the final exam of the summer course. One exam item required the completion of the following definition:

The *union* of an indexed family of sets $\{A_\alpha : \alpha \in \Lambda\}$ is $\{x : ...\}$.

(An acceptable answer is $\{x : x \in A_\alpha$ for some $\alpha \in \Lambda\}$).

As we discussed her work on this item, she complained about having trouble with writing out definitions. She felt that she should receive credit when she "understood" a definition but could not write it out in correct notation.

V: And this one I didn't understand. When he starts talking about families of sets, that throws me. I know what *union* means, but – I wrote that I knew what it meant here [points to a diagram she had drawn in the margin of her exam paper], but I wrote that I'm confused how he's giving –. See, it says he wants the definition of *union*, and I know what that means. I can picture it. But then when he puts it like this [on the exam], I really get confused. It's just the notation.

I: How do you picture it?

V: How do I picture what *union* is? I drew a picture like *A* union this [points to her diagram]. I think of an *x*. The intersection is whatever's in both sets. Like say there's a 1, 2, and 3 in this and a 4 and a 5 in this set. The intersection would be null or empty set, but the union would be 1, 2, 3, 4, and 5 – everything in both sets. See, that's the definition, and I don't –.

I: What if you had 10 of these circles?

V: Oh, well, then that depends on how you wrote it. If you had 10 circles and they were labeled *A*, *B*, *C*, *D*, and *E*, the union of *B*, *C* would be everything in *B* and everything in *C*.

I: What's the union of all 10 of them?

V: Everything, in all 10 of them. See, and I feel like I really understand it. I feel like it's not fair just because the notation throws me I lose 6 points. But I guess that's the idea. I have to be able to understand the notation.

Vicki felt that her concept image of *union* served as an adequate definition and that having to know the notation was a burden added to the task of learning the definition rather than an essential part of the definition. Thus, her perception of proof and rigor influenced what she learned – or did not learn – and how she thought about the concept, and because her perception differed from the professor's, she did not meet his expectations.

3.3. *Concept Images*

In many cases the students were unable to do a proof because they did not understand the theorem or the concepts involved. They could not produce a proof by working formally with logic and definitions but needed intuitive understanding before they could get started (D2).

In both preliminary studies and the main study, the students fervently wanted and clearly needed examples to help them understand mathematical concepts and do proofs. Dr. Pierce observed that the students wanted two types of examples: (a) illustrations of definitions or concepts and (b) worked problems. It was through the examples, particularly those of the first type, presented by the professor and by the textbook that the students were able to build their concept images and, subsequently, their understanding of the definitions and notation.

Even though examples, concept images, and informal approaches were helpful, and often necessary, for *discovering* a proof, they did not guarantee that a student could *write* a correct proof (D3). The students had to know definitions and be able to use them not simply because that was the required level of rigor in the course but because concept images were inadequate for several aspects of the proving process.

1. Concept images lack the language needed to express mathematical ideas. The students often commented that they "understood" a proof, or a step in a proof, but did not know how to say it. Definitions provide the language – the words and symbols – for writing a proof.

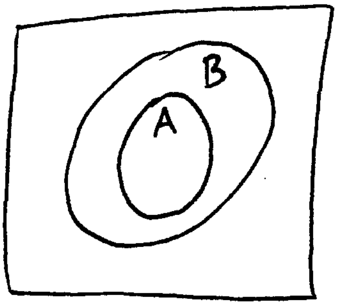

Fig. 3. Ellen's diagram for her proof.

2. Concept images do not supply the individual steps in a proof, whereas definitions often suggest the sequence of individual steps and provide the justification for each step.
3. Concept images do not reveal the logical structure of a proof, as definitions do.

These three points are illustrated below in the episode with Ellen and are discussed further in the subsequent section on concept usage because they accounted for many of the students' failures to do proofs.

Ellen's Set Theory Proof. An item on the first test was a set theory proof:

Prove that if A and B are sets satisfying $A \cap B = A$, then $A \cup B = B$.

Ellen, a mathematics education major who had had several upper-level mathematics courses requiring proofs, was one of several students who gave an intuitive argument. Here is her proof as she wrote it on her test paper. She began by drawing the diagram in Figure 3 in the left-hand margin of her paper.

$A \cap B = A$ says – by definition of intersection – that the members of A and the members of B that are the same are all the members of A. Therefore by definition of subset $A \subseteq B$. If A is a subset of B all of its members are contained in B. When there is a union of a set and it's subset the union then includes the whole set. Therefore $A \cup B = B$.

As had been agreed upon in class, the professor wanted a proof based only on definitions, axioms, previously proved results, and rules of inference. Thus, the proof should have shown $A \cup B \subseteq B$ and $B \subseteq A \cup B$ and explicitly used quantifiers and the precise definitions of *set equality*, *subset*, *intersection*, and *union* to justify each step.

In contrast to the professor's expectation, Ellen's proof was based on her intuitive understanding of set equality, subset, intersection, and union and used her own informal language. She argued informally that $A \subseteq B$

and then claimed that "when there is a union of a set and it's [*sic*] subset the union then includes the whole set" – a claim that was intuitively clear but should have been proven from the definitions and axioms.

Her intuitive notions and the overall strategy of her proof were correct, but she did not use the language and rules of inference that had been agreed upon in class. In the professor's words, she had not learned "the language and culture of how we write these things down." Her concept images were helpful in understanding the proposition and the concepts involved in its proof but were inadequate for presenting an acceptable proof at the required level of rigor. With reference to the three points listed above, the definitions as the professor gave them in class would have provided the language and notation for a proof, the individual steps and their justification, and an overall structure for the proof. Ellen needed to go beyond merely giving an explanation and learn to use definitions in these ways.

The professor's comment about the "language and culture" of mathematics suggests that Ellen not only lacked an understanding of the definitions and how to use them in a proof (language), but also that her perception of proof and rigor (culture) were inconsistent with his. I talked with her about her proof.

I: Why did you write your proof as you did?
E: I thought about it, drew a picture, and then wrote an explanation.
I: Would you have given a proof like Dr. Pierce's if you had realized what he wanted?
E: I didn't think to show each set was a subset of the other.
I: If you had thought of it that way, do you think you could have done it?
E: (hesitates) I don't know. Maybe.
I: What do you think was wrong with your proof? Why did Dr. Pierce not give you full credit?
E: I didn't explain it well enough.

Not only were Ellen's concept images of the set theory terms inadequate for writing a proof, but also her concept image of *proof* as explanation was inadequate for meeting the professor's standards of rigor.

The episodes with Vicki and Ellen illustrate the role of the students' perceptions of proof in their proof writing and learning of mathematics. Although they were both seniors and had previously taken university mathematics courses requiring proofs, neither of them had grasped the significance of precise language and careful reasoning. In contrast to the professor's notion of rigor, they seemed to believe that an explanation is adequate as long as it makes sense, rather than because it follows accepted rules of logic.

3.4. *Concept Usage*

The preceding section argued that the students' concept images were deficient in several operational aspects of the proving process. This section develops that point further by distinguishing three ways of operating with definitions and theorems in doing proofs: (a) generating and using examples, (b) applying definitions within proofs, and (c) using definitions to structure proofs. On many occasions the students were unable to do proofs because they did not know how to operate with definitions in these ways.

3.4.1. *Generating and Using Examples*

Throughout the course the professor stressed the need to generate and use examples for understanding concepts, definitions, theorems, problems, and notation, and for discovering proofs. But the process of generating one's own examples demands cognitive skills different from those involved in studying examples given by the professor or the textbook, and so it was an inability to generate and use examples that particularly distinguished the students' mathematical activity from that of the professor and hindered their progress in understanding concepts and doing proofs (D4).

One reason for the students' inability to generate and use examples is that they had only a limited repertoire of domain-specific knowledge from which to pull examples. In contrast, the professor already possessed knowledge of examples for many of the concepts and theorems. He could use these examples without having to generate any new ones. Furthermore, he knew techniques for generating examples. He told the students to be "cheap", that is, find examples that are as simple as possible, but on several occasions when I observed the students making up examples, they created examples that were more complicated than they could manage.

3.4.2. *Applying Definitions Within a Proof*

A second way of operating with definitions is to use them to suggest or justify individual steps in a proof. Definitions also supply the language, the verbal expression, for the steps in a proof. The students often commented that they had found a proof, they knew how it should go, but they did not know how to say it. In some of these instances it appeared that they lacked the ability to take the language and notation from a definition and use it as a line in the proof.

3.4.3. *Using Definitions to Structure Proofs*

A third way in which students must operate with definitions in doing proofs is to obtain the overall structure of a proof from a definition. By overall structure I mean the organization or skeleton of a proof, particularly how it begins, how it ends, and how the beginning is linked to the ending by rules of logic and a definition, axiom, or theorem. In Ellen's case, the professor wanted a proof based on the definition of set equality, which dictated that two subset relations were to be shown.

As another example, consider the definition of one-to-one: A function f is *one-to-one* if and only if for all x and for all y in the domain of f, if $f(x) = f(y)$ then $x = y$. This definition gives a strategy for proving that a function f is one-to-one: Let x and y be fixed but arbitrary members of the domain of f, assume that $f(x) = f(y)$, and then show, using these assumptions and other available information, that $x = y$. Thus, this form of the definition reveals the structure and logic (i.e., the universal quantifiers and the implication) of a proof that f is one-to-one. In particular, it shows how such a proof should begin (with $f(x) = f(y)$) and end (with $x = y$).

Not all definitions of one-to-one so directly suggest how to prove that a function is one-to-one. Taking the ordered pair perspective of functions, the textbook (Fletcher and Patty, 1988) gives this definition: "A function f is said to be a *one-to-one* function provided that no two distinct members of f have the same second term" (p. 115). This definition may be better for developing a concept image and for checking examples and nonexamples, particularly graphs in the Cartesian plane, whereas the statement that "if $f(x_1) = f(x_2)$ then $x_1 = x_2$" provides a general method, or structure, for proving that a function is one-to-one. Although a mathematician may not notice the distinction between the two equivalent definitions, a novice student who is learning to do proofs may not even see the connection between the two definitions.

In some instances students knew a definition and could explain it informally but could not use the definition to write a proof (D5). Linda, a senior mathematics major, knew that "a function f is *one-to-one* provided that no two distinct members of f have the same second term," and she was able to give examples and nonexamples, but she did not know how write a proof that a function is one-to-one. Even after I led her to the alternative definition – $\forall x_1 \forall x_2 [f(x_1) = f(x_2) \rightarrow x_1 = x_2]$ – and explained its connection with the other definition, she could not tell me how to prove that a function is one-to-one.

In many of the instances where the students appeared not to know how to use a definition to structure a proof, there was another confounding factor: They were confused by the hypothesis of the proposition. For example, consider the following proposition, which was on the final exam:

Let f and g be functions on A. If $f \circ g$ is one-to-one, then g is one-to-one.

All but one student incorrectly attempted to begin the proof with the hypothesis – $f \circ g$ is one-to-one – rather than by assuming that $g(x) = g(y)$

for fixed but arbitrary x and y in A. It is not clear to what extent their failure to prove the proposition was because they did not know how to show that a function is one-to-one, as opposed to being confused by the hypothesis and not knowing how to begin the proof. In either case, however, the students did not know how to use their mathematical knowledge to produce a proof.

4. DISCUSSION

4.1. *Interactions Within the Scheme*

The sequence "Images → Definitions → Usage" within the Concept Understanding box in Figure 1 illustrates that the students' ability to use the definitions in proofs depended on their knowledge of the formal definitions, which in turn depended on their informal concept images. The students often needed to develop their concept images through examples, diagrams, graphs, and other means before they could understand the formal verbal or symbolic definitions. It seems that this reliance on concept images for understanding definitions and notation may diminish as the students move beyond this transition point in their learning of mathematics and become more comfortable with standard notation, mathematical grammar and syntax, and the logical structure of proofs.

The sequence "Images → Usage" illustrates that the students' ability to use the definitions sometimes depended on their informal understanding of the concepts even when they knew the definitions. They were unable to do the proofs by formally manipulating the symbols and language. On the other hand, even well-developed concept images were inadequate for writing a proof. Also, the students seemed less dependent on their concept images when the proofs were rather algorithmic, as were most of the induction proofs.

An observation not illustrated in Figure 1 is that the students' cognitive structures differed from the professor's. Whereas their understanding of a concept appeared to be organized into separate schemata corresponding to their mental images, the different definitions for the concept, and the procedures for using the definitions, the professor appeared to have all of this knowledge organized into one schema. He was able to move easily among the parts of the schema to facilitate a particular task. Also, the superior organization of his knowledge structures reduced the cognitive load, whereas the students often suffered from cognitive overload. Furthermore, in comparison to the students he had more domain-specific knowledge, from which he could select examples and other information, and more general knowledge, which enabled him to apply general mathematical processes to particular proof tasks.

4.2. *Interactions Between the Scheme and the Other Aspects of the Model*

The many arrows emanating from the Concept Understanding box suggest how other aspects of doing proofs depended on the three ways of understanding mathematical concepts.

The arrows pointing from the Mathematical Language and Notation box to the Concept Understanding box show that the students' lack of understanding of the language and notation inhibited their ability to understand, remember, and use the definitions. The professor's flexibility with multiple definitions and notations contrasted with the students' rigid adherence to a single definition or notation. These observations agree with Laborde's (1990) comment that "the teaching of mathematics is faced with the apparent contradiction that language is needed to introduce students to new notions and that language may turn out to be an obstacle to students' understanding" (p. 69). On the other hand, the arrows pointing to the Mathematical Language and Notation box illustrate that the definitions and images played an important role in the students' learning of the language and notation. The definitions provided practice in reading and writing formal mathematics, which helped them learn the words, the grammar, and the symbol system.

Many times throughout the course I observed that the students were stuck at the very beginning of a proof, but frequently they were able to do the proof after I helped them get started. Figure 1 dramatically shows how their inability to get started on a proof was symptomatic of many other difficulties. The sources of those difficulties included deficiencies in all three aspects of concept understanding, a lack of knowledge of logic and methods of proof, and linguistic and notational barriers.

4.3. *Perceptions of Mathematics and Proof*

The data suggested that the students' perceptions of mathematics and proof influenced not so much their ability to do individual proofs as the kind of proof they produced, that is, the level of rigor they considered adequate. Thus, in Figure 1, the Perceptions of Mathematics and Proof box appears at a level above that of individual proofs.

As with other mathematical concepts, the concept-understanding scheme is useful in thinking about proof itself. Early in the course the professor defined proof as a logical sequence of statements leading from a hypothesis to a conclusion using only axioms, definitions, previously proved results, and rules of inference. But his goals were not that the students be able to define proof; rather, he wanted them to learn to recognize a valid proof (concept image) and learn to do proofs (concept usage).

Although the students probably could not give a definition of proof, they had concept images of proof. The concept image of some students was that of proof as explanation, whereas for others proof was a procedure, a sequence of steps that one performs. It was not clear to what extent the students viewed proof as a piece of mathematical knowledge, an object. Dreyfus (1990) noted that no studies have investigated this issue of the transition between proof as a process and proof as an object.

As for the usage aspect of proof, the students learned to do proofs in the course, and they had some limited notions of the purpose of proof, but they probably were not ready to use proof and deductive reasoning as tools for solving mathematical problems and developing mathematical knowledge. This latter claim agrees with Schoenfeld's (1985) observations of the empirical nature of students' beliefs about mathematics and their failure to use deductive reasoning as a mathematical tool.

5. CONCLUSION

Although the short deductive proofs required in the course represent only a narrow aspect of mathematical proof and the process of proving, the findings are significant because they address an important transition point for undergraduate students. Even apparently trivial proofs are often major challenges for them at this point. Until proof is integrated throughout the school and university mathematics curricula in the United States, I believe the abrupt transition to proof will continue to be a source of frustration for undergraduate students and teachers.

The qualitative methodology used in the study and the restricted nature of proof addressed by the course limit the application of the model to other mathematics courses. For the following reasons, however, the concept-understanding scheme may be valid and useful in other mathematical situations. First, the scheme emerged not only from the data collected from the students, but to a large extent also from the professor and the textbook. Dr. Pierce expressed his agreement with the findings and later told me he found the scheme useful in teaching linear algebra.

Second, several students in the transition course had previously taken upper-level courses requiring proofs. All of them said they had relied on memorizing proofs because they had not understood what a proof is nor how to write one. The model presented here reveals at least a portion of what they were lacking and suggests how we might help students recover from similar states of confusion.

Third, although proving requires more complex cognitive processes than those evident in this study, certainly the reliance on careful reasoning, precise definitions and language, and standard methods of proof are necessary in more advanced courses. A retrospective look at the data from the group theory course of the first preliminary study found the scheme there, but the students appeared to be overwhelmed by the necessity of grappling with difficult group theory concepts, problem solving, abstraction, and generalization while learning what a proof is and how to write one. A transition course on mathematical language and proof would have reduced their cognitive load in subsequent upper-level courses while also preparing them for the formal mathematical approach used in those courses.

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