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GROWTH IN MATHEMATICAL UNDERSTANDING: HOW CAN WE CHARACTERISE IT AND HOW CAN WE REPRESENT IT?

ABSTRACT. There has been a variety of approaches to the study of mathematical understanding, and some of these are reviewed before outlining the background to the model we are proposing for the growth of such understanding. The model is explained in detail and illustrated with reference to the concept of fractions. Key features of the model include 'don't need' boundaries, 'folding back', and the complementarities of 'acting' and 'expressing' that occur at each level of understanding. The theory is illustrated by examples of pupils' work from a variety of topics and stages. Finally one of the practical applications of the theory, mapping, is explained in some detail.

BACKGROUND TO THE THEORY

There is currently much practical interest in mathematical understanding. Curriculum reform advocates in many countries cite the need for teaching mathematics with understanding. Conference proceedings and psychological and artificial intelligence literature all exhibit interest in learning and teaching with understanding. Characterising understanding in a way which highlights its growth, and identifying pedagogical acts which sponsor it, however, represent continuing problems.

There has been a wide variety in the approaches to attempting to capture the essence of the phenomenon, and we have reviewed this in detail in Kieren and Pirie (1991) and Pirie and Kieren (1992a). Various categories of understanding, including relational and instrumental, concrete and symbolic, and intuitive and formal, have been proposed (Skemp, 1976; Herscovics and Bergeron, 1988; Schroder, 1987). Alternative views of understanding in relation to cognitive obstacles (Serpinska, 1990) or in terms of mental objects and connections among them (Ohlsson, 1988) have been proposed. Pirie (1988) has called into question the use of categories in characterising the growth of understanding as it can actually be seen by an observer. She observed understanding as a whole dynamic process and not as a single or multi-valued acquisition, nor as a linear combination of knowledge categories.

It was our wish, to better describe this growth of mathematical understanding in the children that we observed in classrooms over time, that led to the development of the ideas for our theory. It was clear to us that the children we were observing exhibited some understanding of mathematics, and so our question became: 'What *is* mathematical understanding?' Our background thinking was further stimulated by the biological theory of cognition in self-referencing systems (Maturana and Varela, 1980, 1987; Tomm, 1989). Over the past three years we have discussed

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our developing theory in a variety of forums (see bibliography). It is a theory of the growth of mathematical understanding as a whole, dynamic, levelled but non-linear, transcendently recursive process (Kieren and Pirie, 1991). This theory attempts to elaborate in detail the constructivist definition of understanding as a continuing process of organising one's knowledge structures (von Glasersfeld, 1987).

In this paper we intend to present the theory with illustrative examples and elaborate on some of its features. We will then suggest possible practical applications for the theory, and look in detail at one of these, namely mapping the growth of a child's understanding.

A MODEL FOR THE THEORY

We first published a description of our theory in 1989. Since then the fundamental structure of the model has not changed but, if you have followed our previous work, you will see that we have altered some of our labels in response to suggestions and reactions to conference presentations (Pirie and Kieren, 1989b; Pirie and Kieren, 1990). When seeking to provide labels for new conceptions one is faced with the dilemma of either choosing existent words which one can hope already convey some of the desired meaning or creating new terminology and then attempting to invest in it associations and connotations that will carry the new ideas to the reader. Both alternatives have inherent weaknesses; words already familiar to the reader may inhibit the accretion of extra meaning and allow the criticism of ideas from an inappropriate stand point. Novel words, on the other hand, bring none of the subtle background that may be needed as a foundation for new concepts. In order, as far as possible, to avoid confusion or misunderstanding in our readers, we have picked the "labels for their key categories following a contiguity relation between the concept ... in mind and one specific of the many facets of meaning ascribed to the word in everyday use". Naturally this both assists understanding of our ideas and unfortunately also "gives rise to the illusion of an easy meta-basis for criticism... against the theory from outside" (Bauersfeld, 1988). We have, for example, used the word 'image' in the labelling of two of the levels. Since evidence at these levels is frequently based on pictorial representation we run the risk that understanding at these levels is judged to be restricted to only this mode of expression, and not seen to also encompass mental imagery. We feel, however, that the concept of mental objects is firmly enough established to be comprehended within our theory. We feel that 'image' is less open to ambiguity than, say, 'idea', which also carries a little of what we wish to describe. On the other hand, the outermost level was originally labelled 'inventing', but this gave rise to criticisms that we emphatically wished to refute. We did not wish to imply that children do not 'invent' at other levels. Indeed they do. What we want to point to is a special, new activity and we feel, with Hadamard (1945), that here "the creation of a word may be and often is a scientific fact of very great importance."

Before defining the terms in Fig. 1 or describing its key properties, we offer

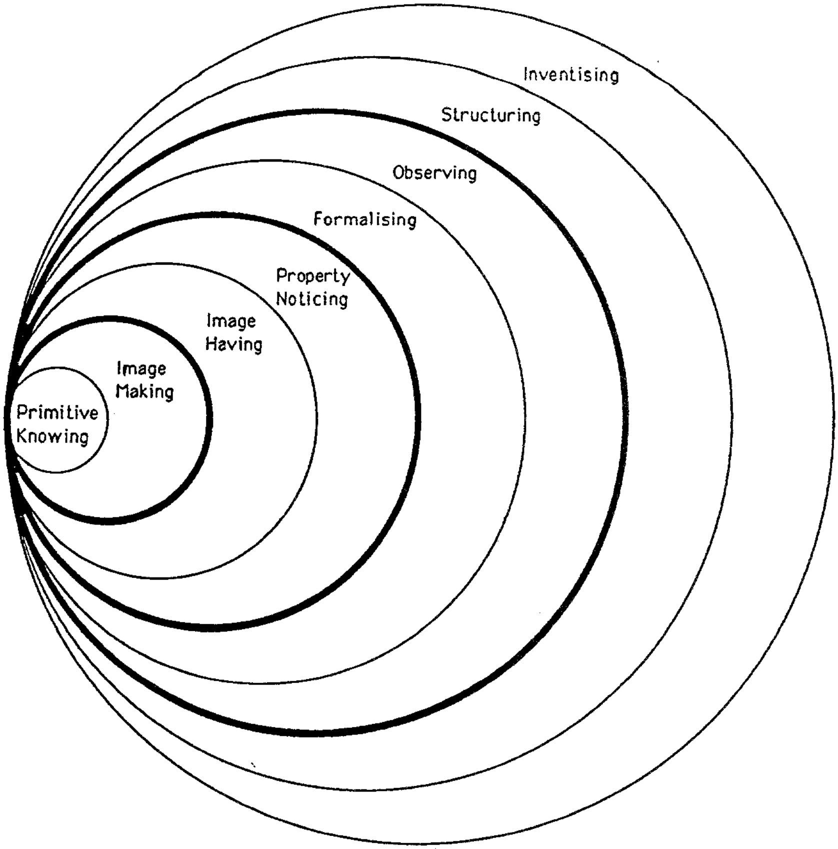


Fig. 1.

a narrative of one student's knowledge building and understanding activities as she came to understand fractions as additive quantities. (The narrative is based substantially on transcripts and student work of a class of twelve year olds. In the story of Teresa below, the work of one student in a particular fraction learning space has been extended to include possible 'observing' and 'structuring' activities.)

Teresa had been introduced in the previous school year to the term 'equivalence of fractions' and to addition of fractions, and had been given and practised using symbolic 'rules' for generating equivalent fractions and for adding them. Prior to the instruction described below, Teresa had exhibited the ability to generate fractions equivalent to 2/3, but, like most of her class, could not 'correctly' add two fractions. She said, "I think you just add the tops and the bottoms." When faced with adding more than two fractions Teresa said: "I don't know how to do that." When tested, Teresa demonstrated that she could physically make models of and identify fractions with any denominator, (1), which confirmed the teacher's assumptions that the students had usable fraction language, but did not understand the addition of fractional quantities. After the work on constructing models of fractions (through folding), Teresa's class was given a kit containing rectangles, based on a common standard sheet as a unit, representing halves, thirds, fourths, sixths, eighths, twelfths, and twenty-fourths. Teresa was given tasks such as the following.

> Using your kit notice that one fourth, three eights, and two sixteenths together exactly cover three fourths or that taken together one fourth, three eighths, and two sixteenths are equal in amount to three fourths. We can write $1/4 + 3/8 + 2/16 = 3/4$. Use your kit to find as many quantities or combinations of quantities that make exactly three fourths as you can. Draw diagrams of your findings. Write fractional number sentences like the one above.

Teresa, working with two partners, engaged in such an activity (2). Very quickly, however, Teresa internalised the activity, coming to think of fractions as numbers which described amounts (3). Teresa saw that many fractions or combinations could make up the same amount. An observer could say that Teresa's idea of addition was 'Fit the addends onto known quantities'. Thus, Teresa now could 'add' $1/3 + 1/6 + 6/12$. She thought of it as $1/3 + (1/6 + 2/12) + 4/12$. On this and many other such items Teresa identified fractional amounts or pieces on which the addends fitted or could be reconfigured to fit. It was later observed (4) that Teresa had developed a persistent and powerful strategy that when faced with complex additive (or subtractive) situations sometimes involving as many as a dozen fractional quantities, Teresa would peruse the addends looking for combinations that made up one, or one half or some other single amount.

Teresa's idea of addition was, of course, not standard and not applicable to many situations. Thus while she knew that $1/2 + 1/3 + 1/4$ was more than one, she said she couldn't "fit them" on anything. Her teacher suggested that she and some of her classmates should see if they could, given two or three fractional pieces, find one other kind of piece, replicates of which would cover all of the given pieces. At first, Teresa could not predict which piece might work, but quickly came to be reasonably skilful at it. This led to a transformation (but as seen in (4) above, not an elimination) of her previous idea of addition. Now Teresa said, "You can do $2/3 + 5/6$ because twelfths fit on both." Very quickly, Teresa, using her knowledge of equivalence, found a way of combining this new idea of addition with her knowledge of equivalence (5). When faced with the question,

> If you have an imaginary fraction kit; it has halves, fourths, fifths, tenths and twentieths, what is $1/2 + 3/4 + 2/5 + 7/10$?

Teresa says: "Twentieths will fit on all of them. Two times ten makes twenty, so one times ten or ten twentieths. Four times five makes twenty so three times five is fifteen twentieths..." Based on this kind of local, context based know-how, the majority of Teresa's classmates could by now 'add' sets of many fractions. Teresa, however, went beyond this rather concrete idea of addition, making statements like: "Addition is easy. You can make up the right kind of fractions just by multiplying the denominators and then just get the right numerators by multiplying by the right amounts. Like if you had sixths and thirds and sevenths, thirds and

$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{12}$	$\frac{1}{24}$
1	0	1	0	0
1	0	0	2	0
1	0	0	1	2
1	0	0	0	4
0	2	0	0	0
0	1	2	0	0
0	1	1	2	0
0	1	1	1	2
0	1	1	0	4
.	.	.	.	.
.	.	.	.	.
.	.	.	.	.

Fig. 2.

sixths go together and then forty seconds work for all because sixths times sevenths give forty seconds. That will be the denominator" (6). Notice that this method is not based on particular pieces but is a method which applies independently of her previous actions. Notice also that Teresa intends that this method work for "all" fractional numbers.

When faced with a situation which involved subtraction, Teresa easily developed strategies to accomplish such tasks varying from using her concrete addition strategy subtractively to making up a method for subtraction (7).

With her classmates, Teresa then worked on the task:

> Using halves, thirds, sixths, twelfths and twenty fourths, make two thirds in as many ways as you can. You can use this chart to help you keep track.

At first Teresa got out her kit and started covering pieces, but she quickly abandoned the kit and started to systematically fill in the chart (8) (see Fig. 2). Once again she had made up and used a method which made no reference to actions; it just followed a symbolic pattern.

At this point, Teresa declared that there should be an exact predictable number of combinations for two thirds or indeed for any fraction (9). She now tried one sixth, one third, and one and then made up a formula which predicted the number of combinations of the fraction set which would make up a given fraction. She tested this by making charts for 4/3, 5/6, and 5/3. She said, "I bet I can predict it for one half, too." Later, Teresa and two colleagues worked on seeing how one could make up and verify general patterns which would relate a given quantity to a fraction set and combinations of that set to the quantity (10). She was on her way to working on partition theory.

We will now use the above narrative as illustration, while we define elements of our theory and describe some properties of it. There are eight potential levels or distinct modes within the growth of understanding for a specific person, on any specific topic and we will illustrate them with reference to Teresa's growth in understanding of additions of fractions.

The process of coming to understand starts at a level we call *primitive knowing*. Primitive here does not imply low level mathematics, but is rather the starting place for the growth of any particular mathematical understanding. It is what the observer, the teacher or researcher assumes the person doing the understanding can do initially. For the growth of initial understanding of addition of fractions, the teacher wished to assume that the students already knew the language and construction of individual fractions. In the story above at (1) he was testing and probing his assumptions about Teresa's basic fraction knowing and capability. Of course, one cannot ever know what this primitive knowledge is in full. From Teresa's point of view it was at least her usable knowledge of fraction words, equivalence, and part-part-whole reasoning.

At the second level, the learner is asked to make distinctions in previous knowing and use it in new ways. In the narrative above, Teresa at (2) used previous part-part-whole knowing to combine fractional quantities into other such quantities. It was the purpose of this activity to occasion Teresa's using of fractions in an additive manner and to record and reflect on those actions. We call this mode of understanding *image making*.

At (3) above we find that Teresa can act additively with fractions without having to act on the objects. We call this activity *image having*. Notice that the original probing by the teacher into Teresa's initial understanding revealed that she did already have an image, albeit an erroneous one, for addition of fractions. This has now been supplanted by a new image formed as a result of the image making activities suggested by the teacher. At the level of image having a person can use a mental construct about a topic without having to do the particular activities which brought it about. Teresa was freed from the need to perform particular physical actions in order to solve fraction addition problems. She now had characterised, developed, and brought forth her sense of the meaning of addition of fractions.

A fourth level or mode of understanding occurs when one can manipulate or combine aspects of ones images to construct context specific, relevant properties. In (5) above we described Teresa using her image of addition as finding subparts which fit and her idea of equivalent fractions to generate a means of performing addition. We call such activity *property noticing*. Notice that Teresa's 'property' is closely tied to her image of fractions – each fraction in a sum is worked on, on its own and then combined. This new 'property' of addition differs from Teresa's image of addition in that Teresa has noticed how her image of addition 'works' and is able to combine aspects of it, structure it and explain this structure. Such property noticing is also evident at (4) where Teresa is seen to have developed a useful additive heuristic based on her image of fitting fractions onto known fractions.

At the following level of understanding, *formalising*, the person abstracts a method or common quality from the previous image dependent know how which characterised her noticed properties. At (6) Teresa is observed to see that addition is something that can be done using only the number concepts and symbols related to fractions. Rather than the addends being thought of in singular image related terms, and addition being carried out dependent on these terms, addition now takes on a formal mathematical character – it is a method which works for any set of fractions without reference to their more physical quantitative meaning. At this point Teresa, and anyone formalising, would be ready for, and capable of enunciating and appreciating a formal mathematical definition or algorithm – in this case for addition. This kind of understanding occurs again in (7) for subtraction and again in (8) when Teresa substitutes building a chart through patterns on addends as numbers rather than recording particular indicated sums which correspond to particular ways of quantitatively making two thirds.

A person who is formalising is also in a position to reflect on and coordinate such formal activity and express such coordinations as theorems. We call such an understanding activity *observing*. Such activity occurs at (9) where Teresa is looking for patterns in her charts or formalisms for combining fractions. Teresa's formula for predicting the number of combinations of her fraction set which would add up to a given fraction, itself an act of *observing*, can be contrasted with an inner understanding activity expressed as "I can get other combinations for two thirds by replacing any addend with equivalent pieces." This we could call a *noticed property*. Both would contrast with an even more general but more concrete *image* expressed as, "Many combinations of fraction-pieces can make two thirds."

Structuring occurs when one attempts to think about ones formal observations as a theory. This means that the person is aware of how a collection of theorems is inter-related and calls for justification or verification of statements through logical or meta-mathematical argument. At (10) for Teresa a statement about partitioning would not be about physical chunks, that would be *image making* or *property noticing*, nor about making partition charts which would be *formalising*. In *structuring* a statement about partitions is a statement about a mathematical structure independent of physical or even algorithmic actions.

As mentioned earlier, the outermost level of the eight in our model we call the level of *inventising*. Within a given topic a person at this level has a full structured understanding and may therefore be able to break away from the preconceptions which brought about this understanding and create new questions which might grow into a totally new concept. At the *structuring* level one can see the rationals as a set of numbers with the form of an ordered pair, a/b. This set of numbers is also seen to be a quotient field. One might now *inventise* by asking: 'What might numbers with the form of ordered quadruples a/b/c/d be like?' It was just such a question which stimulated Hamilton to think about and finally develop quarternions from having the *structured* understanding of the complex numbers.

It must always be remembered that our diagram, given above, is only an attempt to represent our ideas in a 2-dimensional form. It is not 'the model' itself and, indeed, has many drawbacks, although with these caveats in mind the diagram is a useful tool, as we shall show later in this paper, in the mapping of growth of understanding. We need to stress at this point that we do *not* see the

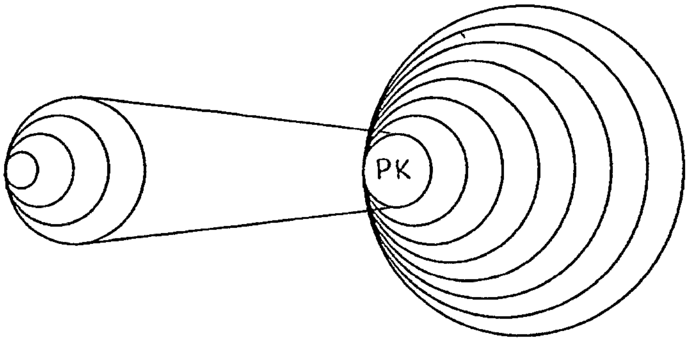


Fig. 3.

growth of understanding as a monodirectional process. In an effort to convey this visually, we have presented the model as a sequence of nested circles or layers, thus emphasising the fact that each layer contains all previous layers and is embedded in all succeeding layers. We see growth as represented by back and forth movement between levels and it is thus that we characterise understanding as a dynamic and organising process. We use the language of 'levels' and 'layers' and certainly there is some underlying hierarchy within the model. Just as the term primitive knowing does not imply low level mathematics, so there is no intention to link the outer levels necessarily with 'better' or 'high level' mathematics.

Earlier in this paper, we noted that primitive knowing was the background mathematical understanding needed to build an understanding of some particular concept. It is therefore possible that a full or partial understanding of that concept could then, in turn, be observed as the primitive knowing for a new mathematical exploration. The model has a fractal-like quality: inspection of any particular primitive knowing will reveal the layers of inner knowings. Using our suggested representation of the model we might illustrate a child using her current understanding of fractions – incomplete as it is – as part of the primitive knowing for the understanding of decimals (Fig. 3).

FEATURES OF THE THEORY

'Don't need' Boundaries

One of the strengths of mathematics is the ability to operate at a symbolic level without reference to basic concepts and this is reflected in a critical element of our theory, observable in the model, and illustrated by the bold rings. Beyond these boundaries the learner is able to work with notions that are no longer obviously tied to previous forms of understanding, but these previous forms are embedded in the new level of understanding and readily accessible if needed. We call these rings the 'don't need' boundaries in order to convey the idea that beyond the boundary one does not need the specific inner understanding that gave rise to the outer knowing. One can work at a level or abstraction without the need to mentally or physically reference specific images. This does not, of course, imply that one cannot return to the specific background understanding if necessary. Indeed quite the contrary is true as will be shown in our discussion of *folding back* and disjoint understanding later in this article. We simply point to the fact that one does not need to be constantly aware of inner levels of understanding.

The first of the 'don't need' boundaries occurs between *image making* and *image having*. When a person has an image of a mathematical idea, she does not need actions or the specific instances of image making. Teresa, with a mental picture, an image of addition of fractions, stopped physically fitting and covering items with her kit. In contrast, property noticing is defined as the result of working with existent images to notice general properties and therefore access across the *image having/property noticing* boundary is essential.

The next 'don't need' boundary occurs between *property noticing* and *formalising*. A person who has a formal mathematical idea does not need an image. Teresa was able to think of fraction addition as combining entities of the form a/b with no reference to actual partitioning and covering. As with the relationship between image having and property noticing, *observing* involves, by definition, focusing on current *formalising*.

A third 'don't need' boundary occurs between *observing* and *structuring*. A person with a mathematical structure does not need the meaning brought to it by any of the inner levels. For example, Teresa would be in a position to prove theorems about addition, division, etc., of ordered pairs without any reference to what a fraction really represents.

Folding Back

The discussion so far has focused attention on the definitions of the levels and their embedded nature, and indeed these are necessary and structurally important to the theory, but a more crucial feature is that of *folding back*. This is the activity, vital to growth of understanding, which reveals the non-unidirectional nature of coming to understand mathematics. When faced with a problem or question at any level, which is not immediately solvable, one needs to *fold back* to an inner level in order to extend one's current, inadequate understanding. This returned-to, inner level activity, however, is not identical to the original inner level actions; it is now informed and shaped by outer level interests and understandings. Continuing with our metaphor of folding, we can say that one now has a 'thicker' understanding at the returned-to level. This inner level action is part of a recursive reconstruction of knowledge, necessary to further build outer level knowing. Different students will move in different ways and at different speeds through the levels, folding back again and again to enable them to build broader, but also more sophisticated or deeper understanding.

This notion fits well with our constructivist beliefs (Pirie and Kieren, 1992b),

Fig. 4.

and is best illustrated with the story of another student, Katia, ten years old and in a different class from Teresa. She has been folding rectangular pieces of paper and drawing pictures to represent cutting up pizzas (image making). From this she has formed some image for fractions (image having). Furthermore, she has noticed the property of equivalence and can construct simple fraction chains such as $1/2 = 2/4 = 4/8 = 8/16 = \ldots$ obtained from doubling (property noticing). She has also realised that like fractions can be combined, that is to say that, as a result of colouring in activities, she knows how to combine say, 3/8ths and 2/8ths to make 5/8ths. In addition she has formalised a part of her image with the statement that "writing any number over any other number will give a fraction, where the bottom number is the folded pieces and the top number is how many you have" (formalising). The question now under discussion is, 'How can one combine non-alike fractions such as halves and thirds?' The strategy of doubling does not achieve a useful equivalence which would enable fraction addition in this situation. Clearly one possible route to a solution would be for the teacher to offer the rule: 'Find a common denominator, and cross multiply to find the numerators and then add the numerators'. This would give Katia an action to perform but not necessarily any new understanding. Actually the teacher asked, "Well what *are* these things called fractions?" Katia's response was, "They came from cutting things up – usually pizzas!" and she then folded back to drawing pizzas (image making) as illustrated in Fig. 4, and re-formed an image for halves and thirds combined now with the already noticed property for creating equivalent fractions. This she did here with the explicit aim of throwing light on her newly posed problem of addition.

Once the pizzas were both divided into 1/6ths then it seemed sensible to put the 3/6ths and 4/6ths together as 7/6ths, or a whole one and an extra 1/6th. After further similar calculations she attempted to formulate an algorithm for herself and offered, "You times the bottoms and add the tops – you times the denominators and add the numberaters" (sic). At this point she was ready and able to accommodate the teacher's rule *with understanding.*

Thus from the level of formalising Katia folded back to image having to make some sense of the operation required at the formalising level. The property of equivalence was then used for the purpose of creating a meaning for addition of fractions. In fact, the original image became enriched by the idea that one can combine, as well as divide up, fractions. It would have been of no value to simply re-call previous actions. Katia needed to re-member and combine existing images to form a new way of looking. This folding back enabled a reconstruction of inner level knowing as a foundation for outer level understanding. This, and other examples will be looked at more closely later in the paper when we discuss the use of 'mappings'. A more detailed account of folding back is given in Pirie and Kieren (1991) and Kieren and Pirie (in press).

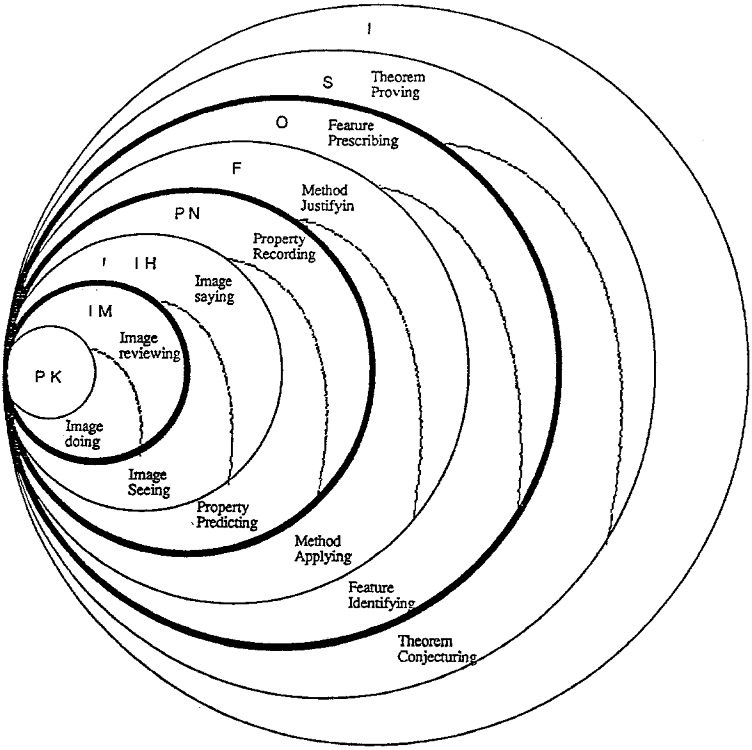
The Complementarities of Acting and Expressing

The final feature of the theory that we wish to mention here is that of the structure within the levels themselves. We believe that each level beyond primitive knowing is composed of a complementarity of *acting* and *expressing* and each of these aspects of the understading growth is necessary before moving on from any level. Furthermore growth occurs through, at least, first acting then expressing, but more often through to-and-fro movement between these complementary aspects. At any level, acting encompasses all previous understanding, providing continuity with inner levels, and expressing gives distinct substance to that particular level.

Currently, we are trying more precisely to define these complementarities at each level and intend now to present in some detail descriptions of those within the image making, image having and property noticing levels. It is important that the reader realise that we see understanding as a process and not as an acquisition or location and that the rings illustrate modes of understanding rather than outwardly monotonic phases. For this reason we have chosen the six verbs, *doing*, and *reviewing, seeing*, and *saying, predicting* and *recording*, as labels for the acting/expressing complementarities within the image making, image having and property noticing rings. The boundaries between these complementarities are represented by dotted lines in Fig. 5.

Once more we have had to choose our terminology with care and will use the medium of a classroom example to illustrate the features we are wishing to define. Nevertheless it is perhaps appropriate first to forestall certain criticism, based on misunderstanding, by elaborating a little on the terms 'acting' and 'expressing'. As will, we hope, become clear, acting can encompass mental as well as physical activities and expressing is to do with making overt to others or to oneself the nature of those activities. Although verbal expression is not strictly necessary we must always remember that it is only through such externalisation that an observer can infer the understanding that the student is constructing. Expressing is not, however, intended to be synonymous with reflecting. Reflection is frequently a component of the acting activity, since it incorporates the process of looking at *how* previous understanding was constructed. Expressing, on the other hand, entails looking at and articulating *what* was involved in the actions.

The classroom under examination this time is that of a group of 14 year olds. The general topic under consideration was quadratic equations and the particular area being explored in the sequence of lessons we are going to look at was that of the graphical representation of such equations. The teacher assumed that the students' primitive knowing would include: evaluating polynomial expressions (at least of the second degree), making tables of values, and graphing points from these tables. The initial task offered to the pupils was the following:



PK - Primitive Knowing
IM - Image Making
IH - Image Having
PN - Property Noticing
F - Formalising
O - Observing
S - Structuring
I - Inventising

Fig. 5.

Consider the function $y = 3x^2 + 1$.
Make a table of values for x and y, x taking values from -3 to $+3$.
Now draw the graph of the function.
Repeat this for the following functions: $y = x^2$, $y = x^2 - 2x$, $y = 2x^2 - 2x$, $y = 2x^2 - 2x - 1$.

The students were observed making tables of values, plotting and joining the points, and then moving on to the next function. Most of them successfully produced the graphs presented in Fig. 6.

Up to this point they had all been engaged in image making; more specifically in the 'acting' aspect of this level of understanding which we term '*image doing*'. They had been performing actions that might lead to the formation of an image for the graph of a quadratic function. In such behaviour one could not see whether the

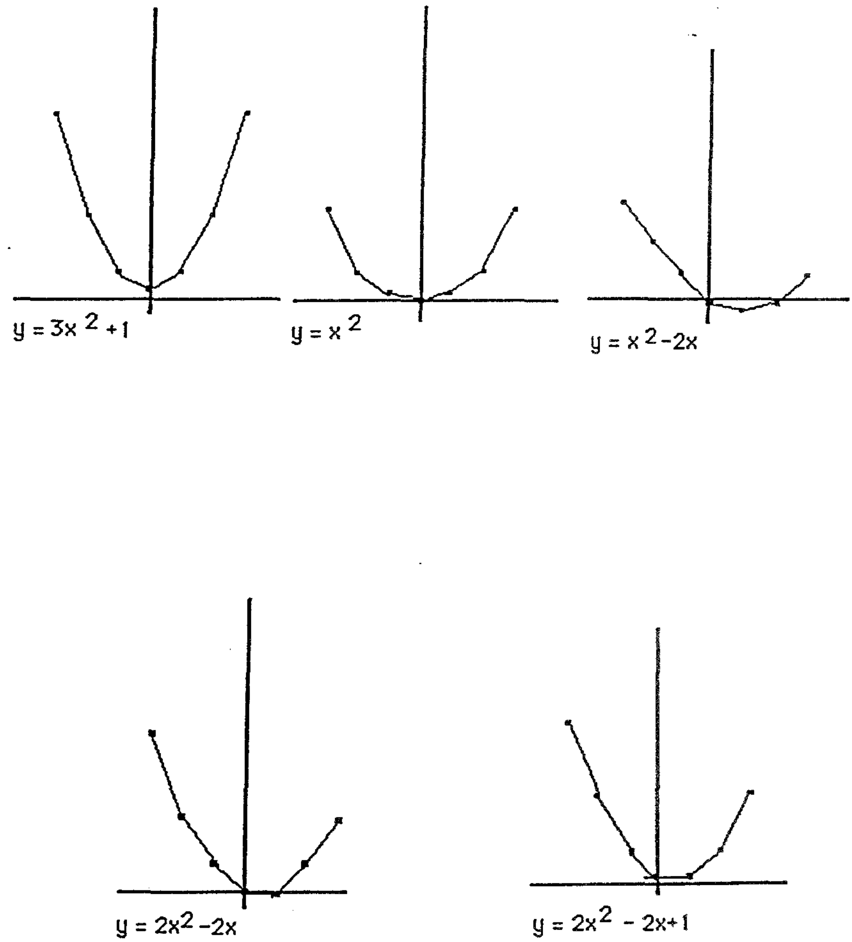


Fig. 6.

students had considered each graph as a whole before moving onto the next. This second activity is the 'expressing' complementarity at this level that we wish to term '*image reviewing*'. In this situation, 'acting' involved joining up the points in the order in which they were calculated, while 'expressing' entailed seeing some order within the activity they were engaged in. We have collected evidence that is leading to the assertion that to be said to understand a mathematical topic by showing image making behaviour, a student must have done image reviewing as well as image doing. Image doing is not enough for sustained understanding.

To probe the students' understanding, once they had had a chance to plot several of the functions given, the teacher added the point $(-2,20)$ to the first graph plotted (this point being within the range plotted (-3 to $+3$) but not on the *drawn* line) and asked the pupils to consider whether it belonged to the graph of

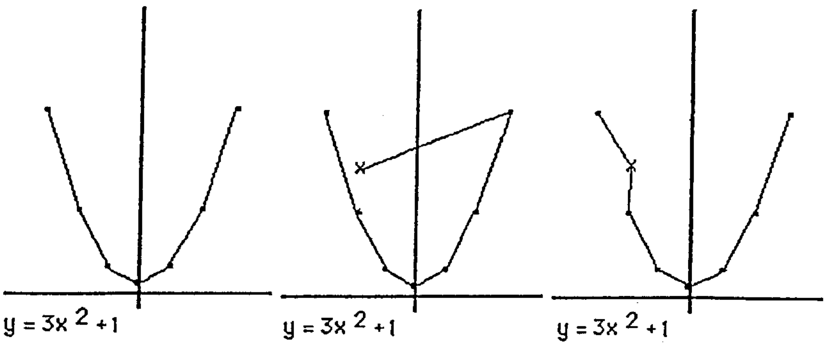


Fig. 7.

$y = 3x^2 + 1$ (Figs 6a,7a). The students who merely joined it to the last point plotted (Fig. 7b), were still *image doing*. They were still following instructions and had not reviewed their work, not connected their activities involving the different graphs in any way. Those students who deleted the appropriate joining line – the line joining (−3,28) and (−2,13) – and connected in the new point (Fig. 7c), could be seen to be engaging in *image reviewing*. They had reviewed their previous work and adapted the new task to fit some tentative idea that they might have about how these graphs should go. They were incorporating the point logically into their plotted values but not releasing it to any formed idea of the shape of a quadratic graph. What we are illustrating is that a person who is simply image doing sees her previous action as completed and rejects returning to it in anything other than a rule-bound way. The image reviewing behaviour allows for the constructive alteration of previous behaviour without yet seeing a pattern.

Those students, however, who responded with statements such as 'that can't be right', 'it can't go there', or who, when having joined up the new point, said 'that doesn't look right' were demonstrating that they had gone further in their understanding and, through reviewing the graphs they had plotted so far, had constructed some image for the graph of a quadratic. They could articulate the fact that the new point did not fit with the image they had formed, although they did not yet say why. We have called this, the 'acting' part of the level of image having, *image seeing*. The complementarity of 'expressing' at this level, *image saying*, is revealed by comments such as 'I thought they should all be U-shaped' or 'we've already got a point for $x = 2$'. The students here are able to say *why* the point does not conform to the image they have. It is interesting to note that the two student remarks given above reveal also that they have formed quite different images from the work they have been doing. The one is related to a visual representation whilst the other concerns an image of the uniqueness of points on a quadratic graph. This example serves, too, to illustrate both the fact that our use of the word 'image' is not restricted to visual images, and that for any topic there will always be a multitude of images formed. It is the interconnecting of these images that leads to the level of property noticing.

x	-3	-2	-1	0	1	2	3
x^2	9	4	1	0	1	4	9
-x	3	2	1	0	-1	-2	-3
y	12	2	2	0	0	2	6

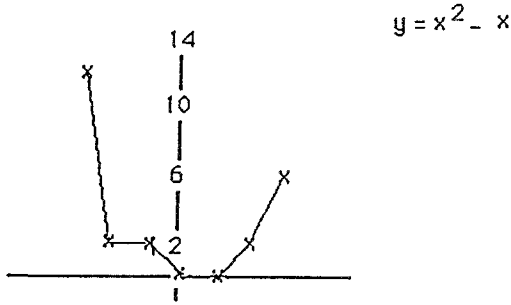


Fig. 8.

The teacher's intervention had been deliberately intended to move the students through image reviewing to image seeing and saying, but, of course, it is the student's response to the situation that determines the effect of the questions for that person (Kieren and Pirie, 1992), and in the case of the pupils who rejected the new point or who could not express why they were not happy with the addition to the graph, the question served to confirm for the teacher that these pupils probably needed to spend more time at the image making stage. In all probability they should be encouraged to draw and review further graphs before being asked to predict features of the graph of the general quadratic function.

To clarify further the images that the students were creating, the teacher produced a table of values for the students to plot, which contained an erroneous calculation producing a point that should not be on the graph (see Fig. 8a). The student who merely plotted the points and joined them up in order (Fig. 8b) was, as above, simply image doing, following the set of instructions provided, without reviewing the outcome. Those who said "That's different" but did nothing further were image reviewing. They had made some unformulated review of their other graphs. Those, however, who responded with comments like: "I would have thought that the point should be here" or "I'll check table because it shouldn't look like that", demonstrated understanding at least at an *image seeing* level.

One student, Julie, remarked: "That is wrong. These are all pointy or flat-bottomed U-shapes", confirming for the teacher that she had moved to the level of image saying. The distinction we are trying to make between image seeing and image saying is that the former – seeing – occurs when a student has 'collected together' previous instances and has a pattern, while the image saying behaviour articulates the features of this pattern. At this level a person is in a position to talk about her actions and carry them beyond the graphing situation. The reader should notice that, even for these inner understandings, these are ways by which students can be expected to judge and defend their behaviour. They can 'organise' even these very informal schemes.

The particular example of Julie is cited to suggest that image having does not imply necessarily having the 'right' or complete picture. Two insights, however, into the images that she has, are afforded to the teacher by Julie's statement. One articulated pattern, that of the nature of the 'bottom' of the graph, is dependent on the way in which the graphs are being drawn using straight lines to join the points together. Julie is relying on her primitive knowing of graph plotting which has, to date, been always linear. The second, that of the overall shape of quadratic curves, is limited by the examples she has worked with so far. An appropriate intervention for the teacher to make at this juncture might be to ask a question to tempt Julie to fold back to further image making activity – further individual graphing. For example, 'Have you tried to graph $y = 1 - x^2$?' might be such a prompt. This new image doing and reviewing would now be informed or at least affected by the image so far seen and articulated, and would challenge the notion held as to the shape of the graph, by producing an inverted U-shape.

To this point we have tried to make distinctions between image doing and reviewing, between seeing and image saying, and between the activities of image making and image having generally. We are saying that a person showing both image doing and reviewing is showing a certain kind of understanding in their actions and we are also saying that a person who is image having is engaging in a qualitatively different kind of understanding activity, in seeing and saying that quadratics are not simply the successful results of graphing activity, but are things with identifiable features.

Returning to the example of Julie, what in reality happened was that the teacher was intrigued by the notion of 'pointy or flat bottoms' and asked her what she meant. The response, "Well it looks like a quadratic which has an odd-number in front of the of x^2 is pointy and an even-numbered one is flat-bottomed", revealed that Julie was in fact at the level of property noticing. Based on the images she had distilled from the graphs she had drawn, she had been engaging in the property noticing 'acting' activity of *property prediction*. She was distinguishing and connecting features of her image to form two classes of graphs – a new kind of understanding activity. The teacher intervention served to extend the 'acting' to the 'expressing' activity we call *property recording*. Recording here need not be written, but must involve articulate expression of some clear form. We have observed in our research several instances of where students have engaged in property predicting without recording or at least consciously making an explicit mental note that a property existed and seemed to 'work'. It seems that at both image having and property noticing levels the 'acting' notions are ephemeral and without the complementarity of 'expressing' do not remain with the student from one session to the next. A lack of 'expressing' activity seems to inhibit the students from moving beyond their previous image.

Notice again that we have selected a property – 'pointy bottoms' and 'flat bottoms' governed by the coefficient of x^2 – which is not a 'usual' property of quadratics and, in fact, may later be proven wrong or incomplete. Including under the rubric of mathematical understanding, mathematical activity which has a very non-standard character or is even 'wrong', might seem unusual, but if one considers the examples in the narrative of Teresa's work at the beginning of this essay or many other examples like it which we have gathered from children and young adults building their own mathematical knowledge structures and organising them into what, to a knowledgeable observer, would be incomplete understandings, it is apparent that such knowing and understanding is not atypical. We are looking at the nature of understanding as an activity and not as a particular content. The teacher, in a situation such as that just described, could be expected to provide the student with the opportunity to defend her 'property' by testing it against new instances which are deliberately chosen to invoke folding back, further image making, and further property noticing behaviour to allow the student to adjust and extend her image. As can be seen, our model of understanding provides teachers and researchers with a language which can enable them to look at the images which students actually 'see and say' rather than assume that students mathematical concepts correspond to given standard mathematics. We are convinced that this complementarity of acting and expressing exists and is necessary at all levels of the model and we are currently collecting data to enable us to illustrate these activities at the outer levels.

Before moving on to look at some of the applications of the theory we need briefly to return to the level of image making and counter a possible observation that this level is ill-defined since one could engage in any activity and call it image doing. We would only wish to consider potentially fruitful activity as evidence of growth of understanding. To illustrate this within the scenario of the classroom considered above, imagine the students, who, not being told explicitly to join the graph in the order of the x values, quite reasonably, from their point of view, produce something akin to that in Fig. 9. They may be engaging in a task which is congruent with their primitive knowing – from previous experience, for them 'graphs' means 'bar graphs' – but is not even image doing with respect to quadratic functions.

Applications of the Theory

With Einstein (quoted in Fine, 1986) we see a theory as: "a self-sharpening tool whose warrants and value in the end rest on this, that they permit the coordination of experience, 'with dividents' [*mit vorteil*]". So what are the 'dividends' that this theory of the growth of mathematical understanding can offer?

We have used it in a variety of learning environments as a tool to observe the mathematical behaviour of students as they work on a single mathematical task and as they build and organise mathematical knowledge structures over periods of time. The theory has enabled us to comment closely on the levels at which different students are making sense of their mathematical activities and thoughts

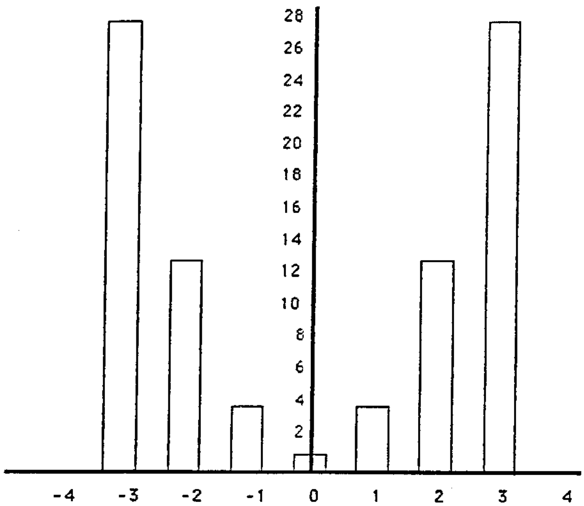


Fig. 9.

(Pirie and Kieren, 1991, 1992a; Pirie and Newman, 1990). Such insight into students' understandings have been used to provide a frame for planning and engaging in mathematics lessons and, in addition, to make observations about curriculum development (Kieren and Pirie, forthcoming).

The scope of this paper allows us to examine only one particular application of the theory in greater detail and we wish to put forward the way in which we have created a technique we call 'mapping' to record the growth of a person's mathematical understanding. Using the layered pictorial representation of the model we aim to produce in diagrammatic form a 'map' of the growth of students' understanding *as it is observed.* This last phrase, 'as it is observed', is important because we make no claims as to what might have gone on 'in the students' heads'. Analysis can only ever be based on what the teacher observes. This notion of mapping entails plotting as points on a diagram of the model, observable understanding acts and drawing continuous or disconnected lines between these points, dependent on whether or not the student's understanding is perceived to grow in a continuous, connected fashion.

To illustrate this notion, we will discuss the work of Richard, who was one of six university mathematics education students engaged for four hours in building a geometry for shapes created by a computer procedure. The students controlled the procedure by inputting three parameters from which the computer generated one member of a potentially infinite set of geometric figures such as those in Fig. 10.

Richard and his partner tried out only a few examples before he said, "Oh, they're just inward spirals." After a very few individual image making acts Richard articulated the image that he had for the geometry and we represent this in a diagram of the model by the line joining points A and B (Fig. 11). The two students then tried a few more examples to generate and test the property that

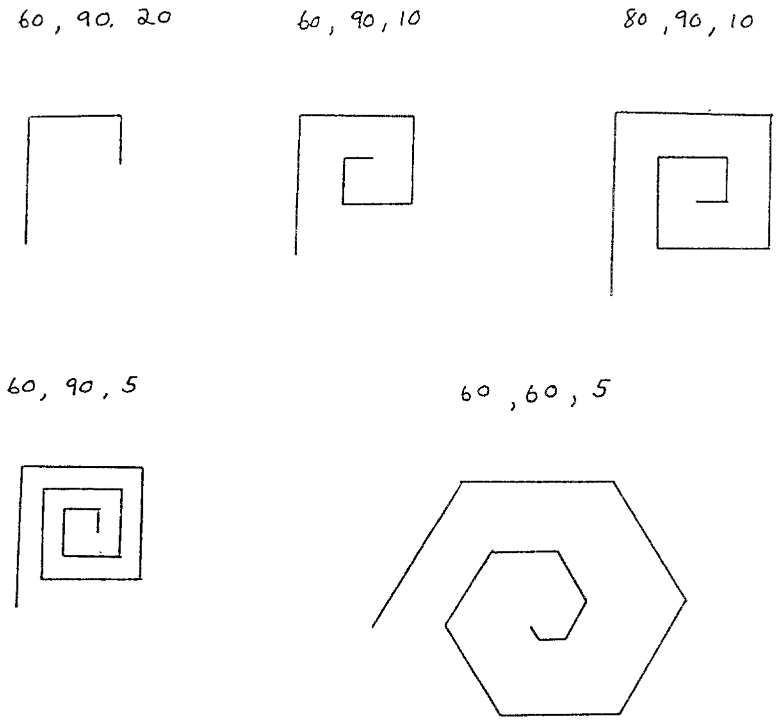
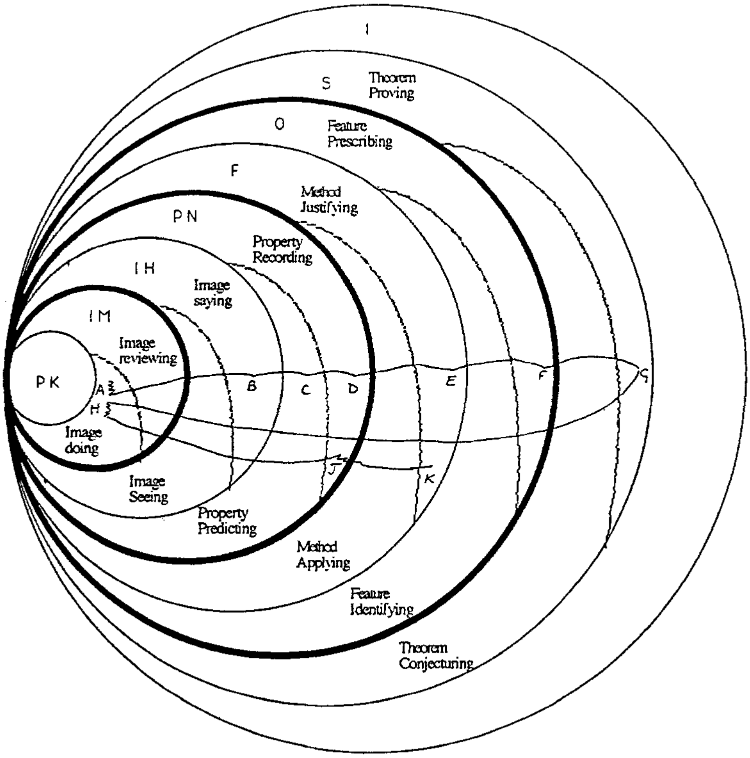


Fig. 10.

an ‘angle’ parameter input of 360/n generated an n-sided polygonal spiral. We represent this property noticing with the points C and D on the diagram. While his partner now proceeded to look for other kinds of shapes. Richard stopped working on the computer. He said: “The program just generates spiral shapes by drawing a line of an input length, then turning right through the input angle. This is just repeated with the length reduced by an input decrement till it stops.” This last remark suggests that he sees these shapes as a class controlled by a *formalised* statement (point E).

Richard then moved away to write-up his mathematics. He noted down the formal *observation*: “the spirals generated by the angle $180 - N$ and $180 + N$ are reflections of one another” (point F) and then set this observation in a mathematical *structure* by writing a short ‘proof’ based on his assumed formal procedural definition (point G). In terms of the levels of the model above, Richard was observed moving quickly and directly from *image making* through the intermediate levels out to *structuring*.

At this moment the teacher drew a square on a piece of paper and asked, “Could this be a member of your set?” To the teacher’s surprise Richard said “No”. The teacher then used the procedure, without allowing Richard to see the parameters that were used as input, to generate a square on the screen. Richard returned to



PK - Primitive Knowing
IM - Image Making
IH - Image Having
PN - Property Noticing
F - Formalising
O - Observing
S - Structuring
I - Inventising

Fig. 11.

the machine and tried out several examples before getting a square. The teacher's question had caused him to fold *back* to *image making* (point H). Eventually he *noticed a new property of* his existing image (point J) and revised his *formalisation* by saying, "Oh I see, the decrement could be zero – of course the program doesn't stop" (point K). Richard's understanding of spirolaterals had quickly grown 'deep' – out to formal, structural levels. The teacher's intervention invoked a folding back to inner level action. One might have thought that he would see the query on the square as a trivial consequence of his formalised understanding but he did not. His images at that point did not suffice to enable him to do this. He needed to reconstruct and enlarge his understanding at an inner level.

It is clear to us that student's maps are not all alike. Some students may, unlike

Richard, spend time creating a broad, rich image before moving outwards to seek properties and formalisations.

If we were to map the growth of understanding of Katia, whom we discussed earlier in this article, we would see a quite different pattern emerging. She spent several lessons in image making activities, moving forward to build up a rich image for fractions that enabled her to construct with understanding of the meaning of equivalence as it occurs in chains created by doubling the numerators and denominators of fractions. We indicate the extensive working at a single level by means of a serrated line as drawn at points A, B, and C in Fig. 12. She then folded back to further drawing and colouring-in activities (D) which led to the additional image that like fractions can be combined by counting the total number of pieces involved (E). Although not expressed in algebraic or even very mathematical language, she is heard to formalise her understanding with a generalised statement defining a fraction (F). The challenge then facing the class was to find a way to combine, or add, fractions which did not have a common denominator. Nothing in Katia's images helped her here. Had the teacher offered her the 'rule' she would have had a way of working at the formalising level, but no image in whose roots the formalising lay, to which she could fold back in later times of lack of understanding. This apparent understanding, which occurs when a student works with information that does not emerge from or become connected to her own constructured knowledge, we term *disjoint* from her existing understanding. We hypothesise, that students will be unable to successfully build further understanding based on this *disjoint* knowing until they have in fact constructed the connection for themselves. We would represent this with an unattached cross (G), to indicate that the understanding at this point was not connected to or based on the student's current understanding. In fact Katia folded back to further image making activities (H, I, J), this time with the added understanding of the images and properties she has already constructed, before moving out to the formalising level again with her personal attempt to express the process of addition (K). As stated earlier, the fact that the explanation was not completely correct does not deny the label of *formalising* for her action. When given an accurate verbal version of the process she had no trouble connecting it to her own meaning and using it with understanding.

We do not yet know whether this difference in maps is person or topic dependent. What is clear, however, is that it is not age related. One pair of students in Richard's class spent the whole of the first two hours simply making images for themselves of what the program could do. Despite repeated interventions from the teacher they resisted the need to record, or possibly even to review, their images and when they returned for the second session they were unable to recapture much of the image that they had 'seen' previously. Eventually these two students moved out to predict and record some of the properties of these images. Even with their broad understanding and multiple images of what the computer program could do, there came a point where their images were insufficient for the growth of understanding at an outer level and some folding back was necessary; while struggling to formalise some of their thinking, they constantly reviewed the

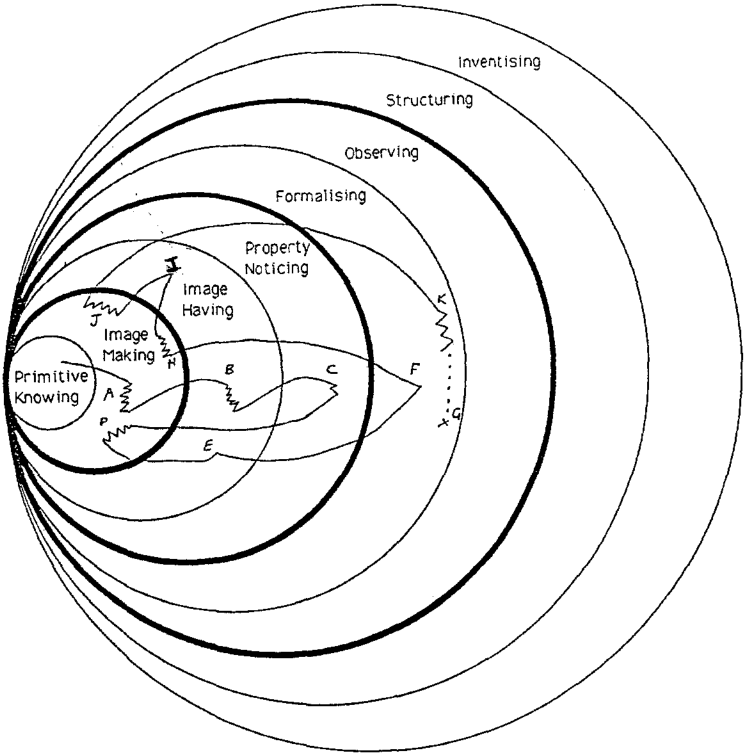


Fig. 12.

properties they had noted, for clues to a general picture and returned to drawing further spirolaterals to confirm or falsify their conjectures. The map of growth of understanding for these students would look something like Fig. 13.

The episode also illustrates the notion that folding back can happen directly to any inner level, as with Richard and Katia, or by re-tracing the path of growth through the intervening levels, as with this second pair of students. The nature of folding back cannot be *generally* prescribed; it is unique to *specific* examples of growth in understanding and to each individual person. Every student will have a singular path for any topic, and yet all paths will involve 'folding back to move out' in their actualisation.

This method of representing students' paths of growth of mathematical understanding has the potential to allow researchers to study in detail the actual nature of this growth either for an individual over several topics, or for many students within the learning of a specified topic. The insight that this could give would be both psychologically and pedagogically valuable to the study of learning.

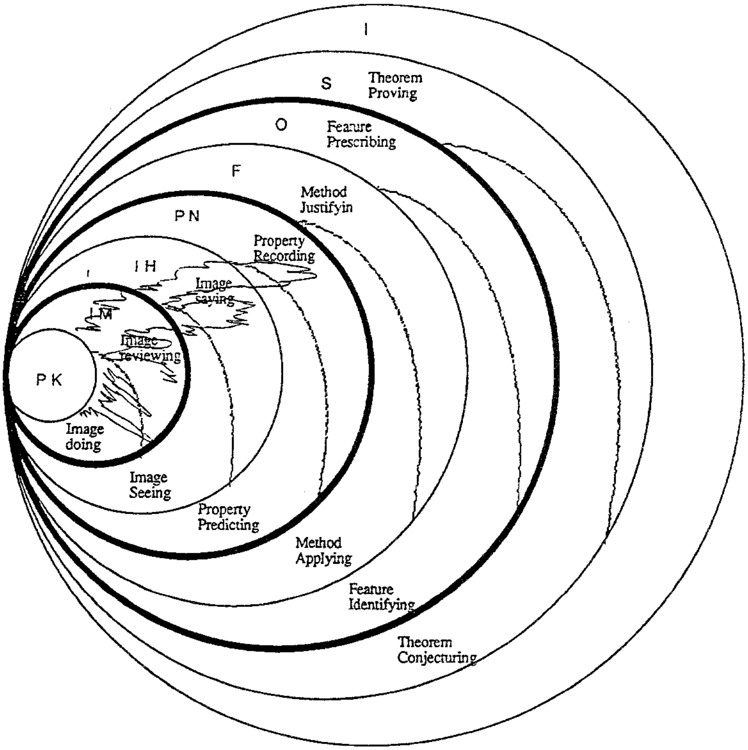


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Fig. 13.

SUMMARY

The purpose of this paper has been to show a theory of the growth of mathematical understanding which is based on the consideration of understanding as a whole, dynamic, levelled but non-linear process of growth. This theory demonstrates understanding to be a constant, consistent organisation of ones knowledge structures: a dynamic process, not an acquisition of categories of knowing.

Its levelled nature has been illustrated through the model of eight embedded rings, each of which represents a level of understanding activity potentially attainable for any particular topic by any specific person. These levels range outwards from the existing knowing that the person brings to the task, through the making and having of an image, to the noticing of properties and formalisation of that image. Reflection on the formalisation leads to observing the thought structures and to their consistent reorganisation. Logical arguments at the next level provide the necessary axiomatic structure to complete the understanding of the topic and leave the knower with the freedom to perhaps mentally alter something within that structure and explore the new field of mathematics thus created. A key tenet of the theory is that outer level knowing does not necessarily mean higher level mathematics. Equally, the converse is also true: high status mathematical topics need to be worked on at the image making level before one can begin to look for an appropriate formalisation or structure.

Other crucial features of the model include the notion of 'don't need' boundaries to explicate the unique power of mathematics to solve problems in non-image related, symbolic ways. The complementarities within each level provide the link between acting and expressing and together contain the necessary wholeness of the very nature of understanding.

This theory of growth of understanding has the built-in dynamic of folding back to move out and such growth can be thought of as a continuous path traced back and forth through the levels of knowing. This growth is a non-linear phenomenon which involves folding back to re-member and to re-construct new understanding. We see it as a non-monotonic pathway across the embedded rings in our model.

FUTURE RESEARCH

A further area for investigation that we are now examining is that prompted by many of the detailed maps of individual students that we have analysed. This is the effect of different kinds of teacher interventions on the paths of growth of understanding of their pupils. We have identified three classes of intervention, *provocative*, which have the effect of moving the student outwards, *invocative*, which have the effect of causing the student to fold back in order to enlarge or alter his image, and *validating*, which allow the teacher to view, or the student to confirm, existing understandings.

We end with a favourite quotation from Maturana:

> At this point there is either much more to say ...
> or nothing.

When we first enunciated our theory, we were not sure which of these options obtained. We hope now that we have demonstrated that there is indeed much more to say about the growth of mathematical understanding.

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