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IS ONE PROOF ENOUGH?
TRAVELS WITH A MATHEMATICIAN OF THE BAROQUE PERIOD[1]

ABSTRACT. It is reasonable to assume that a subject of presumably universal appeal must rely on just one style. In spite of its universality, mathematics employs many styles. In particular, there are many styles of proof. In this paper we present and analyse a number of proofs of a property of the area under an hyperbola due to Gregory of Saint-Vincent, a mathematician of the first half of the seventeenth century. There is a baroque and prolific quality to the architecture of his proofs, and this quality points to a connection between a culture and the discovery of a mathematical theory. An historical perspective shows that, in addition to many styles, the universality of mathematics implies a variety of procedures.

THE ONE-PROOF-ONLY TRADITION

Among intellectuals, mathematicians are often regarded – as they have been since Antiquity – as victims of a curious affliction: they repeatedly try to prove the obvious. While everybody agrees that a formula like $A = \pi R^2$ requires proof, far fewer insist on a formal proof of the theorem that the ratio of the areas of two circles is equal to the square of the ratio of their radii $(A/A' = (R/R')^2)$. This, however, is what is proved in Proposition 2 of Book XII of Euclid's *Elements*. Euclid's proof is very useful because it introduces one of the most powerful tools of ancient geometry – the method of exhaustion – so named far later by the Bruges-born mathematician Gregory of Saint-Vincent. One reason for the general indifference toward some kinds of proof is certainly the existence of easy proofs of particular cases. In our instance, a simple proof is known for two similar triangles: the ratio of the two areas is equal to the square of the scale factor. Therefore, the same is assumed to hold without further discussion for any two similar figures, say, for two circles, where the scale ratio is the ratio of the two radii. It is interesting that even meticulous geometers who copy the rather long euclidean proof in Book XII are quite casual when dealing with other similar figures – they just consider the result obvious. Surprisingly, in some textbooks dealing with the Lebesgue integral and with Lebesgue measure in the plane, the formal (and very easy) proof of the dependence of areas of similar figures on the scale factor is often left out.

This general attitude was summarized in *La logique ou l'Art de penser*, a book first written by Arnauld and Nicole in 1662, in which the cartesian influence is very strong: the authors deplore the common weakness of mathematicians for proving the obvious.[2] Therefore, the famous Jansenist Arnauld is quite casual when discussing the area of a circle in his *Geometry* of 1667 (sometimes called

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la *Géométrie de Port-Royal*). He is satisfied with a brief explanation that uses the method of indivisibles.

According to the one-proof-only approach, proving the same result in two different ways is a mortal sin. As attested by the long book of Arnauld, what is involved here is not only a question of brevity or the desire to avoid prolixity. Providing two (or more) proofs seems, even if indirectly, to imply the existence of different paths to mathematical knowledge. This contradicts the strong implicit assumption that there exists only one natural way, only one authorized sequence of logical steps leading to a correct conclusion. The epistemological background of this one-proof-only tradition is certainly to be found in rational intuitionism. It is therefore instructive to classify mathematicians according to their habit of providing just one proof or more than one. In this respect, Euclid, Descartes and Bourbaki are opposed to the prolific Euler, to Gauss and to Lebesgue. We are aware that when a book becomes a textbook, different proofs of the same result are often presented (commentators on Euclid have assiduously played their part in this natural process). But this does not really change the general picture of a textbook, as the various versions of the same result are generally provided for lemmas only; what is involved is just local simplifications that reduce the number of assumptions. These, then, are variations on a theme that do not generate different theories.

Paradoxically, even the modern (perhaps already outdated) practice of an axiomatic presentation of mathematics has not changed the one-proof-only practice. We know that a given set of axioms leads to a body of theorems which may be quite different from, sometimes even in contradiction to, those deduced from another one. But it is far more unusual for a group of "important" properties to be proved along radically different lines. Even if equivalences are sought, one tends to think that there exists a preferable – natural, logical and economical – way. In school, as well as at the university, we are quite far from accepting the logic of equivalences, which, in a way, offers two different pictures of the same reality – as if we were anxious not to transform mathematics into a sort of game. We all know, for example, that there were until recently two species of teaching mathematicians: those favoring the presentation of integration theory from a functional point of view (i.e., a measure is an element of the topological dual of a certain space of functions) and those preferring the measure (probabilistic) point of view. And, by definition of species, the two viewpoints cannot be fruitfully reconciled! How many times have we heard from students that what was explained in probability theory had nothing to do with what was practiced in functional analysis? Incidentally, the one-way approach is quite opposite to the situation experienced by the research mathematician.

Such examples remind us that mathematics has its schools and its fashions. Thus in many respects mathematics evolves like a culture. In other words, the one-proof-only tradition is just one among many traditions, and certainly not the only way to deal with mathematics.

The historical perspective can help us understand better the development and disappearance of certain "cultural" habits within mathematics. If we admit this cultural approach to mathematics, then, as far as a proof is concerned, it might be interesting to study the practitioners who systematically present different proofs of the same result. Among many possible leading mathematical personalities, the choice of a relatively obscure man, Gregory of Saint-Vincent, working at the dawn of the scientific revolution, may prove more fit for our purpose, in the sense that his mathematical idiosyncrasies were developed over a rather long period of time, became more and more systematic, a sort of mania, and therefore more visible. They are not hidden by the eagerness to adapt one's style to what one wishes to discover, as is often the case among greater mathematicians. I hope that a sort of spectral analysis of some of his proofs may give rise to fruitful questions concerning the practice of mathematics in general. Gregory of Saint-Vincent is to be understood within the baroque period to which he belongs; he differs from, say a man like Arnauld, who belongs to the classical period. Mathematics is not outside the flow of time.

I certainly should pause here to give some historical facts, though I will restrict myself to the bare minimum, and will provide more information while tackling the mathematical proofs provided by our hero.

GREGORY OF SAINT-VINCENT: A TEACHING MATHEMATICIAN

He is essentially a man of one book: *Opus geometricum*. But what a book! It contains more than 1200 pages (in folio), and thousands of figures. It was printed in Antwerp in 1647, but was never republished. One thing about the book immediately stirred some uneasiness: the addition to the title, namely: *quadruturæ circuli*. The engraved frontispiece shows sunrays inscribed in a square frame being arranged by graceful angels to produce a circle on the ground: *mutat quadrata rotundis*. There was uneasiness in the learned world because no one in that world still believed that under the specific Greek rules the quadrature of a circle could possibly be effected, and few relished the thought of trying to locate an error, or errors, in 1200 pages of text. Four years later, in 1651, Christiaan Huygens found a serious defect in the last book of *Opus geometricum*, namely in Proposition 39 of Book X, on p. 1121. This gave the book a bad reputation. In spite of Huygens' 1672 recommendation of the book to the young and eager Leibniz (who while in Paris was seriously reading mathematics), and in spite of Leibniz' clear statement in his description of his own apprenticeship (*'More substantial help came from the famous triumvirs: from Fermat by his invention of a method pro maximis et minimis, from Descartes by his showing how to describe curves of usual Geometry by means of equations, and from Father Gregory of Saint Vincent by his numerous bright inventions*"),[3] our hero was not later appreciated. He survived because of some historians of mathematics, such as M. Cantor,[4] H. Bosmans[5] and J. E. Hoffmann.[6]

Born in 1584, Gregorius a Sancto Vincentio (his Latin name) entered the Jesuit order to be sent to the Roman College, where he showed great mathematical gifts. He was noticed by his professors, men like Clavius to whom we owe the so-called

Gregorian calendar. He returned to Belgium in 1615, and for approximately 10 years worked in Antwerp and Louvain. He was an outstanding teacher (for a small audience). He published very little, for he preserved almost everything he created for the book about the quadrature of the circle which he wanted to be an impressive and lasting monument. It is almost certain that he wrote what we will study in the sequel before 1625. This is indicated by his manuscripts left in Brussels.[7] This is important because at that time Cavalieri's *Geometry of indivisibles* (1635), and Descartes' *Geometry* (1637) were not yet available. We must think of Gregory as a direct heir to Clavius, apparently not interested in the works of the Italian school of algebraists of the 16th century, nor even in the work of Vieta. He should be regarded as a contemporary of Kepler. Unlike Kepler, who had so many difficulties grasping the Conics of Apollonios, Gregory had mastered completely the works of the mathematician of Perga.

ONE PROPOSITION AND TWO PROOFS/ A STRATEGIC REDUNDANCY

But let us turn to his mathematical text. We have selected – how could it be otherwise in such an ocean of words – a famous and original result. In modern terms, Gregory of Saint-Vincent proves that the variable area bounded by an hyperbola, one of its two asymptotes, and two lines parallel to the other asymptote, can be expressed by a logarithmic function. He is the first to establish a link between a curve as simple as the hyperbola and the functional property of logarithms (transformation of a product into a sum). Two propositions are offered, namely Proposition 107 and Proposition 108, yielding apparently the same result as that stated in Proposition 109; they appear in Book VI (*De hyperbola*), around pp. 585–586 of the *Opus geometricum.* We give an English translation of the Latin text and reproduce the original figures.[8] Our translation is close to the original rather than a modern transcription. We will not attempt to reduce the technical difficulties of the original by using modern formulations and terminology. Our purpose is not to admire a result of Gregory and merely give a taste of it. We want to illustrate a certain style of mathematical proof, a style using repetitions.[9] A modern intrusion would corrupt the whole process. We even require from the reader a certain complicity with the author of the early seventeenth century, a certain artistic connivance.

Before stating Proposition 109, a few explanations are needed as we immerse ourselves in the very middle of a book of rather complex structure. An hyperbola is classically defined as the section of a circular cone by a plane which does not intersect the cone's axis and is not parallel to one of its generating straight lines. Like Apollonios before him, Gregory shows that the plane curve has a center. He then works only with one branch, using the two asymptotes for the drawing (without assuming a right angle between the asymptotes). The reduction to plane geometry is achieved by what he considers a characteristic property of an hyperbola (Figure 1): an area like that of the triangle AmM is constant when M traverses a branch of the hyperbola[10] (notice that the role of asymptotes is symmetric).

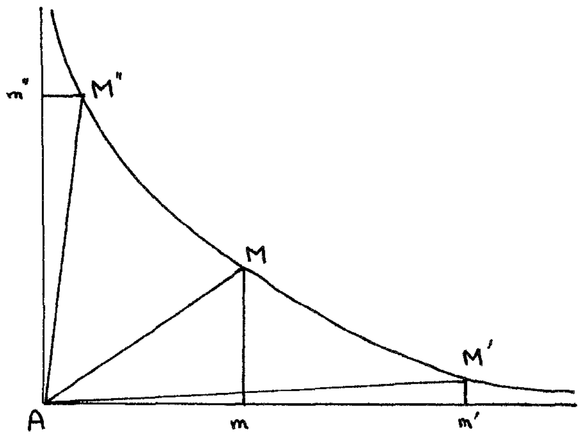

Fig. 1.

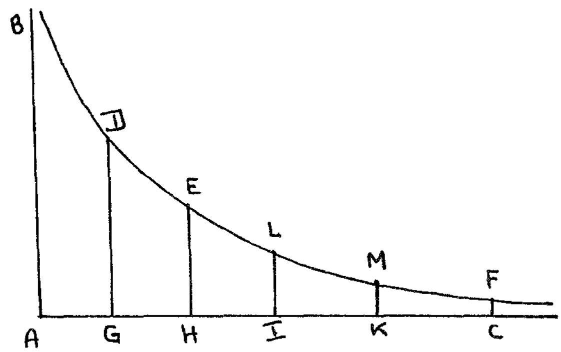

Fig. 2.

PROPOSITION 109. *Let AB and AC be the asymptotes of a hyperbola DEF. Divide AC so that AG, AH, AI, AK, AC are in a continuous proportion. Set GD, EH, LI, MK, FC equidistant from AB. I say that HD, IE, KL, CM are equal segments.*

The following remarks clarify Proposition 109: segment HD means the concave area DGEH under the hyperbola (Figure 2), for "equidistant from" read "parallel to", and continuous proportion stands for the chain of equalities

$$\frac{AG}{AH} = \frac{AH}{AI} = \frac{AI}{AK} = \frac{AK}{AC} = \cdots .$$

In other words, the proposition states that a geometric progression of points G, H, I, K, C, on the axis of abscissae yields an arithmetical progression of areas, DGEH, DGLI, DGMK, DGFC, etc.

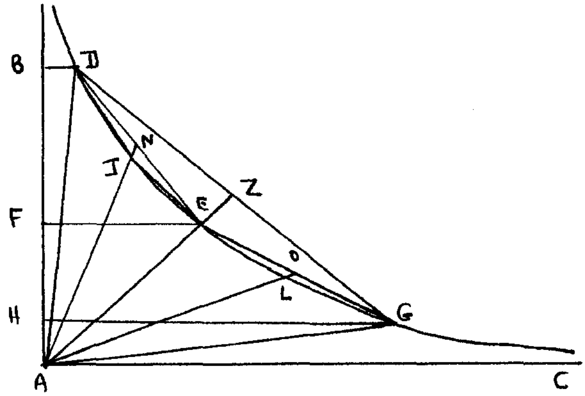

Fig. 3.

If it is important to state the result in such an indefinite way (with no final step in the progression), then it seems clear that a proof is required for just three successive terms. This is just what is involved in each of Propositions 107 *and* 108. Surprisingly, at least for a modern reader, our author does not bother to say that he has two ways of obtaining the same result. He does not apologize for his abuse of our patience. This is not due to bad preparation. Rather, it is part of a general strategy.

Gregory of Saint-Vincent does not consider his two proofs as independent facets of the same problem – they are but two successive steps on the same route. In other words, the Jesuit mathematician prefers to describe precisely the road, the method of discovery, rather than to show the shortest way to the summit. This is an important matter of style, and, like our mentor, we will even have to question the existence of a summit. Is mathematics only a road?

At least there is a road in the case we are examining, and "milestones" along the road can be seen in the form of propositions. To begin with, we state Proposition 106, and need one definition. For an hyperbola, a central conic, a diameter relative to a chord DG is a straight line passing through the centre A and dividing the chord DG into two equal parts DZ=ZG (Figure 3). Then the ordinate F of the intersection E of the diameter with the hyperbola is the geometric mean of the ordinates of B and H, that is the square root of their product. This result will be of use in the sequel.

PROPOSITION 106. *Let AB and AC be the asymptotic lines of the hyperbola DEG. Set DG in an ordinate way along diameter AE.*[11] *Let DE and EG be drawn. I say that the convex segments DIE and GLE are equal.*

Before producing the proof of Proposition 106, we proceed to Proposition 107 for which we do not need another figure, as stated by Gregory himself.

PROPOSITION 107. *Under the same conditions,*[12] *let DB, EF and GH be*

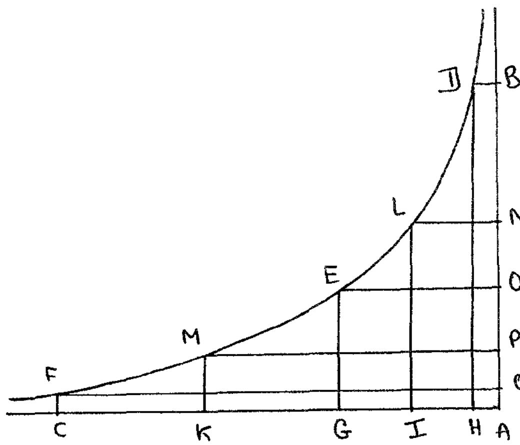

Fig. 4.

equidistant from one of the asymptotic lines. I say that the two concave segments BDFE and GHFE are equal to one another.

The next proposition states the following.

PROPOSITION 108. *Let AB and AC be the asymptotic lines of the hyperbola DEF. Let DH, EG and FC be equidistant from the asymptote AB along a continuous proportion. I say that segment DHGE is equal to segment EGCF.*

From Proposition 106 to Proposition 108, the landscape has changed: we started from an equality concerning convex segments bounded by an hyperbola (Proposition 106) and we ended with concave ones, both in Proposition 107 and 108. Carefully written, the proof provided for the intermediate Proposition 107 explains the transition from convex to concave areas relying on a property of trapezoids inscribed in an hyperbola. This transition is simple, and the fact that the proof is short is an external, but clear hint. Gregory has his own style, which implies not only one repetition, but more than one. He intentionally states Proposition 108, which more or less says the same thing as Proposition 107. However, the proof of Proposition 108 is a long one, which means that this proposition is not a simple consequence of Proposition 107. Moreover, a new figure is drawn, with different letters and with a different aspect. In particular, it is not the same asymptote which plays the main role. Such hints show that something important must be deduced from the comparison between the two propositions.

Before entering into the details of the proofs, we note that, although his main objective was achieved with Proposition 108, Gregory returns to the transition from convex to concave areas in Proposition 110. This is another repetition. It is as if he wanted to answer a persistent query regarding the change from convex to concave segments without adding any rhetorical comment. A simple diagram (Figure 6) summarizes Proposition 110 and shows that this is trivial.

Area (OCD)=Area (OBA), due to the property of an hyperbola. Subtract from

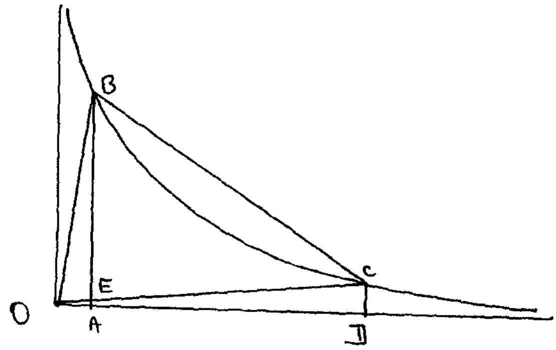

Fig. 5.

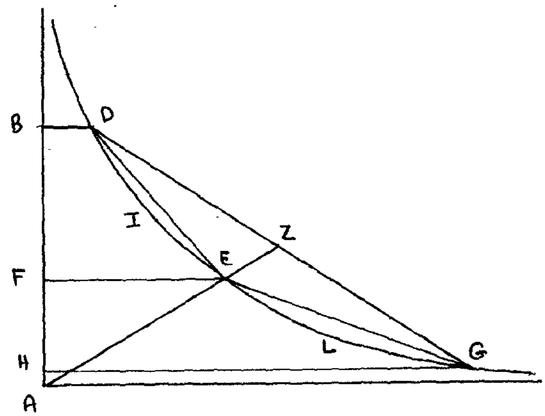

Fig. 6.

both areas the triangular area OEA and add to both the curvilinear area EBC. The concave curvilinear trapezoid ABCD has the same area as the curvilinear triangle OBC. Similarly, the equality of areas holds for the rectilinear trapezoid ABCD and the rectilinear triangle OBC. This equality, obtained by means of the usual technique of adding and subtracting areas – that is the so-called application of areas used in Euclid's *Elements* – provides an easy link between concave and convex segments.

Proposition 107 is also obvious if we know from a study of the hyperbola that the rectilinear trapezoids BDEF and FEGH have the same area. Then the equality of the convex segments DIE and ELG is equivalent to the equality of the curvilinear trapezoids BDEF and FEGH.

Clearly, the aim of the second repetition, from Proposition 107 to Proposition 110, is just to show the transition from convex to concave areas is trivial, and that we gain nothing by achieving it in Proposition 107 – at least if we do not think of a road to follow. Proposition 110 states a "frivolous" property after "hard" ones. It is introduced at this stage to urge us to look back at the road followed.

We see that there exists a strategy guiding redundancy. And Gregory of Saint-Vincent was keen enough to advertise it, somewhere in his book, while explaining the passage from the particular to the general and vice versa. Both ways are permissible and in this respect repetitions are not only unavoidable but also necessary as parts of the exposition of the mathematics: "*In fact, I felt comfortable, from time to time, to do quite often this here, or that there. This because the truth of the same theorem can more than once be more clearly and obviously revealed in many particular cases; that, because from the knowledge of particular cases, one more easily intuits proof of universal theorems*".[13] But we should certainly not restrict such a redundancy strategy to didactical objectives. More is sought.

A BAROQUE ARCHITECTURE: SYMMETRIES AND CONTRASTS

The geometric modification we have noticed in the passage from Proposition 106 to Proposition 108 is just a small part of a deeper change, and we should not focus our attention on this aspect only, as we have been warned by the repetitive procedure in Propositions 107 and 110. We know too that Propositions 106 and 108 yield the same result, which is summarized once in Proposition 109. At least, we are waiting for two different proofs, for example one quite short and one quite long, each according to a distinct pattern. There should be some pronounced differences to justify a repetition! Surprisingly, the two proofs are of approximately the same length and follow a strictly parallel pattern, somewhat like those parallel sentences inscribed in many a Chinese house, where each character in one scroll has its counterpart in another scroll, with the same tone used and the same syntactic role, but with a completely different design and meaning. Under the parallelism, we have to guess the opposition! To emphasize this, and to understand what is meant by Gregory, we have presented the proofs in two columns.

At the beginning of each column, the two very different figures (Figures 7 and 8) are thus in complete contrast with the parallel construction of the proofs of Propositions 106 and 108, whose lines we stated earlier. Symbolically, if the two figures present themselves in a symmetric way for the hyperbola (which is a faithful reproduction of the original illustrations), the segment lines, the true scaffolding for the curve, are completely different.[14] In one case (the right figure) the hyperbola is literally attached to its asymptotes which act as axes of reference. In the other case (the left figure), the asymptotes could almost be ignored, as diameters and chords are far more important. The left figure gives intrinsic points and lines of the hyperbola, while the other one requires external references. Two opposite ways of doing geometry!

In both proofs, it is the equality of two areas which has to be established. Therefore the convex hyperbolic segments DIE and GLE in Figure 7 have their exact counterpart in the concave hyperbolic segments DHGE and EGCF of Figure 8. And Gregory warned us about the equivalence between concave and convex

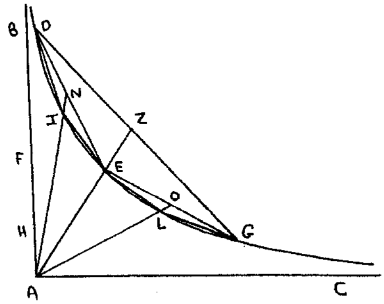

Fig. 7.

Let us divide ED and GE into two equal parts by N and O. Let AIN and ALO be diameters. Join DIE and GLE. We will get maximal triangles DIE and GLE (Prop. 102), equal to one another (Prop. 103), larger than half of the segments in which they are inscribed.

In the same way, if maximal triangles are inscribed in the segments which are left on both sides, one clearly sees that the maximal triangles in the segments ID and IE are equal to the maximumal triangles in the segments EL and LG

This operation can be continued indefinitely in each of the segments, in such a way that triangles subtracted from segment DIE are larger than half of the segment from which they are subtracted and equal to those subtracted from segment GLE, each larger than half of the segment from which they are subtracted. It follows that segment DIE is equal to segment GLE (Prop. 116 of *de Progressionibus*).

Which was to be proved.

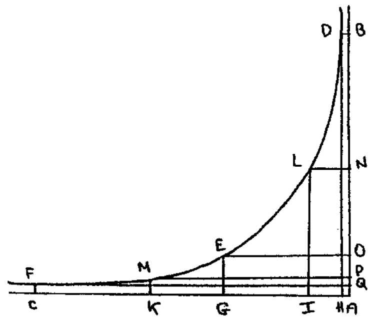

Fig. 8.

Let LI and MK be proportional means between HD, EG, FC. Let DB, LN, EO, MP, FQ be equidistant from the asymptote AC. As DH, LI, EG, MH, FC are continuiting the same ratio, AH, AI, AG, AK, AC are proportional by difference (Prop. 1 of *de progress.*); HI is to IG as AI is to AG, i.e. as EG is to LI. Therefore the parallelograms LH and EI are equal (Prop. 14).

In the same way, because GI is to GK as MK is to EG, parallelograms EI and MG are equal. Similarly, parallelogram FK is equal to MG. Therefore the four parallelograms LH, EI, MG, FK are equal. Thus the two parallelograms LH and EI, subtracted from the segment EGHD are equal to the parallelograms MG and FK subtracted from the segment FCEG. Thus, let DH, LI and EG be proportional means, as well as EG, MK, FC. As above, and in the same way, parallelograms in the segment EGHD are equal to parallelograms in the segment FCEG.

As this can be continued indefinitely, in such a way that parallelograms subtracted from segment DHEG are larger than half of the segment DHEG and equal to those subtracted from segment EGFC, which are also larger than half of the same segment, it is constant that segment FCEG is equal to segment EGDH (Prop. 116 of *de Progressionibus*).

Which had to be proved.

segments. Two parallel ways of doing geometry!

Note that point E (left column) is defined differently from the line EG (right column). Point E is defined *geometrically* as the intersection of the hyperbola and the diameter AZ (DZ=ZG) relative to the chord DG. Line EG is defined *analytically* by a proportion: it is a line parallel to one asymptote, whose length is the mean between DH and FC, i.e.

$$\frac{DH}{EG} = \frac{EG}{FC}.$$

This antagonism between a geometric position and an analytical one is what underlies the symmetry of the two parallel proofs. This antagonism is precisely reflected in Figures 7 and 8 respectively. To each halving of areas there corresponds a point like E in the left column (intrinsic significance) and a computation of means in the right column (extrinsic significance). The geometric situation presented in the left column is then reduced to definitions given earlier by Gregory. This is done with a certain elegance; it is the usual synthetic presentation of geometry. By contrast, the right-hand column is self-contained, as any computation should be, and is therefore contrived. This explains why the right column appears somewhat longer; it is the analytical presentation. The opposition between the two presentations is reinforced by parallel tools: for example, Gregory systematically uses triangles in the left column, and parallelograms in the right.

In the left column, a maximal triangle, like DEG, is used. This simply means a triangle of maximum area when E traverses the hyperbolic arc DG. The point E, a point of intersection, is also the (unique) point of the hyperbola where the tangent is parallel to the chord DG. We keep a geometric property and therefore the use of maximal triangles simplifies the vocabulary of the dichotomy though not the proof. Gregory has to use previous results, for example the equality of the areas of the two rectilinear triangles DIE and ELG. This is not an obvious result, and the geometric proof of Proposition 104, is rather painful.[15] It is then easy to show (second paragraph of the left column) that the maximal triangles in ID, IE, EL, and LG have equal areas, and so on.

For the right-hand column, and for the same purpose, a computation is conducted. The calculation is done using proportion theory only, without recourse to algebraic relations. This makes the task rather clumsy for a modern reader. Without too many changes, we can visualize the path followed. Starting from

$$\frac{LI}{DH} = \frac{EG}{LI} = \frac{MK}{EG} = \frac{EG}{MK},$$

we also get

$$\frac{DB}{LN} = \frac{LN}{EO} = \frac{EO}{MP} = \frac{MP}{FQ},$$

due to the characteristic property of the hyperbola, which can be stated as

$$\frac{AH}{AI} = \frac{LI}{DH} .$$

Thus we also have, only on the axis of abscissas,

$$\frac{AH}{AI} = \frac{AI}{AG} = \frac{AG}{AK} = \frac{AK}{AC} .$$

A classical rule of computation in proportion theory gives

$$\frac{AH}{AI} = \frac{AI - AH}{AG - AI} = \frac{HI}{IG} .$$

But

$$\frac{AI}{AG} = \frac{EG}{LI} .$$

Therefore HI/IG=EG/LI, for which an interpretation in terms of areas is obvious (HI.LI=EG.IG). The rectangle with opposite vertices L and H has the same area as the rectangle with opposite vertices E and I.

Now, in complete analogy with the left-hand column where we obtained the equal areas of maximal triangles ID, IE, EL and LG, we have exhibited four equal areas, namely the areas of the parallelograms LH, EI, MG and FK.

From now on, the two proofs will follow exactly the same pattern. From two given areas equal areas are successively subtracted. At each step, what is subtracted is larger than half of what should be subtracted. For the left column this simply means that the area of a maximal triangle like DIE is larger than half the area of the hyperbolic convex segment DIE.[16] For the right column Gregory claims that the same is true, i.e. the area of the sum of two rectangles EI and LH is larger than half the area of the concave hyperbolic segment EGHD. But he does not even attempt to prove this. And he certainly does not wish to, because he relies on an argument by symmetry with what has been done according to the proof in the left column. This seems sufficient. This is an instance of reasoning by analogy. The trouble is that the result is in general false!

To check this point we revert briefly to modern terminology. For $O < a < b$, Gregory's assertion reduces to

$$\log \frac{b}{a} < 4 \left(1 - \sqrt{\frac{a}{b}}\right) .$$

This cannot be true for too large a value b (and a fixed value a) or for too small an a (and a fixed b). The relation holds if a and b are close enough (for example if $b < 2a$, as is easily shown). In this sense, as the process of dichotomy is performed continuously and the interval diminishes in length, the validity of Gregory's method is secured after a certain stage.

Now we have a better insight into Gregory's style. As we already said, the two parallel proofs are not independent. The first proof "helps" the second one.

But this does not come from a logical argument. The "deduction" stems from a similar reduction; it uses an analogous path. Thus for Gregory it was important, from a stylistic viewpoint, to exhibit a strict parallelism between the two proofs.

But we should not stop at this view. An architecture provoking such symmetries is also intended to underline deep contrasts. Gregory's world is not static, and moves are suggested by apparent symmetries hiding, or rather revealing, strong differences. Like the sculptures of the baroque age! To geometric modes in the first proof there correspond analytical ones in the second. To line segments attaching the hyperbola to its asymptotic lines in the right figure, the analytic apparatus, there correspond line segments[17] within the hyperbola itself in the left figure – the intrinsic geometry. The comparison between the first and second proofs helps us to go from classical geometry to analytic geometry. This transition is not realized – we have already mentioned the absence of polynomial equations. The algebra underlying this analytic geometry is simply the algebra deduced (or constructed) from the theory of proportions. But a modern reader can anticipate a new perspective, the perspective of Cartesian geometry.

In a rather subtle way, by a similar appeal to Proposition 116, proved long before in Book II, Gregory of Saint-Vincent makes the two proofs parallel in their conclusive step. As can be deduced from the sole title of this book (*De progressionibus*: about progressions), this proposition has only an analytical sense. It is stated without geometric ideas, for arbitrary magnitudes, in the spirit of Book V of Euclid's *Elements*. For a modern reader it is easily interpretable as a limit result, and a modern transcription will be sufficient for our purpose.[18]

Proposition 116 of Book II of the *Opus Geometricum* provides us with a stability result. It states that if it is possible to successively subtract from two given magnitudes X and Y magnitudes $x_1, x_2, \ldots, x_n$, and $y_1, y_2, \ldots, y_n$, respectively, in such a way that

$$\alpha = \frac{x_n}{y_n}$$

for all integers n, and that

$$X - (x_1 + x_2 + \cdots + x_n) > x_{n+1} > \tfrac{1}{2}(X - (x_1 + x_2 + \cdots + x_n)) ,$$

$$Y - (y_1 + y_2 + \cdots + y_n) > y_{n+1} > \tfrac{1}{2}(Y - (x_1 + x_2 + \cdots + x_n)) ,$$

then $X/Y = \alpha$. In particular, if $\alpha = 1$, then the magnitudes X and Y are equal.[19] This is precisely the case for the proofs already analysed, where X and Y are areas, whether of convex segments (Proposition 106) or of concave ones (Proposition 108).

A modern reader is not at all impressed by the result in Proposition 116. If $R_n = X - (x_1 + x_2 + \cdots + x_n)$, where the x_i and X are positive real numbers (magnitudes), then the inequalities $R_n > x_{n+1} > R_n/2$ imply the inequality

$$R_{n+1} < \frac{R_n}{2} .$$

Therefore $\lim_{n\to\infty} R_n = 0$, and so

$$X = \sum_{n=1}^{\infty} x_n \,.$$

Similarly, $Y = \sum_{n=1}^{\infty} y_n$, and if $y_n = \alpha x_n$, then we deduce easily that

$$Y = \alpha \sum_{n=1}^{\infty} x_n \,,$$

or

$$Y = \alpha X \,.$$

The techniques employed by Gregory of Saint-Vincent are precisely those which would lead to a theory of limits. Gregory stayed within the values of geometric progressions and their limits; recall that he uses no polynomial algebra.[20]

Like any artist, Gregory does not exhibit the scaffolding of his achievements. We are the ones to do the deconstruction. Of course, we are guided by his logical hints and his stylistic signals. But Proposition 109 is not the end of Book VI of the *Opus Geometricum*: there are 140 more propositions to come! We may feel lost within this architecture which plays the game of new structures hidden by mirroring elements.

A ROAD OR A MAZE?

The same reproach can often be levelled at authors who do not attack a definite problem. The more proofs they provide, the more desperately we try to find the way through the maze they seem to have built.[21] This is probably because in mathematics we are accustomed to contemplating completed structures – generally self-sufficient – and are far less ready to acknowledge open buildings, structures under construction without blueprints delivered in advance. A new method has a far better chance of being accepted if it solves explicitly an old conjecture; we even know that the method may be more important than the conjecture it settles. But, at least, we feel freer to circumscribe the whole method by grasping the problem involved.

Gregory of Saint-Vincent offers no completely solved problems. He meanders from geometric methods to analytical tools with no decisive questions to settle. If we may intrude for a short while into psychology, we could even assert that his strange and obstinate search for squaring the circle plays the role of a limited, if spurious, horizon, which (at least theoretically) may put an end to his investigations, and thus prevents his analysis from appearing endless, even if this is contradicted by the very length of the *Opus Geometricum*, perhaps the longest text ever published in mathematics.

But let us go back to Proposition 109, which states a remarkable functional property of the hyperbola: areas transform geometric progressions into arithmetical ones. A name ought to be given to such behaviour. Naturally we may think that logarithms should enter the picture here. But Gregory never mentions logarithms, a term coined in 1614 by Napier with this property in mind.[22]

Even if he ignored logarithms, we naturally think that he should have used a name for the stated property of the hyperbola. The fact that he did not raises the question as to what he was aiming at. The answer cannot be a simple one.

Clearly we have seen a route, the analytic path. In the proof of Proposition 108, for example, the parallelism led from a maximal triangle to a geometric mean, considered as an operation on abscissae. But there was already a name for it – geometric mean – which moreover has a geometric significance; there was no need of another name. Conversely, the absence of a known geometric interpretation of the property of hyperbolic areas left what has been found without a name!

We must realize how strange the result must have appeared to Gregory. He certainly knew that he had invented something of the same value as Archimedes' squaring of a parabolic segment. But he also knew that his result was of a different kind. It was not a quadrature, even in an extended Greek sense, as no precise square figure was proved equivalent to the convex hyperbolic segment; there was simply an equality between two unknown areas (the convex or concave hyperbolic segments). It is with good reason that we interpret the result as a step into integral calculus. We immediately perceive that a new world was coming into being, a world which could not be reduced to the old geometric one, in the sense that it could not be reduced to polynomial functions, a world which was worth investigating for its own sake, a world where no boundaries were *a priori* visible or assignable.

Was it possible to do something other than browse through the new landscape? Gregory did not build a maze. Confronted by so large a domain, he could only lay out some tracks. His behaviour is very much like that of his Jesuit colleague, Father Athanasius Kircher when confronted with new domains, whether they were Chinese characters or monsters. In both cases, no framework or general structure being available, a sort of classification was proposed which was neither rational nor experimental. Repetitions were constant.

So they were used in *Opus Geometricum*, sometimes for good reasons (and we have tried to justify the purpose around Propositions 109 in Book VI), sometimes for lack of other ways to acknowledge a common pattern. Had we space enough, we could investigate a curious vicious circle in Gregory's work on the hyperbola.[23] Vicious circles, by the way, are not uncommon in the work of authors dealing repetitively with the same properties, presented under various facets. The image of a road might then give way to the nightmare of a maze!

CONCLUSION IN A METAPHORIC FORM: A JOURNEY

At a certain early stage of his endless book, Gregory confesses that he has been haunted for a long while by a common phrase appearing often in Euclid and Archimedes: "*Et hoc semper fiat*" (And this should be done over and over again). He adds: "*Titillavit me haec particula et coëgit cogitatione circa haec versari*"[24] (This short expression titillated me to the point where I would obsessively return to it over and over again). The Greek mathematicians used iteration to measure areas and this is exactly what Gregorius a Sancto Vincentio did in the hyperbolic case, justifying the whole process and entering a new territory. The word "iteration" derives precisely from the Latin *iter*, a journey. Gregory in his whole *Opus geometricum* describes a journey, or more exactly, we travel with him. Let us keep this metaphor of a journey. Starting from a known territory – the Conics – which we investigate with him, we reach another land: comparisons of areas which cannot be computed, a land as yet unknown. We have no timetable and no maps. In this new world we pass more than once through the same places, though not in the same mood. How could we coin names for this largely unknown land using only words of known territories?

Like every traveller, Gregory of Saint-Vincent dreamt of destinations, some quite magical ones. At the end of Columbus' voyage there could only be India! Squaring the circle could have been Gregory's India. Trisection of an angle or duplication of the cube could have been his other aims: classical fancies. However, they were not reached! The new-found world could prove richer, though perhaps not for the first traveller. Therefore the leisurely journey – how could it be called otherwise when it is so long, with no precise destination? – does not turn out to be a happy one. Sad tones are perceptible: "*Benevolent reader, from the last proposition you clearly see where the previous propositions aimed at*". A clear statement that this was not clear before. "*To be able to find a way, thanks to progressions and hyperbolae, to include two proportional means between two given quantities*". This is precisely equivalent to the duplication of the cube! Unfortunately the duplication was not reached". *Had it been possible to solve the last proposition with the same success as we solved the previous ones, we could have succeeded.*" No luck! "*In a certain way, waiting for more, we have traced a path; if you wished, from just one last effort, you could try to get access to where we still could not reach*".[25]

Finally at the end of this trip with Gregory of Saint-Vincent, some will be more convinced that they are right in presenting mathematics with precise aims and therefore in a fixed way. Others will still be happy with idling among results and will refuse the one-proof-only style. History cannot bring a sanction to the present time: it serves as an eye-opener. It is always a journey, even at a leisurely pace.

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NOTES

1. This work was carried out during a period as guest professor at the Institut de physique théorique, Université de Louvain.
2. Arnauld and Nicole (1683, 4è partie, chap. IX. De quelques défauts qui se rencontrent d'ordinaire dans la méthode des Géomètres).
3. Leibniz, 1686; Gerhardt, 1863, Vol. V, pp. 226–233.
4. Cantor, 1913, pp. 892–896). In his famous book on the history of mathematics (Montucla, 1799–1802), the author does not seem to appreciate the Jesuit mathematician. He was not impressed with him when he wrote his earlier book on the quadrature of the circle (Montucla, 1754.
5. Bosmans, 1901–1902, pp. 22–40.
6. Hoffmann, 1942.
7. 17 large volumes of manuscripts for which Hermann van Looy (1980, 1984) attempted a chronology. See also E. Sauvenier-Goffin, 1951. A careful study of the content of the manuscripts, in relation to the published book, would certainly reveal processes of thought in the mathematics of the early seventeenth century and may then complement an interesting study of Whiteside (1968). In particular this is true of the part devoted to the *ductus*.
8. Apparently, the *Opus geometricum* was never translated. Short excerpts exist in French, English and German (for instance in Kästner, 1800, Vol. III, pp. 225–247). I have translated into French the whole book VI (with some other propositions) and this will appear with a commentary: *Une algèbre de raison au XVIIème siècle: la quadrature de l'hyperbole par Grégoire de Saint-Vincent.*
9. To use history of mathematics for didactical purposes, at least for theoretical investigations in didactics, we have to pay the price and thoroughly analyse the data (Dhombres, 1978).
10. We could say that this triangle's property plays the part of a Cartesian equation $xy = a$. But Gregory does not make use of such algebraic notations. We should therefore resist the temptation to translate everything he says in terms of algebraic relations (more on this later). At least, the triangle property shows that any geometric progression of points on one asymptote gives rise to another geometric progression of points on the other asymptote (related to the first by an inverse ratio). It is to this kind of algebra that Gregory restricts himself.
11. Positaque ad diametrum AE ordinatim DG.
12. Iisdem positis. Which also means that the figure of reference should be the same as in Proposition 106.
13. Scholion provided at p. 119 of the *Opus geometricum.*
14. This is not the place to investigate more carefully the role played by images in the *Opus geometricum*, and their link with the general organization of knowledge during the sixteenth century. Images played many roles, a mnemotechnical one for example. This was remarkably well explained by J. D. Spence (1980) when dealing with Matteo Ricci, another Jesuit mathematician educated at the Roman college and sent to China where he translated into Chinese the six first book of Euclid's *Elements* (in the Latin version of Clavius, 1574). Images had symbolic roles and they were used extensively by the Catholic Counter-Reform led by the Jesuit Fathers. Images, in the sciences, might be seen as the interface between a hermeneutic conception, so strong in a man like Giordano Bruno and the rationalist tradition, embodied in a Francis Bacon. When browsing in Brussels through the manuscripts of Gregory of Saint-Vincent, one is immediately struck by the frequency with which figures like 7 and 8 appear in the notes and in the more mature texts. They even turn up with various commentaries! They are found even in the oldest folios and they must have been a recurrent image in Gregory's mind (for example ms 5785, p. 201 (red ink); ms 5784, p. 205; ms 5773–5775 at Proposition 124, or at Proposition 124, or at Proposition 150, Proposition 389 and Proposition 424).

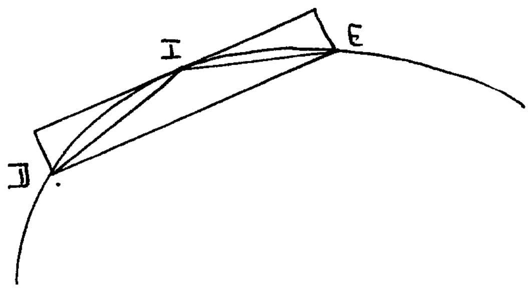

Fig. 9.

15. This result could easily be obtained via an analytical proof. Since the abscissae (or ordinates) of the three triples D, I, E–E, L, G–D, E, G each form a continuous proportion, this is also true of D, I, E, L, G. However, Gregory of Saint-Vincent does not wish to adopt an analytical style so early. He clearly waits for a more important issue. This underlines his strategy and the clear opposition between Propositions 106 and 108.
16. Since the tangent in I is parallel to the chord DE, this result is almost obvious if we complete the rectilinear triangle. Thus, two times the triangle area DIE provides the area of a rectangle, which is larger (due to inclusion) than the curvilinear area of the convex hyperbolic segment DIE.
17. To be faithful to the original figure, segments BD, FE and HG must be drawn. But they do not appear in the proof, contrary to the Euclidean habit of naming all lines. Quite symptomatically, the original figure confuses B and D.
18. We add this remark as a sort of dialectical complement to a previous note, in order to reassure those who may fear dealing with history of mathematics because they should then have to restrict themselves to the knowledge of one remote period. When studying old texts, we may certainly use today's achievements, for example to cut through some parts. It all depends on what we are looking for. An historian has no pretension to bring back the past in all its purity. In the present paper, we do not try to deconstruct all of Gregory's methods. We have selected one topic: his repetitious style. Thus we feel free to summarize in a modern way his thinking about limits (or "terms" to use his own vocabulary).
19. An analysis of the proof, and a translation, is to be found in the book already mentioned (J. Dhombres, *Une algèbre des raisons...*).
20. Our aim is not to present the complete work of Gregory. We focus on a few points concerning proofs, and have even given up a large part of the usual scholarly apparatus.
21. As they have a clear aim, we are not considering here authors who systematically provide as many proofs as they can of a given result. We know for instance books devoted to all known proofs of Pythagoras theorem, or of a fixed point theorem, etc. These books belong to the traditional genre of encyclopedias.
22. A logarithm transforms geometrical means, defined by a ratio or logos into arithmetical mans, defined by a number of arithmos.However, a Jesuit student of Gregory, Antonio da Sarasa, was soon (1649) to observe the coincidence between the property of the hyperbola and logarithms. This paper, written in a very rhetoric Latin, is translated in J. Dhombres, *Une algèbre de raison*, op. cit. For the functional property of logarithms, see Aczél–Dhombres (1989).
23. He wishes to present a converse of Proposition 109. If straight lines cut equal areas under the hyperbola, then abscissae must follow a geometrical progression. Under a continuity law, this result is, by the way, quite obvious. Gregorius aims at further results and investigates properties of the exponential function: this is a true inverse of logarithmic properties, at least if we think in functional terms. But in one proof, he makes use of a particular case of what he appears to be proving some propositions further!
24. *Opus geometricum*, Book 2, *De progressionibus*, p. 51.
25. *Opus geometricum*, Book 6, *De hyperbola*, p. 601.

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