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PRINCIPLES FOR THE DESIGN OF TEACHING

ABSTRACT. In this introductory article, after some initial discussion of an appropriate approach to mathematics as a curriculum subject, we sketch a theory for designing teaching, based on mathematical activity, situations, tasks, and interventions, exposing and resolving cognitive conflicts, changes of structure and context, feedback, reflection and review. We next review the main psychological principles underlying this theory, then consider some examples of teaching designs in the light of the theory. Thus we open the discussion of the theme of this issue, which continues with the fuller discussion of other examples in the remaining articles.

INTRODUCTION

Aims of Mathematical Education

Mathematics arises from the attempt to organize and explain the phenomena of our environment and experience. It has been expressed thus:

> Mathematics is ... an activity of organising fields of experience.
>
> H. Freudenthal (1973, p. 123)

> Mathematics concerns the properties of the operations by which the individual orders, organises and controls his environment.
>
> E. A. Peel (1971, p. 157)

These descriptions are somewhat non-specific, though Peel is clearly referring to basic mental operations such as classifying, comparing, combining, representing. A more specific characterization of mathematics is given by Gattegno:

> To do mathematics is to adopt a particular attitude of mind in which what we term *relationships per se* are of interest. One can be considered a mathematician when one can isolate relationships from real and complex situations and later on when relationships can be used to create new situations in order to discover further relationships.
>
> Teaching mathematics means helping one's pupils to *become aware of their relational thought*, of the freedom of the mind in its creation of relationships; it means encouraging them to develop a liking for such an attitude and to consider it as a human richness increasing the power of the intellect in its dialogue with the universe. (1963, p. 55)

Other authors have offered more controversial descriptions, which emphasize the logical aspects. One may recall Russell's "the subject in which we never know what we are talking about, nor whether what we say is true". For the purposes of guiding curriculum construction, we find the positive descriptions more helpful.

Education is normally seen as a forward-looking, purposeful activity, the aim of which is to develop pupils' capacities and knowledge so as to equip them

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more fully for adult life. This crude view is modified by the recognition that the aim of life-enhancement is available to some extent immediately, and indeed the concept of continuing education assumes that adults too can continue to derive benefit from educational experiences.

In terms of the mathematical curriculum, one is attempting to give pupils experiences of organizing and interpreting significant areas of their experience by the use of mathematical ideas and activities in a way which equips them to continue to do this in adult life. It follows that classroom activity should reflect the way in which mathematical experiences arise in adult life as well as providing genuine mathematical experiences for pupils in their own immediate situation. This does not, of course, rule out important ancillary activities, such as the memorizing of important data or the practising of frequently needed skills, but it does set them in place as subsidiary to the main mathematical activity of inquiry.

In designing lessons and building a curriculum, one needs to consider three aspects: the nature of the mathematical activity, the conceptual content, and the nature of learning.

Mathematical Activity

Most uses of mathematics involve a cycle of mathematization, manipulation, and interpretation — that is, recognizing in the given situation the relevance of some mathematical relationship, expressing this relation symbolically, manipulating the symbolic expression to reveal some new aspect, and interpreting this new aspect or giving some fresh insight into it in the given situation. A relatively complex case is the use differential equations for measuring the vibrations of a string leading to the prediction of a set of normal modes of vibration — a fundamental and a sequence of harmonics. A more elementary case is the use of the concept of multiplication and the corresponding algorithm to determine the cost of a certain weight of goods at a given unit price.

It may happen that the transformation of the mathematical representation, which is the middle part of this cycle, gives rise to some new relationships or procedures which can then take their place in the body of mathematical knowledge. In the examples quoted, this may consist of new methods of solving the differential equations or a recognition of the inverse nature of multiplication and division. We might think of the former type of activity — the making and using of a mathematical representation of reality — as the typical *applied* mathematical activity, and the latter as pure mathematics. (The Dutch workers use the terms "horizontal" and "vertical" mathematization for these processes.)

Traditional mathematics instruction has assumed that the part of this process which most needs teaching is the middle part — that is, the algorithms for calculation and the methods of solving the equations. Even from early times, these methods were the precious treasures handed on from one generation of priests or clerks to the next. Historically, there was progress in the way the computations were schematized, from the abacus or calculation table with lines and stones, to the abacus on paper and lastly the column algorithms. But traditional mathematics lessons have consisted of the demonstration (sometimes with explanation) of a single method followed by practice with a variety of different numbers. Converting fractions to decimals or percentages, performing operations on directed numbers, and solving proportion problems have all been dealt with in this way. When the problem structure varies from the standard type (for example, when givens and requireds are rearranged), it has been assumed that there is a need for further specific teaching, rather than the extension or adaptation of the basic idea by the pupils themselves to apply to the changed situation. In geometry, the traditional model has been to demonstrate the *proof* of a theorem and then to set exercises requiring identification, in a more complex diagram, of a figure, to which the conditions of the theorem apply, and to use the result of the theorem to deduce some new property of the diagram. What is clear now, however, is that the initial recognition of the mathematical relations in the situation and their representation also present difficulties as serious as those met in the manipulation phase, and that these have not usually been adequately treated. To use an analogy, the emphasis has been on learning to use tools and not on making furniture; and when the latter is attempted, it demands strategic capabilities — concerning planning, designing, costing, choosing materials, and selecting tools — which have not been developed.

Thus, the pupils' main lesson experience should be of genuine and substantial mathematical activities, which bring into play general mathematical strategies such as abstracting, representing, symbolizing, generalizing, proving, and formulating new questions. These are the activities which embody the *raison d'être* of the subject. Alternating with these should be the learning of the particular concepts and skills needed for the exploratory activity.

Learning and Its Outcomes

Given that the pupils are to be offered activities which embody the characteristic mathematical strategies, and which embrace the major concepts of the subject, what needs to be done to turn these into *learning* experiences? That is, what can we do to make it more likely that the pupils will actually perform better when they meet these or similar tasks again? Learning is not just success in the present task but improvement in capability. This factor has been neglected in some recent pedagogies, which assume that a sequence of gently graded problem-solving tasks results in learning. Our research (see Bell, 1993, pp. 115–137) has shown that such improvements are short-lived unless positive steps are taken to make the gains more permanent.

Skills, Concepts, and Strategies: Exploratory and Focused Activities

The learning of mathematics involves the development of a number of somewhat different types of acquisition; in our discussion of mathematical ability, we have drawn attention to the need for learning general mathematical strategies, as well as particular concepts and skills (Bell, Costello, and Küchemann, 1983). To some extent these all figure in every mathematical activity, but if one is concerned, as we are, to promote substantial learning, it is helpful to focus at any one time on some particular aspect, to raise it to the level of consciousness, and to offer appropriately intensive experience. Thus, to develop the strategy of generalizing one would offer a set of experiences, of different types, in different contexts and in different conceptual fields, but each requiring the forming and expressing of some generalization, and one would draw attention to the characteristic features of the process. Particular skills or concepts would be dealt with in a similar way. It is thus appropriate to alternate general exploratory activities with work focused on related specific concepts, strategies, or skills.

In some respects, the requirements conflict. For example, in an activity aimed at developing the ability to carry through an investigation, in which one follows up each discovery by choosing an appropriate question to tackle next, one cannot control which concepts and skills will be involved in the work as it progresses. Conversely, when the aim is to work on some particular concepts and skills, it is necessary for the discussion to be guided so as to explore the various aspects of those concepts; one cannot at the same time allow the inquiry to take its direction from what appears the most relevant question to ask next.

The lesson sequences described in the following sections are mainly concept focused rather than strategy focused. However, the starting tasks could alternatively be developed into strategy- or skill-learning sequences by altering the focus of attention, thus choosing different aspects for repetition as the task is developed into a learning sequence.

PRINCIPLES FOR DESIGNING TEACHING

Our design principles are the following. First one chooses a *situation* which embodies, in some *contexts*, the concepts and relations of the *conceptual field* in which it is desired to work. Within this situation, *tasks* are proposed to the learners which bring into play the concepts and relations. It is necessary that the learner shall know when the task is correctly performed; hence some form of *feedback* is required. When *errors* occur, arising from some *misconception*, it is appropriate to expose the *cognitive conflict* and to help the learner to achieve a resolution. This is one type of *intervention* which a teacher may make to assist the learning process.

Another general mode of intervention is in adjusting the *degree of challenge* offered to the learner by the task; the extent to which the task itself provides this flexibility is a significant task feature. The next requirement is for ways of developing a single starting task into a multiple task, bringing the learner to experience a rich variety of relations within the field. Typically, this can be done by making *changes of element* (e.g., type of number), *structure*, and *context*. The degree of *intensity* of this complex of learning experiences is an important factor. *Reflection* and *review* are other key principles; they imply the perception and study not only of the basic concepts and relations within the tasks but also of the properties of the different types of problem within the field and of the methods of solution found — meta-knowledge of the tasks and of the activity.

In the following sections, we shall first review the evidence for the psychological principles on which this design theory is based, then discuss some examples of curriculum designs to establish how they relate to these principles. Evidence of the success of these developed teaching sequences will be considered, where it exists.

UNDERLYING PSYCHOLOGICAL PRINCIPLES

We shall consider *connectedness*, *structural transfer across contexts*, *feedback*, *reflection and review*, and *intensity*.

Connectedness

A fundamental fact about learned material is that richly connected bodies of knowledge are well retained; isolated elements are quickly lost. This well-established principle was demonstrated by Bartlett (1932); he gave people a number of story passages to read and then asked them at various intervals of time to recall the story. He found that, as time went on, the recalled stories become smoothed; details incongruous with the general picture were lost, and additional details fitting into the general pattern were unconsciously invented. Similar trends were observed when they attempted to recall and remake drawings. It is this principle which explains the importance of discussion which explores the relations surrounding a concept from all possible points of view. Two pictorial examples of this smoothing effect are shown in Figure 1.

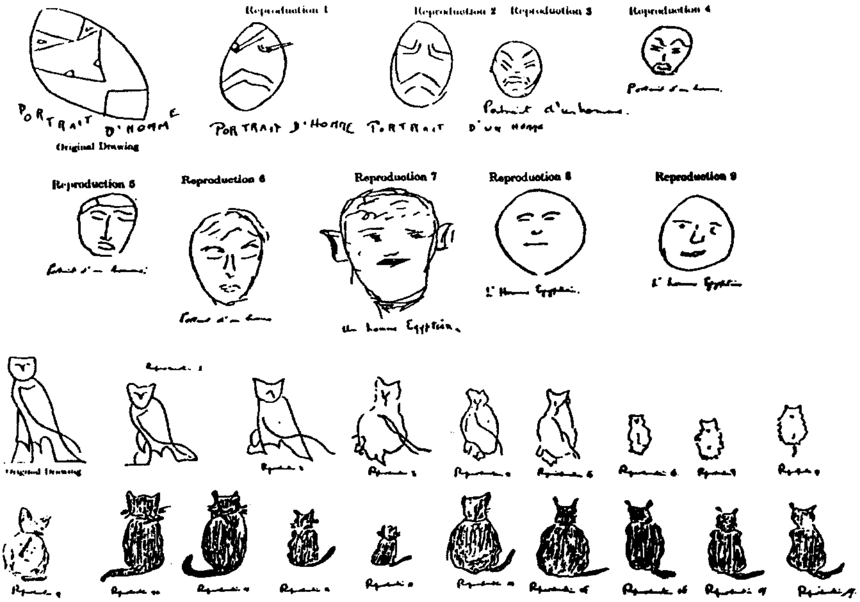

Figure 1. Two series of drawings from memory (Bartlett, 1932, pp. 178–181).

The connectedness principle is also amply attested by the difficulty of reorganizing a well-established cognitive structure to accommodate new material. For example, the principles that addition and multiplication make numbers bigger and subtraction and division make them smaller — formed from experience with natural numbers — persist when pupils are otherwise working correctly in the extended field of directed numbers or rational numbers. The importance of *cognitive conflict* is shown in both the Gelman experiment described below and in our own diagnostic teaching research reported later in this issue.

Connectedness also facilitates the retrieval of items from memory. It is generally accepted that long-term memory is of virtually unlimited capacity; the limitation on its effectiveness concerns the *retrieval* of desired material, and this clearly works best when the stored material has many interlinks and these are organized in a helpful way — for example, in hierarchies which the person can use when searching.

There is also considerable evidence that children use and invent methods of their own to perform tasks for which they have actually been taught standard methods in school. Their methods appear to be constructed from elements of well-connected and well-assimilated knowledge, and it is clear that in these cases the taught methods have not been similarly well assimilated.

Plunkett (1979) quotes several examples of calculations performed with understanding in a non-standard way. The calculation "What's 213 take away 188?" is solved: "Well, it's 12 up to 200 and 13's 25". Jones (1975) made a more substantial analysis of the methods by which the subtraction 83 – 26 was correctly evaluated. Three of these methods had been taught as standard. Twenty-five children successfully used one or other of these; but fifty children obtained the correct answer by one of fourteen other methods that had been developed without being taught.

Somewhat similar results are quoted by McIntosh (1978), who describes a variety of self-devised methods that depend on conceptual understanding and are reasonably efficient, at least for the individual concerned. A particularly large range of different procedures was adopted for the subtraction 431–145. An example is: "Took 1 from 431. Took 45 away from 430 made 385. Then added 1 to make 386. Finally I took 100 away answer 286".

The variety of thought-processes is surprising, and sometimes amusing, and contrasts strongly with the narrow path which the learning of standard techniques might be expected to provide.

A study of how effectively children were helped, in typical classroom settings, to build connections between concrete mathematical experiences and formalizations of the relevant principle was made by the project Children's Mathematical Frameworks based at Kings (formerly Chelsea) College, London (Johnson, 1989). The transition from concrete experiences to formalization of the mathematical principle was followed through in seven topics, with children aged between 8 and 13. Two examples were (1) area — moving from covering with squares to the formula length times breadth — and (2) fractions — forming and using the principle of equivalence numerically, after experience with divided regions. Typically, pupils were able to adopt the formalized principles but not able to relate them to the concrete experience. The degree of mismatch between the teaching and the pupils' state of knowledge was summarized as follows: typically, of six pupils taught, two understood already, two failed to understand both before and after, and the remaining two learnt some use of the principle, but not necessarily successfully and generally not with insight. This strongly points to the need for beginning lessons with tasks that allow the pupils to use and to show their existing knowledge. The teacher would build on this knowledge and help the pupils to *develop* their own methods, rather than expect the children to leave this knowledge aside and attempt, possibly unsuccessfully, to pick up a new method. At the same time, any errors and misconceptions which the pupils show must be dealt with.

A particularly important kind of connectedness is logical implication. It is often assumed (especially by those who have acquired facility in mathematics) that if a certain relation is known then so are all its logical consequences — at least if the chain of connection is not too long. If this were true the learning of mathematics would not present the difficulties it does. That this is not so for most people implies both that pupils should be persistently encouraged to stretch small amounts of knowledge as far as possible by exploring implications and that they should be made aware of as many such relations as possible.

Structure and Context

It is also a common assumption that once mathematical ideas are understood, the recognition of them in fresh contexts does not present any great difficulty. In fact, structural knowledge tends to be tied to the context in which it is learned and is not easily transferred.

The effect of familiarity of context and the difficulty of transfer of structural knowledge across contexts have been shown vividly in a sequence of experiments begun by Wason (1966). In its original form, the task used was as follows: subjects are presented with four cards, showing respectively A, D, 4, 7. It is known that every card has a letter on one side and a number on the other. Subjects are then given the rule "If a card has a vowel on one side, then it has an even number on the other side" and are told "Your task is to say which of the cards you need to turn over in order to find out whether the rule is true or false."

To solve this task, it is necessary to turn over A and 7 in order to check that the combination of vowel with odd number does not occur. The other two cards do not matter: from the rule, D can have an even *or* an odd number on the other side; the 4 can have a vowel *or* a non-vowel. Thus the key to the task is to consider only those cases which might falsify the rule. In fact, what most subjects do is attempt to *confirm* the hypothesis; this is equivalent to turning over the A and the 4 cards. Out of 128 university students, only five chose the two correct cards (Johnson-Laird and Wason, 1970).

In Wason's original rule, the elements are fairly abstract and the relationship between them is arbitrary. Later studies examined the effect of adopting a more realistic guise. Their findings suggest that the difficulty of the task is not so much due to its logical structure *per se* but to its content or mode of presentation. The first of these studies was by Wason and Shapiro (1971) who used the rule "Every time I go to Manchester, I travel by train" and presented subjects with four cards showing *Manchester, Leeds, Train, and Car*. They found that 10 out of 16 subjects correctly chose to turn over the cards Manchester and Car, compared to a control group of 16 subjects given the original task, of whom only two chose the correct cards.

An even greater increase in facility was found by Johnson-Laird, Legrenzi, and Legrenzi (1972) who used the rule "If a letter is sealed, then it has a 50 lire stamp on it". Subjects were asked to imagine they were post-office workers sorting letters; they were presented with the five envelopes shown below (see Figure 2) and instructed to "select those envelopes that you definitely need to turn over to find out whether or not they violate the rule".

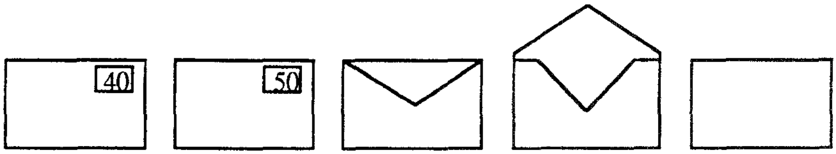

Figure 2. Letter task used by Johnson-Laird et al. (1972).

Subjects were given two such realistic tasks and two symbolic tasks (in which the rules were similar to Wason's original rule, but envelopes were used rather than cards). Again, the realistic tasks proved substantially easier than the symbolic — 22 out of 24 subjects getting at least one correct, as against 7 out of 24. It was also noted that there was a striking lack of any direct form of transfer between types of task and that on subsequent questioning only two subjects acknowledged that the underlying logical structures were the same. Other researchers make similar observations, adding strong support to the view that it is the context rather than the logical structure which is crucial to solving the tasks.

This work casts doubt on the traditional practice of teaching a mathematical concept or method, offering a mixed collection of applications, and assuming that the recognition of the relevance of the mathematics to the application does not present serious difficulty. It suggests, rather, that there should be extensive exploration of the structural relations within one familiar context, then repetitions of the study in another familiar context, as learners look for signs that the structural aspects are the same. Further contexts may be explored, moving learners towards unfamiliar ones and ensuring that those in which subsequent application is desired receive specific attention. Tasks which deliberately ask for the transfer of a given structure to a new context have also been found valuable — for example, "Here are some questions on number lines; what would they be in the context of money transactions?".

More direct evidence of failure to connect in-school and out-of-school experiences is provided by some recent ethnographic studies. Carraher, Carraher, and Schliemann (1985) studied the calculations used on the street by young Brazilian vendors and compared them with what was being done in school. One typical boy calculated the price of 10 coconuts at 35 *cruzeiros* each as follows: 3 is 105, then 3 more, 210, and another 3, 315, and 1 more, 350. In school, these boys were quite unable to perform similar calculations by the methods taught. Lave (1988) also found great differences between adults' competence in working out the best buys in the supermarket and their corresponding work in school.

Lave used the term *situated cognition* to describe the way these competences were linked with contexts. However, such evidence is perhaps unnecessary to convince teachers, who will be quite sufficiently familiar with, for example, pupils' apparent inability to recognize in a science lesson a piece of mathematics which they have undoubtedly studied in mathematics class, and the frequent confession by adults of their inability to see the relevance of much of their school mathematics to anything outside the classroom.

Feedback

The next learning principle is that the pupil should know immediately when she has correctly solved the given problem.

The importance of this is clear if one considers for a moment the learning of a pupil who works an exercise, getting some answers wrong but not knowing this until her book is returned, some days later, marked "7/10" — which she may regard as pretty good. The effect of this work has been to *reinforce* the three wrong responses as effectively as the seven correct ones. Even end-of-lesson provision of answers has the same effect unless time is provided to correct the errors, identify their cause, and take steps to avoid their future occurrence.

An experiment which shows vividly the positive effectiveness of immediate feedback in a conflict situation is that of Gelman (1969). The aim was to train five-year-old children to achieve the Piagetian conservations of length and number — that is, to recognize when two lines were of the same length, even though displaced relative to each other, and to know when two rows of objects had the same number, even though they might not cover the same space. The children were given 32 sequences, each consisting of six trials. Two of the sequences are shown schematically in Figure 3; they are actually presented using objects. The other problems were variations of this with respect to colour, size, shape, starting arrangement, and combinations of quantity.

For the length training, in each trial children were asked to point to two configurations which had the same length and to two which had different lengths. They were told immediately, "Yes, that's right" or "No, that's not right", but nothing more; the correct choice was not indicated. Sixteen six-trial problems were presented to the children on each of two successive days, alternating length and number problems. On the third day, the post-test was given, consisting of conservation tests on the concepts of number (trained), length (trained), liquid quantity (untrained), and substance (untrained). The post-test was repeated two to three weeks later. Two control groups were used, one of which had the same problems but without feedback of correctness, the other using problems in which the differences between the the configurations were obvious and

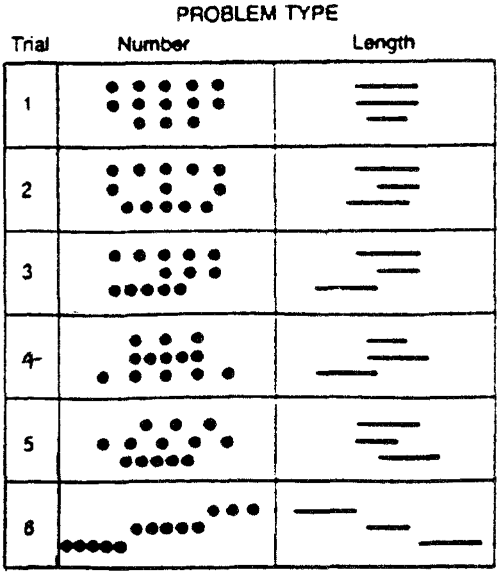

Figure 3. Number and length discrimination problems (Gelman, 1969).

qualitative, not quantitative — for example, two toy lions and one toy cup. The results showed that the trained children produced 90 to 96 percent correct responses in the post-tests on number and length and about 55 percent correct in the transfer to liquid quantity and substance. The non-feedback group scored 20 to 30 percent correct on number and length, and virtually nothing on the transfer tasks. The second control group scored almost zero throughout. In the delayed post-tests, the 90 to 96 percent responses were maintained, while the scores on the transfer tests increased from 55 to about 70 percent. It was also observed that the quality of explanations improved between the two post-tests. Gelman attributes the success to the large numbers of trials and the alternation of number and length sequences, as well as to the immediate feedback.

In conflict-discussion lessons (see Bell, 1993, pp. 115–137) feedback is an integral part of the process of discussion in pairs, in groups, and in the class as a whole. In other tasks, including games, the mode is predict and check (for example, with calculator or number line). In certain practice tasks (e.g., scale reading), the *sum* of each group of five answers is given as a check. In the Logo microworld tasks (e.g., Sutherland, 1993, pp. 95–113), feedback is in terms of succesful achievement of the desired picture on the computer screen.

Reflection and Review

Exploring relationships and resolving conflicts through discussion are in themselves reflective activities. Here we imply something more — a more global reflection on the process of performing the task and identifying the crucial steps, and on the new knowledge gained and how it fits into one's existing body of knowledge. This development of *awareness* is important in labelling the newly gained knowledge in memory in such a way as to make it accessible on relevant future occasions.

Direct experimental evidence of learning with and without specific acts of reflection is hard to find, even though it makes the difference between behaviourist conditioning and constructivist theories of learning. For Piaget, the key learning act was "reflective abstraction" and Schoenfeld (1985) has shown that successful problem solvers monitor their solving activity. There is, however, a body of evidence showing how the kind and amount of learning depend on the mental orientation brought to the learning activity.

Craik and Lockhart (1972) review a number of studies in which subjects were oriented to process material in different ways, and they argue that it is "depth" and "elaborateness" of the processing that determine how well material is subsequently recalled. In one such study (Tresselt and Mayzner, 1960), subjects were presented with a series of words and asked to (a) cross out vowels, (b) copy the words, or (c) judge whether the word was related to the concept "economics." On a subsequent free-recall test, four times as many words were remembered under condition (c) and twice as many under condition (b) as under condition (a). In another study (Shulman, 1971), subjects had to scan a list of words for features that were either structural (e.g., words containing the letter A) or semantic (e.g., words denoting living things). On a subsequent, unexpected recognition test, performance in the meaningful (semantic) condition was significantly better than in the structural condition, even though scanning time was about the same for both. A similar result was found by Hartley (1980) in a study involving student's recall of statistical material. In this experiment, polytechnic students were presented with a text on probability and asked questions at either a syntactical level ("underline any words with more than six letters") or a semantic level ("either paraphrase parts of the material or provide examples of the propositions"). Three post-tests were used: free-recall, cued-recall, and a comprehension test. The two "semantic" groups were much the superior on all post-tests: in the free-recall test they recalled sentences whilst the syntax group recalled words and short phrases. The two semantic groups recalled about the same amount of material, but the "paraphrase" group was superior on free-recall and the "example" group on comprehension. The experiment suggests that different tasks engage and develop different cognitive structures which result in different learning outcomes. The implications are that teaching should emphasize meaning, not particular verbal forms or formulae, and that asking for examples induces deeper-level processing than does asking for paraphrasing.

Intensity

It is well known that repetition is an essential element in learning. Questions remain concerning the effects of the degree of variety in the set of tasks; some of these are implicit in the foregoing discussion. There is clearly no general answer to this question, but there is evidence in some experiments that what might be regarded as excessive repetition has resulted in striking gains. It is also clear that while repeated *memorization* tasks may produce short-lived results, intensive *insight-demanding* tasks produce longer-term gains.

Brownell and Chazal (1935) showed that two months of daily drill on addition facts increased speed of direct recall, but the effect soon faded. Also, the work made no improvement to the pupils' methods for working out other facts from known ones (e.g., 9 + 8 is two 8's [16] and 1, so is 17).

Gelman's strikingly successful conservation training, discussed above, employed a large number of similar problem sequences, with controlled structural and contextual variation. The purpose in this case was to establish two new concepts by discrimination. In another experiment in which the finding of gradients of graphs was taught by a Gagné-type learning program, involving the step-by-step building of subskills, some students were required to answer three consecutive problems correctly in each unit before moving to the next, while others had only one problem to answer correctly. The first group took 25 percent more time, but achieved 50 percent more learning (Trembath and White, 1979).

Our own diagnostic teaching experiments (Bell, 1993, pp. 115–137) have shown that *intensity* is related to successful learning. Games provide such experience in that many trials with limited variation are involved and a discussion focused on one or a few points is itself an intensive experience. Our results suggested additinally that the most vigorous and intensive dicussions resulted in the greatest amount of learning. However, it is clearly not extensive repetition alone which has the effect: this property was possessed also by the contrasting teaching materials which we used. The presence of feedback and a high level of personal engagement are important.

TEACHING DESIGNS

We shall now elaborate somewhat the design theory sketched briefly above before considering some actual examples.

Situation, Task, and Intervention

The situation needs to have high face-validity — to be as close as possible to actual situations. This reduces the problem of transfer. For example, if we wish pupils to be able to operate mathematically in real-life situations, then such situations must be dealt with in class. Next, within the situation, a *task* containing key concepts is proposed, either by the teacher or, preferably, in discussion with the class. The task is attempted by the pupils and only when their initial responses have been made does the teacher intervene to offer hints or help towards a solution. This helps to ensure that the new knowledge will be well embedded in the pupils' existing cognitive structure. By contrast, a demonstration-plus-exercises method risks failing to make any contact with the pupils' actual knowledge, as was a real danger in the Children's Mathematical Frameworks project, referred to above (Johnson, 1989). It also ensures that those aspects of the problem which do present difficulty will be dealt with — for example, the problem-formulating stage or the construction of an equation to represent the problem.

Thus the first principle of lesson design is to *begin* by offering the target tasks — that is, appropriately challenging tasks that typify situations we want pupils to be able to deal with in the future. The second principle is *flexibility* — of task and of intervention. Flexibility of task is necessary to ensure that all pupils in the class, with their varied knowledge and abilities, can find a suitable challenge within it; flexibility of intervention means that the teacher should negotiate with the pupil in the course of the task, adjusting the challenge to keep it at an appropriate level.

Careful minimal interventions also encourage learners to stretch their existing knowledge as far as possible and thus to extend and enrich their conceptual network of facts, relationships, and their implications, rather than to assume that each new twist of the situation requires an additional element of instruction.

Reflection and discussion, with the learners expressing their perceptions of the situation in as many different ways as possible, help to connect the new knowledge firmly with the old; they also provide for the expression and sharing of different understandings on the part of different learners.

Feedback

In some cases feedback is intrinsic to the task; in other cases, a predict-and check-mode may be possible — for example, a mental calculation may be compared with a calculator result. In some games, the opponent may challenge a doubtful answer. In sets of written exercises, the sum of five successive answers, or something similar, may be given as a check. Sometimes definite feedback of correctness is difficult or impossible, but provision can be made for discussion of the task in pairs, or in a group, so that at least some errors and misconceptions can be detected.

Changes of Structure and Context

The development of a single task or question into a set which covers the field of relevant relationships in the situation can be achieved in several ways. The initial task can be varied by interchanging the roles of the given and the required information; the type of element (e.g., small/large whole numbers, decimals) can be changed. Reversals, such as asking pupils to make up their own questions, and to cover the set of possible structures, are possible, so are role-reversals such as "marking homework", where pupils are asked to detect and explain errors.

The principles above imply that this should be done within a single situation and context and then repeated in a second one, and so on, carefully controlling the perceptual distance between one context and the next so as to achieve eventual availability of the structure in any context, but without provoking breakdowns by demanding too great a jump. It may be an appropriate task, as part of this process, to ask specifically for the translation of a given question into a new context, such as "If this number line question were put in a money context, what would it be?".

Review activities need to be built in at the end of a cycle of exploration. They may take various forms, such as a class discussion pooling the various results and/or methods which different pupils or groups have found; a self-check test, possibly including a self-assessment against a list of objectives within the topic; a story or written report of work done or results found; the making of a "concept map" of the various concepts, relations, facts within the topic — identifying their place in the broader field of mathematics. Looking back is not of itself an attractive activity for most pupils, but some of these possibilities may be attractive and effective.

Some degree of *intensity* of experience is necessary for effective learning, whether of facts to be memorized, concepts to be understood, or strategies to be acquired. *Creative activities* — such as requests to design similar tasks of their own, to make up some similar questions, or to write a story around a given situation or a report of the outcomes of an exploratory activity — demand reflection on the material and the mental reorganization of it. They create new connections and so establish the material more firmly in memory as well as provide more accessibility to it. They can also provide personal creative satisfaction.

Differentiation by Individualization or Flexible Tasks

Traditional teaching does not take much account of differences between the pupils in a particular class with respect to their speed of learning or to their previous knowledge. Considerable differences typically exist, even in classes which have been formed to be as homogeneous as possible. One solution to this problem which has been identified on quite a wide scale is to provide a system of individual learning material, booklets, or worksheets through which pupils may progress at their own rate. Such schemes are in some places quite popular, but research shows that they tend to produce short-lived and superficial learning, probably due to the drastic reduction in the teacher's role as mediator and interpreter and the reliance on the printed word and diagram as almost the sole means of communication with the learner (Bassford, 1988; Bell and Bassford, 1989; Bell and Birks, 1990; Bell, 1993; Birks, 1987). In contrast, those methods which we have found to be effective involve the crucial management of the learning situation by the teacher. These methods all employ *flexible tasks* in which the particular questions addressed are determined initially in discussion between the teacher and the pupils. Typically, also, the pupils generate further questions and challenges for themselves and each other, thus posing questions at an appropriate level for themselves and ensuring that they are all working at their personal frontier of knowledge.

FIRST EXAMPLE — BUS NUMBERS

Using the Social Situation

Exploiting interaction in the classroom is an implicit feature of the lesson aspects we have discussed. It is a more explicit feature of some of the work done by French researchers, which will be discussed later. The situation depicted in Figure 4 is one of several used by van den Brink (1974) to introduce addition and subtraction to first graders with the use of an arrow notation. The sequence begins with active games in the playground and in the classroom, with rides on the school bus, discussion of the number of children on the bus, and the number getting on and off at stops. Later problems are presented pictorially and, later still, diagrammatically. Thus the "task" begins as a (dramatic) *situation*, within which certain aspects are identified for particular study.

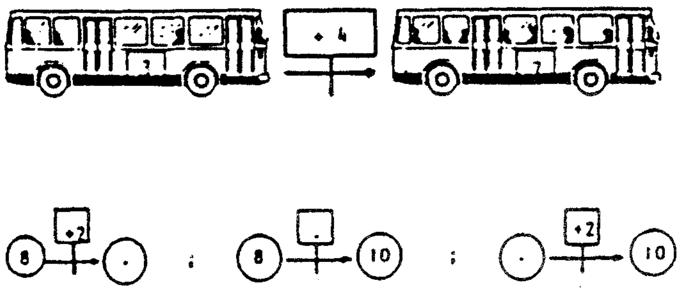

Figure 4. Bus-number problems (van den Brink, 1974).

Structural Variation

Note that both addition and subtraction are involved together from the beginning, as is natural in this situation. Also, the natural concept is not that of a binary operation but of an initial state, a change, and a final state, and indeed of the repetition of this cycle as the bus moves on from one stop to the next. Other situations (e.g., double-decker buses) are used to exemplify other types of addition such as the binary union of two sets.

The children usually solve the problem by counting. For all types of addition, this is straightforward. For the first type of subtraction, the change-unknown problems, it is almost as easy, as one has only to keep tally of the number of number-words counted: from eight to nine, then ten, is one, two more. The initial-state-unknown problem is considerably harder, to conceptualize as well as to calculate, and for many children it would be impossible except for its attachment to the real context — how many passengers were there on the bus before it reached the stop. The solution requires a reversal of the sense of the "plus two"; two must be taken away from ten. Once this is appreciated, the calculation requires counting back two numbers from ten; most children manage this quite well. An alternative approach is by trial: "Where would I be if in two more places I would reach ten? I think, about eight. I'll try it. Eight — nine, ten. Yes."

Thus, within this situation, problems are posed covering the three different additive structures which may be symbolized as ST(S), S(T)S, (S)TS — that is, as an initial state transformed into a final state, the brackets indicating which element is unknown. The structures [SS](S) and [S(S)]S, where two states combine to give a third, as in the binary union of sets, do not occur in this bus-stop situation, but they may if attention is focused on double-decker or articulated buses. It is helpful if the children can recognize quite consciously the different types of problems; we have found that this is in general easier for them than we expected, at least for older pupils.

Errors and Misconceptions

The most common numerical error at this point is to count both starting and finishing numbers — in this case, to count eight, nine, ten and take three as the solution. Apart from numerical errors, van den Brink lists four major conceptual errors which occur in these problems. These all concern the roles played by the numbers and the relations among them. The first two both appear to stem from the difficulty of co-ordinating two items of data. Shown pictures of a partly full bus and an empty one, and asked how many passengers got off, some children simply count *all* the empty spaces, thus failing to take account of the number actually on the bus before the stop. Similarly, when asked, in another problem, of how many *got on*, some count the total number shown as being *on* the bus. The second difficulty arises from confusion about the different roles of the different items of information. This occurred most frequently when the problems were represented diagrammatically rather than in full pictorial form as shown in Figure 5.

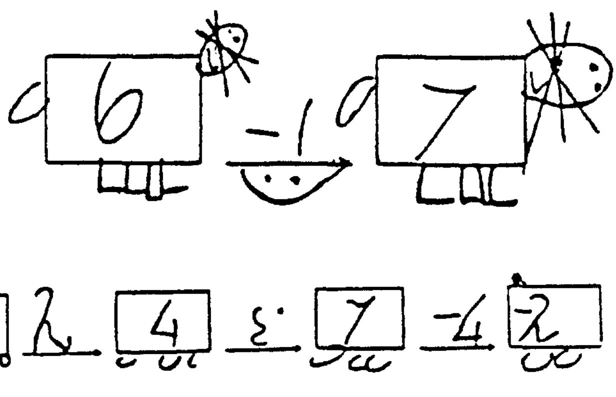

Figure 5. Children's scripts, bus problems (van den Brink, 1974).

Both of the illustrations in Figure 5 are problems made up by children. For the first, Raymond explained, "then the cat ate six. Then he got two more chunks" (pointing to the dots under the arrow). "He took one away. Then he had seven". Here Raymond has made a sum which is somewhat more complex than the usual ones, since the increase and decrease occur together, without any recording of the intermediate state. The second problem shows a more complete loss of the meaning of the representation: Kim's explanation was that all the numbers in the bus-boxes are, in fact, changes and some of the arrow numbers actually represent numbers on the bus. This is a typical example of loss of meaning; the connection of the diagram to the real situation has been lost and needs to be remade. This is, of course, the "treatment" for all of these conceptual errors — to get the children to re-create the real situation which the problem describes and to think about the problem in that context.

Choice of Situations and Tasks

First there is the *choice of a real context as a starting point.* This is important from several viewpoints. Most obviously, it adds interest and secures fuller involvement of the pupils. But equally it helps them to relate mathematics and reality, so that they are more able to use their school experience to add insights into their daily environment. When making bus journeys, they will know that they can, if necessary, find out how many people got on or off simply by counting the number of people on the bus before and after and performing a subtraction. This gives a modest sense of that greater power over the environment which is one of the most fundamental aims of mathematical education. It also enables significant learning to continue in some of the many hours of the day and week outside the mathematics classroom.

Second, the problems contain immediately a *variety of structures*, but remain a single context for some time, before taking up similar kinds of problems in a different context. This promotes the learning of inter-relationships and avoids the degeneration of learning to identifying those child-invented rules which enable the immediately given problems to be solved without any depth of understanding. This phenomenon is widespread when limited sets of problems are given. For example, a set of problems all involving addition leads to the rule "Pick out the two numbers and add them", whereas a mixed set involving both addition and subtraction (and possibly other operations too) makes it necessary to recognize features of the situation which indicate the operation. In contrast, compare the task "Fill the gaps in 3/4 = 6/8 = ?/12 = 12/? ...". This allows correct answers to be obtained simply by continuing the sequences on top and bottom, without any understanding of equivalence of fractions. A real context allows such pedagogical hazards to be dealt with by "thinking what it means in the real situation".

Choice of Other Contexts

Situations of a similar kind, but based on skittle games (knock down three, how many are left?) or marbles (had five, won three, how many are there now?), are seen by van den Brink as going along with the bus problem. He notes that each of these presents somewhat different sets of structures and of number sizes (e.g., in skittles there are always complements of nine; in marbles there is greater emphasis on changes and combinations of changes). Thus a progression through these situations provides both consolidation and extension of the earlier work.

Focus on Awareness of Structures and on Possible Misconceptions

In this situation, we know that the initial state-unknown problems are hard and that confusion of state and change occurs and is related to assumptions of starting or finishing empty or full. It is therefore important that problems be included that give rise to these errors and that they be exposed, discussed, and corrected. We have mentioned above that it appears that *awareness of the problem structures*, and how different errors arise in them, can be learnt by children; how to facilitate this at different ages is an important research question.

This mode of problem does not provide direct, immediate feedback to the learners on the correctness of their response. "Retelling the story in a different way" might be possible — effectively checking the solutions to the harder problem structures by redescribing the solved situation in "forward" terms. Thus the solution to the reverse bus problem, $? + 2 = 10$, might be checked by restating it as "eight on the bus, two got on, then ten were on altogether". Checking with neighbours certainly happens, but at this age (about 5 or 6) it is also necessary that the teacher look at the children's book and hear the children's account of it to monitor their understanding and detect error and misconceptions.

In this situation, after the introduction of the story and some oral problems, the pupils solve several given problems in writing, then make up a few similar ones of their own. Van den Brink has taken this creative principle a state further in getting the pupils to make up pages of an arithmetic book "for the children who will do this next year". Three such pages were prepared at intervals through the year, and finally the pupils were asked to arrange these in order of difficulty. This provoked intensive reflection on the work which had been done, on which problems were easier and which harder, and why, and which types should be grouped together, and so on.

Chain-sums, where the bus makes several stops, are also included. The situation was introduced first by means of active games in the playground, with rides on the school bus, and in the classroom, with discussion of the numbers of passengers getting on and off. The pupils were then given problems on a worksheet; initially, the buses were drawn, later the problems were presented systematically. The pupils similarly began by drawing the buses and writing in the numbers, later becoming more schematic. Mistakes were detected by the teacher's marking of the written work. These appeared later in the year when the problems were presented without re-enactment of the game. The mistakes consisted mainly of breakdowns in conceptualization of the problem — for example, mistaking numbers *on* the bus and numbers *getting on*; these were corrected by thinking back to the real situation.

Closely related structurally to the bus situation are some problems used in our own work on "change" problems with negative numbers (see Bell, 1993, pp. 115–137). These include movements up and down the popular music charts and the football league tables, and a "world weather" situation, in which journeys around the world involve changes in temperature. As in the bus situation, all three kinds of change problems occur (the unknown may be the final state, the change, or initial state). Arrow diagrams similar to van den Brink's are used and an aim of our work is to develop in the children an awareness of the different types of problems.

Reflection and Discussion

When competence has been acquired in solving the main types of problems, it is possible to reflect on the different types. One may note, for example, that additions and subtractions can cancel each other out or that when one asks how many people were on the bus *before* a certain number got on or off, this "thinking backwards" requires the opposite operation to the one stated in the question. (For example, if there are three *more* after the stop, there were three *less* before.)

SECOND EXAMPLE — RATES

A unit of work from the Nottingham Diagnostic Teaching Project (Bell et al., 1985) provides an example to compare with that of van den Brink. Entitled "Rates" (Bell and Onslow, 1987; Onslow, 1986), it concerns problems involving two extensive quantities and their quotient (an intensive quantity) — in particular, problems of price (quantity, cost, unit price), speed (distance, time, speed), and miles per gallon. Research using tests and interviews shows that many pupils' solutions to these problems fail because of the influence of numerical misconceptions, such as "multiplying makes bigger" or "small/big is impossible", and also because distinguishing the roles of the two quantities in a rate such as kilometres per minute or pence per gram is a serious obstacle (Bell, Fischbein, and Greer, 1984; Bell et al., 1989). A representative task, aimed at the latter difficulty, is entitled The Thrifty Thackers (Figure 6).

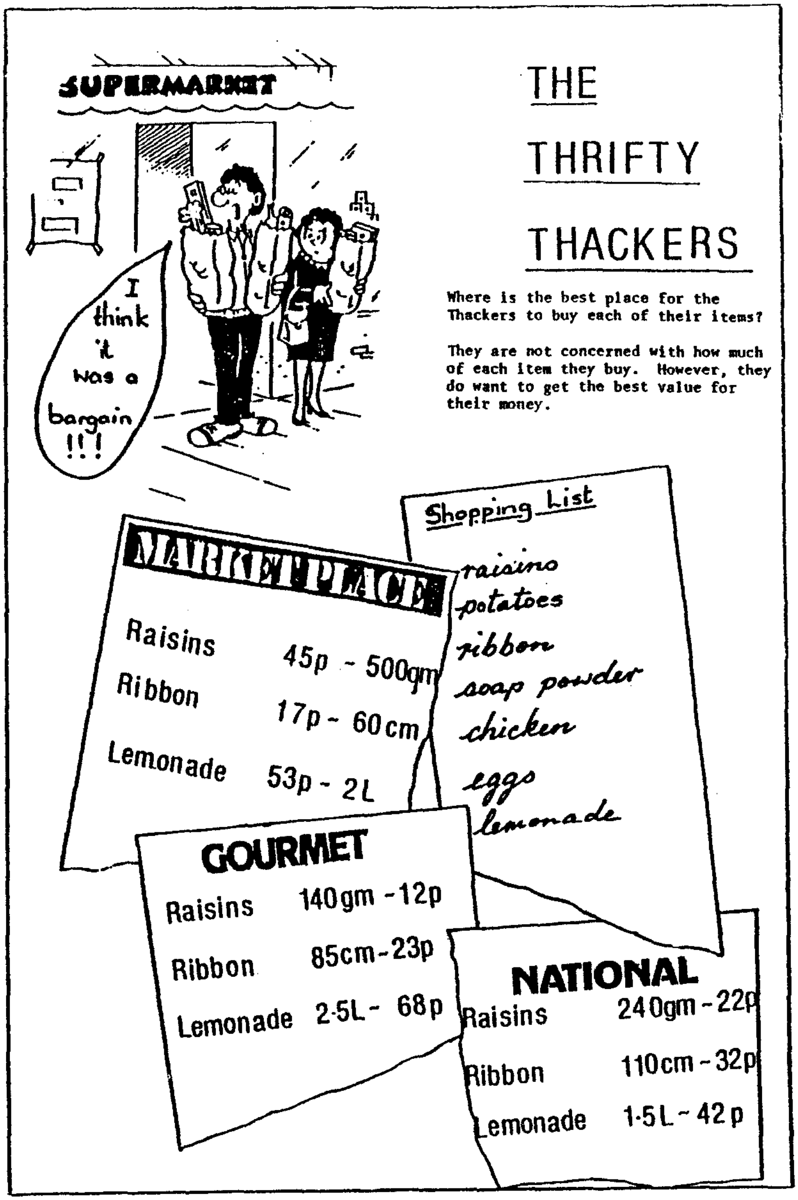

Figure 6. Task from rates teaching unit (Onslow, 1986).

Here the concepts of pence per gram and grams per penny for each item provide the means of deciding the question, though the comparisons in some cases can be made by more global rates (e.g., amounts for 10 pence). The pupils are asked, after discussion of the various ways of solving the problem, to study the possibility of using both pence per gram *and* grams per penny in each case and to observe the relation between these quantities. Thus by sharpening the potential conflict, attention is drawn to the area of possible confusion and the distinction is clarified. The request to make up their own questions was not put to the pupils in this lesson, but to *construct* a set of data like this would be an appropriate and interesting challenge. A separate lesson was devoted to making up questions from the starting points illustrated in Figure 7.

The pupils in question were not used to this task; they found it hard, and many attempts failed. But it was clearly valuable in exposing the difficulties and drawing attention to the essential characteristics of such problems.

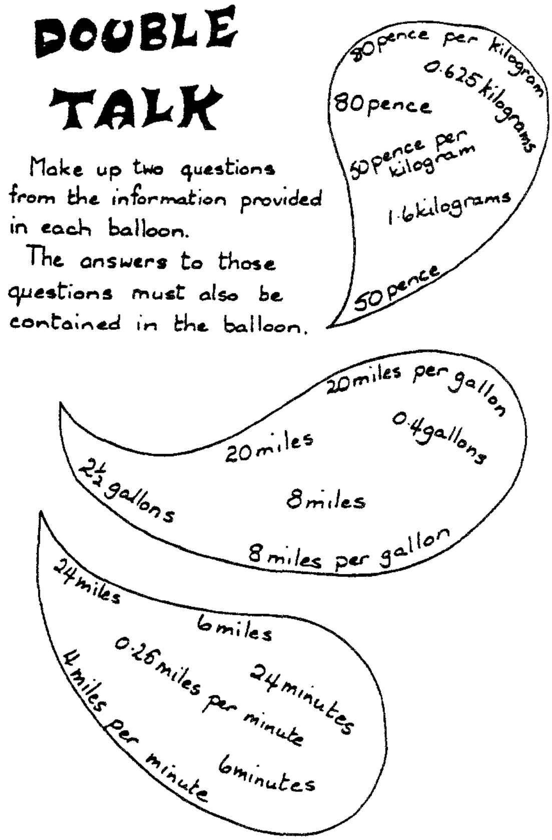

Figure 7. Making up questions task (Onslow, 1986).

A somewhat different question-generating activity has been tried in some experiments conducted by French researchers (on learning of linear functions or rates). In this case, a small amount of data was presented and pupils, working in groups, were asked to make a list of some possible questions which might be answered from this data. From each group's list, one question was chosen, by the teacher, for the group to work on. Thus the generation of questions by the pupils was used, in this case, to provoke the pupils into a more global and extensive consideration of the problem and the relationships in it than would be obtained if they were simply given problems to solve (Rouchier, 1980).

The *Rates* unit also contained two games; one of these is shown in Figure 9.

Another type of game is Directed Scrabble. In this, players place cards to make complete "words" on the table, each word connecting at some point with the existing display. The thinking required — for example, "3rd... Down 7.... 10th", or as shown in Figure 8, "–6, up 4, –2" — brings the crucial concepts into play. Any misconceptions which exist show up in wrong card placement and thus are challenged and corrected. The game may be played either with the ordinal position cards (3rd, 4th, etc.) or with directed number cards; the "move" cards are the same in each case.

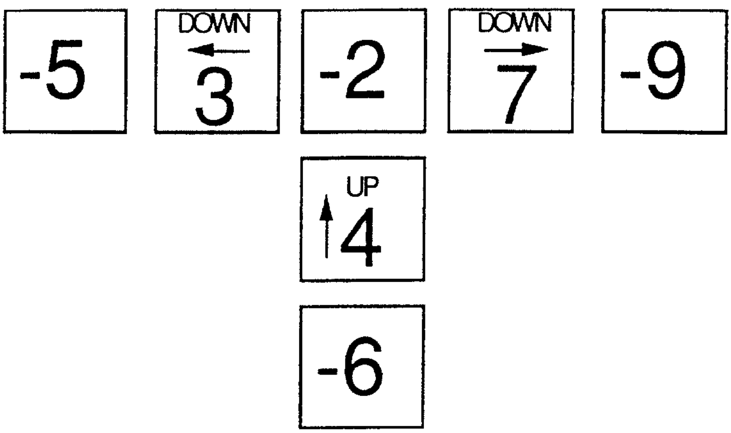

Figure 8. Directed Scrabble game

These games, focusing on the crucial thought patterns, including common misconceptions, may be used repeatedly, in appropriate units of work, and thus make further contributions to long-term development. They can also emphasize common structures in different contexts, as when the isomorphism is recognized between the ordinals and the negatives in the two versions of Directed Scrabble.

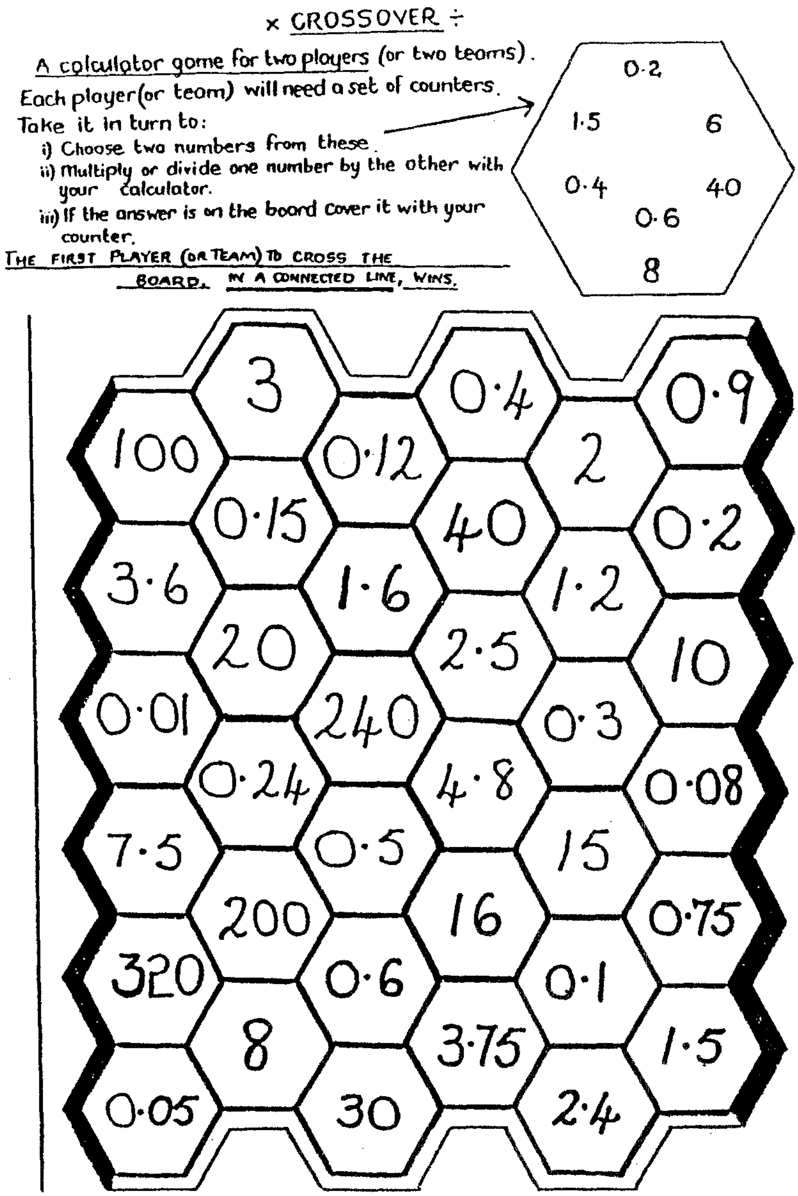

Figure 9. Number game (Onslow, 1986).

Cognitive Conflict

All the activities described here are designed to focus sharply on the key concepts in the field in question and thus help new concepts to be formed by refinement of the more primitive, cruder concepts which precede them. In some cases, studies of conceptual development show the persistence of applying concepts appropriate to one field in an enlarged field where they produce inconsistencies and errors (e.g., multiplying makes bigger). In these and other cases, notations or other perceptual features may lend themselves to misinterpretations.

The presence of such misconceptions, which may be more or less deeply rooted, may be shown by consistent patterns of error or by statements made in the course of discussion. Previous research may also have indicated the likelihood of the existence of certain misconceptions in the topic in question. If students are to overcome these misconceptions, the teaching must expose and discuss them. Results of the Nottingham research show that (1) conflict teaching, which deliberately exposes misconceptions, is more effective than a "positive only" method which warns against the misconceptions and so avoids them; and (2) that learning from games and other concept-involving activities is very much enhanced by intensive discussion in which the concepts and principles become articulated explicitly by the pupils (Bell, 1986-87, 1992a, 1993; Bell et al., 1985; Swan, 1983a, 1983b).

OTHER EXAMPLES — USING THE SOCIAL SITUATION

Our methods make considerable use of the social aspects of classroom life; this will be clear already. But there are some further ways of exploiting this situation which deserve mention; these examples come from work done by French researchers.

In the first case, a two-person number-guessing game has been designed so that the urge to find a winning strategy causes the pupils to develop the concepts desired. In the second activity, the different results obtained (in an angle-measuring task) by different pupils are displayed, thus presenting a cognitive conflict to be resolved. In a third case, briefly mentioned, the demand for different groups of children to communicate information (about the thickness of several kinds of paper) forces them to develop concepts of number pairs which are effectively rational numbers.

The game "Whose Number Is Larger?" is intended for pupils (seven and eight years old) who know and can write numbers at least in tens and units, and probably in hundreds too, but who are not yet fluent in interpreting the numeration system in relation to ordering. The two players are each given, privately, a card containing a number. Their task is to find who has the larger number by asking each other questions, in writing. There is a single message pad and one pencil. Each player writes his or her question. Then each writes the other's question, and so on, until one of them announces that he or she knows whose number is bigger. The only question forbidden is "What is your number?". In this situation, pupils normally begin by trying to guess their partner's number, then progress to "Is your number bigger than ... ?" — eventually trying to avoid revealing their own number. In the course of developing a strategy for this, they find it necessary to become aware of the function of the different digits. Questions such as "Has your number got three digits?" and "Is it in the 30s?" are asked. Thus the urge to improve the game strategy leads to the development of the concepts desired; the conditions of the game have been carefully designed to achieve this effect (Comiti and Bessot, 1987).

The second example is of a short lesson sequence on the *Angle Sum of a Triangle* (Balacheff, 1990). This aims to expose pupils' latent intuition that small triangles (and small-armed angles) have smaller angles than larger ones and that thin triangles do not conform in order to motivate a need to prove that the sum must be exactly 180°.

Members of the class (12 year olds) are first asked to draw a triangle and measure its angles. The results are displayed on the blackboard and divergences discussed (see Figure 10). It is agreed in the class that since different triangles were used, different results are not surprising. Next, everyone has a copy of the *same* triangle, which they measure, and again the results are collected. Pupils are challenged to construct triangles which have small and large values for the angle sum. Some of them make the attempt. It becomes clear that certainty is not achieved and that some more definite type of demonstration is needed. (This is subsequently offered by the teacher.) In the course of this work, problems of measuring angles of small triangles, necessitating lengthening arms — and considering the justification for this procedure — bring into play any existing misconceptions about angle size, space between arms, and arm length and thus contribute to the long-term development of angle concepts. And it is clear that the social situation of the classroom has been manipulated to establish the relevance of public discussion, agreement, and demonstration.

					X									
					X									
					X									
					X									
					X		X							
		X			X	X	X							
X	X	X	X	X	X	X	X	X	X	X	X	X	X	X
106	172	176	177	178	180	181	182	184	185	186	189	190	198	231

Figure 10. Class results for angle sums of triangles (Balacheff, 1990).

Other experiments by French researchers have used communication situations to provoke (1) proof-construction by requiring one pair of pupils to prepare to explain and justify their problem solution to another pair (Balacheff, 1982); and (2) concept-formation by requiring, for example, one group of pupils to request another to specify, in writing, which of five types of paper, of different thicknesses, they require (e.g., "23 sheets of paper 'A' make 0.6 cm [Brousseau, 1981]).

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