# NONSEMISIMPLE 1:1 RESONANCE AT AN EQUILIBRIUM

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**Abstract.** Consider a Hamiltonian system of two degrees of freedom at an equilibrium. Suppose that the linearized vectorfield has eigenvalues $i\alpha, i\alpha, -i\alpha, -i\alpha (\alpha \in \mathbb{R}, \alpha > 0)$ and is not semisimple. In this paper we discuss the real normalization of the Hamiltonian function of such a system. We normalize the Hamiltonian up to 4th order and show how to compute its coefficients. For the planar restricted three body problem at $L_4$ the coefficient that plays an important role in the investigation of the qualitative behaviour of periodic solutions near the equilibrium is explicitly calculated.

## 1. Introduction

Consider a Hamiltonian system of two degrees of freedom $dz/dt = X_H$ with Hamiltonian function $H$, which has an equilibrium point at the origin. Furthermore suppose that the linearized vectorfield $X = DX_H(0)$ has two purely imaginary eigenvalues of multiplicity two and is not semisimple. In view of the multiplicity of the eigenvalues we say that $X_H$ is in $1:1$ resonance. However the nilpotent part of $X$ distinguishes our case rather strongly from the better known resonances $k:l$ with $k \neq l$. Another feature of this resonance is that it arises at the bifurcation between elliptic and hyperbolic equilibrium points (see Section 6).

The general real normal form of a $4 \times 4$ nonsemisimple infinitesimal symplectic matrix with eigenvalues $i\alpha, i\alpha, -i\alpha, -i\alpha$ $(\alpha \in \mathbb{R}, \alpha > 0)$ is given by

$$\begin{pmatrix} 0 & -\alpha & 0 & 0 \\ \alpha & 0 & 0 & 0 \\ \rho & 0 & 0 & -\alpha \\ 0 & \rho & \alpha & 0 \end{pmatrix}, \qquad \rho = \pm 1. \tag{1}$$

Its corresponding quadratic Hamiltonian function on $(\mathbb{R}^4, \omega)$ in coordinates $x_1, x_2$ and corresponding momenta $y_1, y_2$ is equal to

$$H_2(x, y) = \alpha(x_1 y_2 - x_2 y_1) - \tfrac{1}{2}\rho(x_1^2 + x_2^2). \tag{2}$$

Here $\omega = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$ is the standard symplectic form on $\mathbb{R}^4$.

The normal form (1) can be found in the paper of Williamson (1936) who determines the possible normal forms for a linear Hamiltonian system under real symplectic changes of variables. Burgoyne and Cushman (1974) give an algorithm for finding the normalizing real symplectic transformation. Also Roels and Louterman (1970) give an algorithm for normalizing linear Hamiltonian systems. However they construct a complex normal form using complex symplectic transformations. As Burgoyne

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and Cushman (1974) pointed out, normal forms under complex symplectic transformations do not determine the real normal form.

In the following section we will start with a review of the general normal form theory for Hamiltonian functions with the origin as an isolated critical point. Then in Sections 3 and 4 we will derive the normal form up to order 4 and we show how to compute the coefficients of the terms of degree 4 in this normal form from the Taylor expansion (up to degree 4) in the original coordinates. In these sections we will assume that the quadratic terms of our Hamiltonian are in the normal form given by (2), that is, the linearized vectorfield $X$ has corresponding matrix given by (1).

In Section 5 we consider the Hamiltonian function $H$ corresponding to the equations of motion of the planar restricted three body problem at the Lagrange equilateral libration point $L_4$. A. Deprit (1966) gives the Taylor expansion for this Hamiltonian function $H$ at $L_4$. For a special value $\mu = \mu_1$ of the mass parameter $\mu$ the matrix of the linearized vectorfield has purely imaginary eigenvalues and is not semisimple. Therefore it can be brought in the form given by (1) (Burgoyne and Cushman, 1974). We have $\alpha = \frac{1}{2}\sqrt{2}$, $\rho = -1$. We apply the results of Section 4 to calculate the coefficient $a_2$ of one fourth order term ($a_2$ turns out to be a positive rational number). Emphasis is laid on this specific coefficient because further study (work in progress with R. Cushman and J. J. Duistermaat) reveals that the sign of $a_2$ determines the qualitative behaviour of the system.

The important role of this coefficient is also clear from the article of Meyer and Schmidt (1971). They analyze the existence and stability of periodic solutions in the vicinity of the equilibrium for a system at or nearby the resonance under consideration in the present paper and observe that the sign of $a_2$ determines whether the periodic orbits remain or disappear when passing through resonance. Also Roels (1975) considers the problem of the existence of periodic orbits at the resonance situation. Henrard and Renard (1978) showed that also in his approach one has to consider two different cases according to the sign of this coefficient. Both Meyer and Schmidt and Roels use the complex normal form which has the disadvantage mentioned before.

Finally in section 6 we construct a versal deformation (Arnol'd, 1972) of the 4-jet of the normalized Hamiltonian. The versal deformation of the quadratic part is a two parameter deformation. One of these parameters describes the detuning in the ratio of fundamental frequencies of the linear system. We discuss the dependence of this two parameter normal form on the mass parameter $\mu$. It is proved that the family in the planar restricted three body problem is generic in the sense that the derivative of the detuning parameter with respect to $\mu$ is nonzero.

The notation used here is basically that found in Abraham and Marsden (1978).

## 2. Normal Forms

In this section the normal form of a Hamiltonian function $H$ on some real symplectic vector space $(V, \omega)$ will be discussed. Suppose that the Taylor series of $H$ about zero begins with quadratic terms $H_2$. Our special interest is in the case where the matrix of the linear system corresponding to $H_2$ cannot be diagonalized. First we do some linear algebra which enables us to give a proper definition of the normal form.

Let $V$ be a finite dimensional real vector space. A linear transformation $S: V \to V$ is called *semisimple* if each $S$-invariant linear subspace of $V$ has a complementary $S$-invariant subspace.

LEMMA 2.1. *If* $S: V \to V$ *is semisimple then* $V = \text{im } S \oplus \ker S$.

*Proof*. Since $\ker S = \{v \in V | Sv = 0\}$ is $S$-invariant and $S$ is semisimple it follows that there is an $S$-invariant $W$ with $V = \ker S \oplus W$. Therefore $\text{im } S = SV = SW$. Because $S$ is invertible on $W$ and $W$ is $S$-invariant we have $SW = W$. □

The *S-N decomposition* of a linear map $A: V \to V$ is the decomposition $A = S + N$ with $S$ semisimple, $N$ nilpotent and $NS = SN$ (Hoffman and Kunze, 1961, ch. 7).

LEMMA 2.2. *Suppose the S-N decomposition of a linear map* $A: V \to V$ *is given then*

(i) $\ker A = \ker S \cap \ker N$.

(ii) $\text{im } A = \text{im } S \oplus \text{im } N \cap \ker S$.

*Proof*. Because $N$ is nilpotent there is a $m \in \mathbb{N}$ with $N^m \neq 0$ and $N^n = 0$ for $n \in \mathbb{N}$, $n > m$. Therefore on im $S$ we have $(S+N)^{-1} = S^{-1}[I - (S^{-1}N) + (S^{-1}N) - - \cdots (S^{-1}N)^m]$. Thus $A$ is invertible on im $S$. Furthermore we have im $S$ and ker $S$ both $S$- and $N$-invariant. Thus $\text{im}(A \mid \text{im } S) = \text{im } S$, $\text{im}(A \mid \ker S) = \text{im}(N \mid \ker S) = \text{im } N \cap \ker S$ and $\ker(A \mid \text{im } S) = 0$; $\ker(A \mid \ker S) = \ker(N \mid \ker S) = \ker S \cap \ker N$. Using Lemma 2.1(i) and (ii) follow. □

COROLLARY 2.1. *Suppose* $Y$ *is a complement of* $\ker S \cap \text{im } N$ *in* $\ker S$, *i.e.* $\ker S = Y \oplus \ker S \cap \text{im } N$. *Then* $Y$ *is a complement of* $\text{im } A$ *in* $V$, *i.e.* $V = \text{im } A \oplus Y$.

*Proof.* Using Lemma 2.1 and 2.2 we have $V = \ker S \oplus \text{im } S = Y \oplus \ker S \cap \text{im } N \oplus \oplus \text{im } S = Y \oplus \text{im } A$. □

In the following let $(V, \omega)$ be a $2m$ dimensional symplectic real vector space with co-ordinate functions $(x, y) = (x_1, \ldots, x_m, y_1, \ldots, y_m)$, $\omega = \sum_{i=1}^{m} dx_i \wedge dy_i$. Let $P_n(V, \mathbb{R})$, $n \in \mathbb{N}$ be the vector space of real homogeneous polynomials on $V$ with degree $n$, and let $\mathscr{F}P(V, \mathbb{R})$ be the vector space of real formal power series on $V$.

Suppose $H = H_2 + H_3 + H_4 + \cdots$ is the real valued formal power series on $V$ of a Hamiltonian function $H$, $H_n \in P_n(V, \mathbb{R})$. We suppose $H_2$ to be in normal form, that is, the matrix of the linear vectorfield $X$ is in its real symplectic normal form. In other words we suppose that we are given the $S$-$N$ decomposition of $X$. Let $\mathscr{L}, \mathscr{S}$, and $\mathscr{N}$ be the Lie derivatives of elements of $\mathscr{F}P(V, \mathbb{R})$ with respect to $X$, the $S$-part of $X$ and the $N$-part of $X$ respectively. Clearly $\mathscr{L} = \mathscr{S} + \mathscr{N}$ and $\mathscr{L}, \mathscr{S}$ and $\mathscr{N}$ map $P_n(V, \mathbb{R})$ into itself. Let $\mathscr{L}_n, \mathscr{S}_n$ and $\mathscr{N}_n$ be the restrictions of $\mathscr{L}, \mathscr{S}$ and $\mathscr{N}$ to $P_n(V, \mathbb{R})$. Then $\mathscr{L}_n = \mathscr{S}_n + \mathscr{N}_n$ is the $S$-$N$ decomposition of $\mathscr{L}_n$. For $n \geq 2$ let $Y_n$ be a complement of $\ker \mathscr{S}_n \cap \text{im } \mathscr{N}_n$ in $\ker \mathscr{S}_n$, that is $Y_n \oplus \ker \mathscr{S}_n \cap \text{im } \mathscr{N}_n = \ker \mathscr{S}_n$. Using Corollary 2.1 we then have $Y_n \oplus \text{im } \mathscr{L}_n = P_n(V, \mathbb{R})$. We can now formulate the following definition: the Hamiltonian $H$ is in *normal form up to order* $k$ if $H_n \in Y_n$ for $3 \leqslant n \leqslant k$. Note that for $X$ semisimple $Y_n = \ker \mathscr{L}_n$.

Let $\mathscr{L}_f$ denote the Lie derivative with respect to the Hamiltonian vectorfield $X_f$, with Hamiltonian $f \in \mathscr{F}P(V, \mathbb{R})$. In other words $\mathscr{L}_f$ is the Lie derivative *generated* by $f$. It is clear that $\mathscr{L}_f$ maps $\mathscr{F}P(V, \mathbb{R})$ into itself. Define the operator

$$\exp \mathscr{L}_f = \sum_{n \geqslant 0} \frac{1}{n!} \mathscr{L}_f^n.$$

Then the transformation $\phi_f : v \to w(v, w \in V$, vectors with components in $\mathscr{F}P(V, \mathbb{R}))$ defined by

$$w = \phi_f(v) = \exp \mathscr{L}_f(v) \tag{3}$$

is a symplectic transformation. Note that $\phi_f^t(v) = \exp t\mathscr{L}_f(v)$ is the flow of the vectorfield $X_f(v) = \mathscr{L}_f(v)$.

COROLLARY 2.2. *Suppose that* $H \in \mathscr{F}P(V, \mathbb{R})$ *is in normal form up to terms of degree* $n - 1 \geqslant 2$. *Then we can find a function* $F \in P_n(V, \mathbb{R})$ *such that the symplectic transformation given by* (3) *with generator* $F$ *brings* $H(w)$ *into normal form up to terms of degree* $n$.

*Proof.* Let $H_k$ be the homogeneous term of order $k$, in the evaluation of $H$. We have that $\phi_F^t(v)$ is the flow of the vectorfield $X_F$ and thus

$$\frac{\mathrm{d}}{\mathrm{d}t} H_k(\phi_F^t(v)) = \mathscr{L}_F H_k(\phi_F^t(v)).$$

The solution of this differential equation is

$$H_k(\phi_F^t(v)) = \exp t\mathscr{L}_F H_k(\phi_F^0(v)) = \exp t\mathscr{L}_F H_k(v)$$

which for $t = 1$ gives

$$H_k(\phi_F(v)) = \sum_{m \geqslant 0} \frac{1}{m!} \mathscr{L}_F^m H_k(v) \tag{4}$$

(Deprit, 1969). Thus the first term in $H_k(\phi_F(v))$ is of order $k$, the second of order $n + k - 2$ and the others are of higher order. Therefore $(H \circ \phi_F)_k(v) = H_k(v)$ for $2 \leqslant k \leqslant n - 1$, that is, the terms of degree $k \leqslant n - 1$ are unaffected and remain in normal form. Furthermore we have $(H \circ \phi_F)_n(v) = H_n(v) + \mathscr{L}_F H_2(v)$. Decomposing $H_n(v) = G_n(v) + H_n'(v)$ with $G_n(v) \in \operatorname{im} \mathscr{L}$ and $H_n'(v) \in Y_n$, we may choose $F$ such that $G_n(v) = \mathscr{L}F(v) = -\mathscr{L}_F H_2(v)$. Then we have $(H \circ \phi_F)_n(v) = H_n'(v) \in Y_n$. □

## 3. A Special Case

Consider a Hamiltonian function $H = H_2 + H_3 + H_4 + \cdots$ on $(\mathbb{R}^4, \omega = \mathrm{d}x_1 \wedge \mathrm{d}y_1 + \mathrm{d}x_2 \wedge \mathrm{d}y_2)$ starting with the quadratic polynomial

$$H_2 = \alpha(x_1 y_2 - x_2 y_1) - \tfrac{1}{2}\rho(x_1^2 + x_2^2); \quad \alpha > 0, \rho = \pm 1. \tag{5}$$

In this section we will derive the normal form of $H$ up to order 4, that is, we will determine a basis for $Y_3$ and $Y_4$.

The matrix of the linear vectorfield $X$ corresponding to $H_2$ is given by (1). Hence its $S$-$N$ decomposition is $S + N$ with

$$S = \begin{pmatrix} 0 & -\alpha & 0 & 0 \\ \alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & -\alpha \\ 0 & 0 & \alpha & 0 \end{pmatrix}, \quad N = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \rho & 0 & 0 & 0 \\ 0 & \rho & 0 & 0 \end{pmatrix}.$$

Thus

$$\mathscr{S} = -\alpha x_2 \frac{\partial}{\partial x_1} + \alpha x_1 \frac{\partial}{\partial x_2} - \alpha y_2 \frac{\partial}{\partial y_1} + \alpha y_1 \frac{\partial}{\partial y_2}, \tag{6}$$

$$\mathscr{N} = \rho x_1 \frac{\partial}{\partial y_1} + \rho x_2 \frac{\partial}{\partial y_2}. \tag{7}$$

We will now introduce the complex co-ordinates $(z, \zeta) = (z_1, z_2, \zeta_1, \zeta_2)$ given by $z_1 = x_1 + ix_2, z_2 = y_1 + iy_2, \zeta_1 = x_1 - ix_2, \zeta_2 = y_1 - iy_2$. This defines a complex linear isomorphism $(x, y) \mapsto (z, \zeta)$ from $\mathbb{C}^4$ into itself, which transforms any polynomial $f$ in $(x, y)$ to a polynomial $\tilde{f}$ in $(z, \zeta)$. It maps real points $(x, y)$ to $\{(z, \zeta) \in \mathbb{C}^4; \zeta = \bar{z}\}$ and a real polynomial $f$ to an $\tilde{f}$ such that $\overline{\tilde{f}(z, \zeta)} = \tilde{f}(z, \zeta)$. In the sequel we will write $\zeta = \bar{z}$, with the convention that $\bar{z}$ is treated as a variable independent of $z$. In these $(z, \bar{z})$ co-ordinates $\mathscr{S}$ and $\mathscr{N}$ become

$$\tilde{\mathscr{S}} = i\alpha\left( z_1 \frac{\partial}{\partial z_1} - \bar{z}_1 \frac{\partial}{\partial \bar{z}_1} + z_2 \frac{\partial}{\partial z_2} - \bar{z}_2 \frac{\partial}{\partial \bar{z}_2} \right), \tag{8}$$

$$\tilde{\mathscr{N}} = \rho\left( z_1 \frac{\partial}{\partial z_2} + \bar{z}_1 \frac{\partial}{\partial \bar{z}_2} \right). \tag{9}$$

If we use the notation $(l, k) = (l_1, l_2, k_1, k_2)$ for the monomial $z_1^{l_1} z_2^{l_2} \bar{z}_1^{k_1} \bar{z}_2^{k_2}$ every real homogeneous polynomial of degree $m$ on $\mathbb{R}^4$ in complex conjugate co-ordinates $(z, \bar{z})$ may be written as

$$\sum_m c_{lk}(l, k) \quad \text{with} \quad c_{lk} = \bar{c}_{kl}.$$

Here $\sum_m$ denotes summation over all possible combinations $l_1, l_2, k_1, k_2$ with $l_i \geqslant 0$, $k_i \geqslant 0$ and $l_1 + l_2 + k_1 + k_2 = m$. We have

$$\tilde{\mathscr{S}}(l, k) = i\alpha(l_1 + l_2 - k_1 - k_2)(l, k) \tag{10}$$

Thus the kernel of $\tilde{\mathscr{S}}$ acting on the space of real homogeneous polynomials of degree $n$ in complex conjugate co-ordinates is spanned by the monomials $(l, k)$ with $l_1 + l_2 + k_1 + k_2 = n$ and $l_1 + l_2 = k_1 + k_2$. Consequently $2(l_1 + l_2) = 2(k_1 + k_2) = n$. In particular when $n$ is odd ker $\tilde{\mathscr{S}}_n = \{0\}$. Hence $Y_3 = \{0\}$, that is, the normal form of $H$ is void of third order terms. For $n = 4$ the kernel of $\tilde{\mathscr{S}}_n$ is spanned by the following monomials:

$$(2,0,2,0),(0,2,0,2),(1,1,1,1),(2,0,1,1),(1,1,0,2),(2,0,0,2),$$
$$(1,1,2,0),(0,2,1,1),(0,2,2,0).$$

This implies that the kernel of $\mathscr{S}_4$ is spanned by the real 4th degree polynomials:

$$e_1=(2,0,2,0)=(x_1^2+x_2^2)^2, \qquad e_2=(0,2,0,2)=(y_1^2+y_2^2)^2,$$
$$e_3=(1,1,1,1)=(x_1^2+x_2^2)(y_1^2+y_2^2),$$
$$e_4=\mathrm{Re}\,(1,1,2,0)=(x_1^2+x_2^2)(x_1y_1+x_2y_2),$$
$$e_5=\mathrm{Im}\,(1,1,2,0)=(x_1^2+x_2^2)(x_1y_2-x_2y_1),$$
$$e_6=\mathrm{Re}\,(0,2,1,1)=(y_1^2+y_2^2)(x_1y_1+x_2y_2),$$
$$e_7=\mathrm{Im}\,(0,2,1,1)=(y_1^2+y_2^2)(x_1y_2-x_2y_1),$$
$$e_8=\mathrm{Re}\,(0,2,2,0)=(x_1y_1+x_2y_2)^2-(x_1y_2-x_2y_1)^2,$$
$$e_9=\mathrm{Im}\,(0,2,2,0)=2(x_1y_1+x_2y_2)(x_1y_2-x_2y_1).$$

Using (9) we find the image of $\tilde{\mathscr{N}}\,|\ker\tilde{\mathscr{S}}_4$:

$$\tilde{\mathscr{N}}(2,0,2,0)=0,$$
$$\tilde{\mathscr{N}}(0,2,0,2)=2\rho(1,1,0,2)+2\rho(0,2,1,1),$$
$$\tilde{\mathscr{N}}(1,1,1,1)=\rho(2,0,1,1)+\rho(1,1,2,0),$$
$$\tilde{\mathscr{N}}(1,1,2,0)=\rho(2,0,2,0)=\overline{\tilde{\mathscr{N}}(2,0,1,1)},$$
$$\tilde{\mathscr{N}}(0,2,1,1)=2\rho(1,1,1,1)+\rho(0,2,2,0)=\overline{\tilde{\mathscr{N}}(1,1,0,2)}.$$
$$\tilde{\mathscr{N}}(0,2,2,0)=2\rho(1,1,2,0)=\overline{\tilde{\mathscr{N}}(2,0,0,2)}.$$

Therefore the action of $\mathscr{N}$ on the basis $\{e_1,\dots,e_9\}$ of $\ker\mathscr{S}$ in $P_4(\mathbb{R}^4,\mathbb{R})$ is

$$\mathscr{N}e_1=0, \qquad \mathscr{N}e_4=\rho e_1, \qquad \mathscr{N}e_7=\rho e_9,$$
$$\mathscr{N}e_2=4\rho e_6, \qquad \mathscr{N}e_5=0, \qquad \mathscr{N}e_8=2\rho e_4,$$
$$\mathscr{N}e_3=2\rho e_4, \qquad \mathscr{N}e_6=\rho e_8+2\rho e_3, \qquad \mathscr{N}e_9=2\rho e_5,$$

which implies that the image of $\mathscr{N}\,|\ker\mathscr{S}_4$ in $P_4(\mathbb{R}^4,\mathbb{R})$ is spanned by $\{e_1, e_4, e_5, e_6, e_8+2e_3, e_9\}$. Hence a choice of $Y_4$ is the span of $\{e_2, e_7, c_1e_8+c_2e_3\}$ where $c_2-2c_1\neq 0$. Because $Y_4$ is not unique, the normal form of $H$ is not unique. If we take $-c_1=c_2=\frac{1}{2}$ then $Y_4$ is spanned by $\{e_2, e_7, \frac{1}{2}(e_3-e_8)\}$. Because $\frac{1}{2}(e_3-e_8)=(x_1y_2-x_2y_1)^2$ we have as the fourth order terms in our normalized Hamiltonian

$$H_4=a_1(x_1y_2-x_2y_1)^2+a_2(y_1^2+y_2^2)^2+a_3(y_1^2+y_2^2)(x_1y_2-x_2y_1). \tag{11}$$

## 4. Computation of $a_2$

On the symplectic vector space $(\mathbb{R}^4, \mathrm{d}\xi_1 \wedge \mathrm{d}\eta_1 + \mathrm{d}\xi_2 \wedge \mathrm{d}\eta_2)$ consider the Hamiltonian function

$$H(\xi, \eta) = \alpha(\xi_1\eta_2 - \xi_2\eta_1) - \tfrac{1}{2}\rho(\xi_1^2 + \xi_2^2) + H_3(\xi, \eta) + H_4(\xi, \eta) + \dots \tag{12}$$

In Section 3 we found the normal form of $H$ up to order 4. In this section we will compute the coefficient $a_2$ in (11) in terms of the coefficients of $H_3$ and $H_4$. The results of this section will be used in Section 5 where this coefficient $a_2$ in the real normal form of the planar restricted three body problem at $L_4$ is explicitly computed.

The procedure will be as follows. First we will compute the function $F \in P_3(\mathbb{R}^4, \mathbb{R})$ that generates the transformation $\phi_F$ which normalizes the Hamiltonian given by (12) up to order 3 (see Corollary 2.2). This transformation is uniquely determined because in Section 3 we found that $\ker \mathscr{S}_3 = \{0\}$. Hence by Lemma 2.2 $\mathscr{L}_3^{-1}$ exists on $P_3(\mathbb{R}^4, \mathbb{R})$ and thus $F = \mathscr{L}^{-1}H_3$. Furthermore the effect of transforming $H$ by $\phi_F$ is that it adds a fourth order term to $H$. Collecting terms of the desired form leads to $a_2$.

THEOREM 4.1. *The symplectic transformation $\phi_F$ defined by* (3) *with generator $F \in P_3(\mathbb{R}^4, \mathbb{R})$ such that $\mathscr{L}F = H_3$ gives*

(i) $(H \circ \phi_F)_3(x, y) = 0$,

(ii) $(H \circ \phi_F)_4(x, y) = H_4(x, y) + \tfrac{1}{2}\mathscr{L}_F H_3(x, y)$. (13)

*Proof.* (i) follows from Corollary 2.2 and the fact that $Y_3 = \{0\}$. Using (4) we have

$$(H \circ \phi_F)_4(x, y) = H_4(x, y) + \mathscr{L}_F H_3(x, y) + \tfrac{1}{2}\mathscr{L}_F^2 H_2(x, y).$$

Furthermore we have

$$\tfrac{1}{2}\mathscr{L}_F^2 H_2(x, y) = \tfrac{1}{2}\mathscr{L}_F(\mathscr{L}_F H_2(x, y)) = -\tfrac{1}{2}\mathscr{L}_F(\mathscr{L}F(x, y)) = -\tfrac{1}{2}\mathscr{L}_F H_3(x, y)$$

using the fact that we have chosen $F$ in such a way that $\mathscr{L}F = H_3$. Recall that $\mathscr{L}$ is the Lie derivative generated by $H_2$. This proves (ii). □

Next introduce the complex conjugate co-ordinates defined in Section 3. Let $\tilde{H}(z, \bar{z})$ denote the real function $H(x, y)$ in complex conjugate co-ordinates. Then Hamilton's equations in these co-ordinates are

$$\frac{\mathrm{d}}{\mathrm{d}z}\begin{pmatrix} z \\ \bar{z} \end{pmatrix} = X_{\tilde{H}}(z, \bar{z}),$$

where

$$X_{\tilde{H}}(z, \bar{z}) = 2\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \mathrm{d}\tilde{H}(z, \bar{z}).$$

Note that the factor 2 appears because our complex conjugate co-ordinates are not complex symplectic. [One can find easily co-ordinates that are complex symplectic, for instance:

$$u_1 = -(1/\sqrt{2})z_2, u_2 = (1/\sqrt{2})z_1, v_1 = (1/\sqrt{2})\bar{z}_1, v_2 = (1/\sqrt{2})\bar{z}_2.$$

However working in these co-ordinates is not convenient.] Define the Lie derivative of $f(z, \bar{z})$ with respect to $X_{\tilde{H}}(z, \bar{z})$ by

$$\begin{aligned} L_{\tilde{H}} f(z, \bar{z}) = \mathrm{d} f(z, \bar{z}) X_{\tilde{H}}(z, \bar{z}) = 2\frac{\partial \tilde{H}(z, \bar{z})}{\partial \bar{z}_2}\frac{\partial f(z, \bar{z})}{\partial z_1} - \\ - 2\frac{\partial \tilde{H}(z, \bar{z})}{\partial \bar{z}_1}\frac{\partial f(z, \bar{z})}{\partial z_2} + 2\frac{\partial \tilde{H}(z, \bar{z})}{\partial z_2}\frac{\partial f(z, \bar{z})}{\partial \bar{z}_1} - \\ - 2\frac{\partial \tilde{H}(z, \bar{z})}{\partial z_1}\frac{\partial f(z, \bar{z})}{\partial \bar{z}_2}. \end{aligned} \tag{14}$$

In the following we use the same notation as in Section 3. Let $P_n((z, \bar{z}), \mathbb{C})$ denote the space of homogeneous complex valued polynomials in $(z, \bar{z})$ co-ordinates with complex coefficients. A complex basis of $P_n((z, \bar{z}), \mathbb{C})$ is $M_n = \{(l, k); l_1 + l_2 + k_1 + k_2 = n, l_i \geqslant 0, k_i \geqslant 0, i = 1, 2\}$. The following properties are easily obtained:

PROPERTY 1. Let $\tilde{H}_k(z, \bar{z})$ be the real homogeneous polynomial $H_k(x, y)$ in complex conjugate co-ordinates. If $(l, k)$ is a monomial of $\tilde{H}_k$ then $(k, l)$ is also a monomial of $\tilde{H}_k$ and $c_{kl} = \bar{c}_{lk}$.

PROPERTY 2. If $G(z, \bar{z}) \in P_n((z, \bar{z}), \mathbb{C})$ then $\overline{[\partial G(z, \bar{z})]/\partial z_i} = [\partial \overline{G(z, \bar{z})}]/\partial \bar{z}_i$. Moreover if $G(z, \bar{z})$ is some real homogeneous polynomial in complex conjugate co-ordinates then $\overline{[\partial G(z, \bar{z})]/\partial z_i} = [\partial G(z, \bar{z})]/\partial \bar{z}_i$.

PROPERTY 3. $\mathscr{L}_{\tilde{H}} = \tilde{\mathscr{L}}_H$.

PROPERTY 4. $\mathscr{L}_{\tilde{H}(z, \bar{z})} \tilde{G}(z, \bar{z}) = (\widetilde{\mathscr{L}_H G})(z, \bar{z}) = \mathscr{L}_{H(x, y)} G(x, y)$.

Using Properties 3 and 4 it follows that:

$$\tilde{F}(z, \bar{z}) = \tilde{\mathscr{L}}^{-1} \tilde{H}_3(z, \bar{z}) \tag{15}$$

$$\tilde{\phi}_F = \exp \tilde{\mathscr{L}}_F = \exp \mathscr{L}_{\tilde{F}} = \phi_{\tilde{F}}$$

and

$$(\tilde{H} \circ \phi_{\tilde{F}})_4(z, \bar{z}) = \tilde{H}_4(z, \bar{z}) + \tfrac{1}{2}\mathscr{L}_{\tilde{F}} \tilde{H}_3(z, \bar{z}). \tag{16}$$

From Property 2 it follows directly that on $M_n$ the operators $\tilde{\mathscr{S}}$ and $\tilde{\mathscr{N}}$ and thus

$\tilde{\mathscr{L}}$ and $\tilde{\mathscr{L}}^{-1}$ commute with complex conjugation, that is,

$$\overline{\tilde{\mathscr{L}}^{-1}(l,k)} = \tilde{\mathscr{L}}^{-1}\overline{(l,k)} = \tilde{\mathscr{L}}^{-1}(k,l).$$

So to find $\tilde{F} \in P_3((z,\bar{z}),\mathbb{C})$ we have only to determine the image of the set $M_3^* = \{(l,k); l_1 + l_2 + k_1 + k_2 = 3; l_i \geq 0, k_i \geq 0, l_1 + l_2 > k_1 + k_2\}$ under $\tilde{\mathscr{L}}^{-1}$. This we do in the following way. Using (8) and (9) we have

$$\tilde{\mathscr{S}}(l,k) = i\alpha\delta_{lk}(l,k) \quad \text{where} \quad \delta_{lk} = (l_1 + l_2 - k_1 - k_2) \tag{17}$$

and

$$\tilde{\mathscr{N}}(l,k) = \rho l_2(l_1+1, l_2-1, k_1, k_2) + \rho k_2(l_1, l_2, k_1+1, k_1-1). \tag{18}$$

Consider the decomposition $\tilde{\mathscr{L}} = \tilde{\mathscr{S}} + \tilde{\mathscr{N}}$. Since $\tilde{\mathscr{S}}\tilde{\mathscr{N}} = \tilde{\mathscr{N}}\tilde{\mathscr{S}}$ we have

$$\begin{aligned}\tilde{\mathscr{L}}^{-1} &= (I + \tilde{\mathscr{S}}^{-1}\tilde{\mathscr{N}})^{-1}\tilde{\mathscr{S}}^{-1} = \\ &= [I - \tilde{\mathscr{S}}^{-1}\tilde{\mathscr{N}} + (\tilde{\mathscr{S}}^{-1}\tilde{\mathscr{N}})^2 - (\tilde{\mathscr{S}}^{-1}\tilde{\mathscr{N}})^3]\tilde{\mathscr{S}}^{-1},\end{aligned} \tag{19}$$

using the fact that $\tilde{\mathscr{N}}^4 = 0$ on $P_3((z,\bar{z}),\mathbb{C})$.

We use the expression (19) together with the formulas (17) and (18) to find the image of the set $M_3^*$ under $\tilde{\mathscr{L}}^{-1}$. We then also know the image of $M_3$ under $\tilde{\mathscr{L}}^{-1}$, that is, the image of the basis-elements of $P_3((z,\bar{z}),\mathbb{C})$. We may express $\tilde{\mathscr{L}}^{-1}$ as the matrix given in Table I. One can find the image of the $i$th monomial in the front column by adding the monomials in the top row, giving each monomial the coefficient found in the $i$th place of the corresponding column.

Let

$$\tilde{H}_3(z,\bar{z}) = \Sigma_3 h_{lk}(l,k), \tilde{F}(z,\bar{z}) = \Sigma_3 f_{lk}(l,k)$$

be the polynomials $H_3(x,y)$ and $F(x,y)$ in complex conjugate co-ordinates. Because $\tilde{\mathscr{L}}^{-1}\tilde{H}_3 = \tilde{F}$ we may use Table I to express the coefficients of $\tilde{F}$ in those of $\tilde{H}_3$. Let $(l,k)$ be the monomial on the $j$th place in the top row. Then the corresponding coefficient $f_{lk}$ is the sum of the $h_{lk}$ corresponding to the monomials in the front column, with the coefficients found in the $j$th column.

Now we are in a position to compute the coefficient $a_{0202}$ of the monomial $(0,2,0,2)$, which in real co-ordinates is $(y_1^2 + y_2^2)^2$, in the expression given by (16). We have using Property 2 and (14):

$$\tfrac{1}{2}\mathscr{L}_{\tilde{F}}\tilde{H}_3(z,\bar{z}) = 2\,\mathrm{Re}\left(\frac{\partial\tilde{H}_3(z,\bar{z})}{\partial z_1}\frac{\partial\tilde{F}(z,\bar{z})}{\partial\bar{z}_2}\right) - 2\,\mathrm{Re}\left(\frac{\partial\tilde{H}_3(z,\bar{z})}{\partial z_2}\frac{\partial\tilde{F}(z,\bar{z})}{\partial\bar{z}_1}\right), \tag{20}$$

where

$$\begin{aligned}\frac{\partial\tilde{H}_3(z,\bar{z})}{\partial z_1} &= 3h_{3000}(2,0,0,0) + 2h_{2100}(1,1,0,0) + 2h_{2010}(1,0,1,0) + \\ &+ h_{1020}(0,0,2,0) + 2h_{2001}(1,0,0,1) + h_{1110}(0,1,1,0) +\end{aligned}$$

TABLE I
Transformation matrix for $\tilde{\mathscr{P}}^{-1}$

| | 3000 | 2100 | 1200 | 0300 | 0030 | 0021 | 0012 | 0003 | 2010 | 2001 | 1110 | 1101 | 0210 | 0201 | 1020 | 0120 | 1011 | 0111 | 1002 | 0102 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 3000 | $\frac{-i}{3\alpha}$ | | | | | | | | | | | | | | | | | | | |
| 2100 | $\frac{\rho}{9\alpha^2}$ | $\frac{-i}{3\alpha}$ | | | | | | | | | | | | | | | | | | |
| 1200 | $\frac{2i}{27\alpha^3}$ | $\frac{2\rho}{9\alpha^2}$ | $\frac{-i}{3\alpha}$ | | | | | | | | | | | | | | | | | |
| 0300 | $\frac{-2\rho}{27\alpha^4}$ | $\frac{2i}{9\alpha^3}$ | $\frac{\rho}{3\alpha^2}$ | $\frac{-i}{3\alpha}$ | | | | | | | | | | | | | | | | |
| 0030 | | | | | $\frac{i}{3\alpha}$ | | | | | | | | | | | | | | | |
| 0021 | | | | | $\frac{\rho}{9\alpha^2}$ | $\frac{i}{3\alpha}$ | | | | | | | | | | | | | | |
| 0012 | | | | | $\frac{-2i}{27\alpha^3}$ | $\frac{2\rho}{9\alpha^2}$ | $\frac{i}{3\alpha}$ | | | | | | | | | | | | | |
| 0003 | | | | | $\frac{-2\rho}{27\alpha^4}$ | $\frac{-2i}{9\alpha^3}$ | $\frac{\rho}{3\alpha^2}$ | $\frac{i}{3\alpha}$ | | | | | | | | | | | | |

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 2010 | $\frac{-i}{\alpha}$ | | | | | | | | | | | |
| 2001 | $\frac{\rho}{\alpha^2}$ | $\frac{-i}{\alpha}$ | | | | | | | | | | |
| 1110 | $\frac{\rho}{\alpha^2}$ | | $\frac{-i}{\alpha}$ | | | | | | | | | |
| 1101 | $\frac{2i}{\alpha^3}$ | $\frac{\rho}{\alpha^2}$ | $\frac{\rho}{\alpha^2}$ | $\frac{-i}{\alpha}$ | | | | | | | | |
| 0210 | $\frac{2i}{\alpha^3}$ | | $\frac{2\rho}{\alpha^2}$ | | $\frac{-i}{\alpha}$ | | | | | | | |
| 0201 | $\frac{-6\rho}{\alpha^4}$ | $\frac{2i}{\alpha^3}$ | $\frac{4i}{\alpha^3}$ | $\frac{2\rho}{\alpha^2}$ | $\frac{\rho}{\alpha^2}$ | $\frac{-i}{\alpha}$ | | | | | | |
| 1020 | | | | | | | $\frac{i}{\alpha}$ | | | | | |
| 0120 | | | | | | | $\frac{\rho}{\alpha^2}$ | $\frac{i}{\alpha}$ | | | | |
| 1011 | | | | | | | $\frac{\rho}{\alpha^2}$ | | $\frac{i}{\alpha}$ | | | |
| 0111 | | | | | | | $\frac{-2i}{\alpha^3}$ | $\frac{\rho}{\alpha^2}$ | $\frac{\rho}{\alpha^2}$ | $\frac{i}{\alpha}$ | | |
| 1002 | | | | | | | $\frac{-2i}{\alpha^3}$ | | $\frac{2\rho}{\alpha^2}$ | | $\frac{i}{\alpha}$ | |
| 0102 | | | | | | | $\frac{-6\rho}{\alpha^4}$ | $\frac{-2i}{\alpha^3}$ | $\frac{-4i}{\alpha^3}$ | $\frac{2\rho}{\alpha^2}$ | $\frac{\rho}{\alpha^2}$ | $\frac{i}{\alpha}$ |

$$+ h_{1011}(0,0,1,1) + h_{1200}(0,2,0,0) + h_{1002}(0,0,0,2) + $$
$$+ h_{1101}(0,1,0,1)$$
$$= \Sigma_2 (i+1) h_{(i+1)jkl}(i,j,k,l)$$

and

$$\frac{\partial \tilde{H}_3(z,\bar{z})}{\partial z_2} = \Sigma_2 (j+1) h_{i(j+1)kl}(i,j,k,l),$$
$$\frac{\partial \tilde{F}(z,\bar{z})}{\partial \bar{z}_1} = \overline{\frac{\partial \tilde{F}(z,\bar{z})}{\partial z_1}} = \Sigma_2 (i+1) \bar{f}_{(i+1)jkl}(k,l,i,j),$$
$$\frac{\partial \tilde{F}(z,\bar{z})}{\partial \bar{z}_2} = \overline{\frac{\partial \tilde{F}(z,\bar{z})}{\partial z_2}} = \Sigma_2 (j+1) \bar{f}_{i(j+1)kl}(k,l,i,j).$$

Since $(0,2,0,2)$ is the product of the monomials $(0,2,0,0)$ and $(0,0,0,2)$, $(0,0,0,2)$ and $(0,2,0,0)$, $(0,1,0,1)$ and $(0,1,0,1)$, the coefficient of $(0,2,0,2)$ in (20) is

$$\begin{aligned} c_{0202} &= 2\,\mathrm{Re}(3h_{1200}\bar{f}_{0300} + h_{1002}\bar{f}_{0102} + 2h_{1101}\bar{f}_{0201} - 3h_{0300}\bar{f}_{1200} - \\ &\quad - h_{0102}\bar{f}_{1002} - 2h_{0201}\bar{f}_{1101}) \\ &= 2\,\mathrm{Re}\Big( -\frac{i}{\alpha} h_{1002}h_{0201} + \frac{i}{\alpha} h_{1200}h_{0003} + 2\frac{i}{\alpha} h_{1101}h_{0102} + \\ &\quad + \frac{i}{\alpha} h_{0102}h_{0210} - \frac{\rho}{\alpha^2} h_{0102}h_{0201} - \frac{i}{\alpha} h_{0300}h_{0012} - \frac{\rho}{\alpha^2} h_{0300}h_{0003} - \\ &\quad - 2\frac{i}{\alpha} h_{0201}h_{0111} - 4\frac{\rho}{\alpha^2} h_{0201}h_{0102}\Big) \\ &= \frac{4}{\alpha}[\mathrm{Re}(h_{1002})\,\mathrm{Im}(h_{0201}) + \mathrm{Im}(h_{1002})\,\mathrm{Re}(h_{0201})] - \\ &\quad - \frac{4}{\alpha}[\mathrm{Re}(h_{1200})\,\mathrm{Im}(h_{0003}) + \mathrm{Im}(h_{1200})\,\mathrm{Re}(h_{0003})] - \\ &\quad - \frac{8}{\alpha}[\mathrm{Re}(h_{1101})\,\mathrm{Im}(h_{0102}) + \mathrm{Im}(h_{1101})\,\mathrm{Re}(h_{0102})] - \\ &\quad - 10\frac{\rho}{\alpha^2}[\mathrm{Re}(h_{0201})]^2 - 10\frac{\rho}{\alpha^2}[\mathrm{Im}(h_{0201})]^2 - \\ &\quad - 2\frac{\rho}{\alpha^2}[\mathrm{Re}(h_{0300})]^2 - 2\frac{\rho}{\alpha^2}[\mathrm{Im}(h_{0300})]^2. \end{aligned} \tag{21}$$

Thus the coefficient $a_{0202}$ of $(0,2,0,2)$ in (16) is

$$a_{0202} = h_{0202} + c_{0202}. \tag{22}$$

## 5. Calculation for the Restricted Three Body Problem

In this section we will consider the planar restricted three body problem in a rotating co-ordinate frame at the critical mass-ratio of Routh that is for the mass-parameter $\mu$ we have $\mu = \mu_1 = \frac{1}{2}(1 - \frac{1}{9}\sqrt{69})$.

According to Deprit (1966) the Hamiltonian function describing this system with the origin at the equilateral Lagrange equilibrium point $L_4$ can be written as

$$H(\xi, \eta, p_\xi, p_\eta) = H_2(\xi, \eta, p_\xi, p_\eta) + H_3(\xi, \eta) + H_4(\xi, \eta) + \cdots$$

with

$$H_2(\xi, \eta, p_\xi, p_\eta) = \tfrac{1}{2}(p_\xi^2 + p_\eta^2) - (\xi p_\eta - \eta p_\xi) + \omega_{20}\xi^2 + \omega_{02}\eta^2,$$
$$H_3(\xi, \eta) = \omega_{30}\xi^3 + \omega_{21}\xi^2\eta + \omega_{12}\xi\eta^2 + \omega_{03}\eta^3,$$
$$H_4(\xi, \eta) = \omega_{40}\xi^4 + \omega_{31}\xi^3\eta + \omega_{22}\xi^2\eta^2 + \omega_{13}\xi\eta^3 + \omega_{04}\eta^4,$$

where

$$-\Omega(\xi, \eta) = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \omega_{pq}\xi^p\eta^q$$

is the potential energy of the system. For the coefficients we have

$$\begin{aligned}
16\omega_{30} &= -10\gamma\cos 3\beta + 3\zeta\cos(\alpha + \beta),\\
16\omega_{21} &= -30\gamma\sin 3\beta + 3\zeta\sin(\alpha + \beta),\\
16\omega_{12} &= 30\gamma\cos 3\beta + 3\zeta\cos(\alpha + \beta),\\
16\omega_{03} &= 10\gamma\sin 3\beta + 3\zeta\sin(\alpha + \beta);
\end{aligned} \tag{23}$$

$$\begin{aligned}
128\omega_{40} &= -18 + [20 + 35\cos 6\beta]\delta,\\
128\omega_{31} &= 140\delta\sin 6\beta,\\
128\omega_{22} &= -36 - 210\delta\cos 6\beta,\\
128\omega_{13} &= -140\delta\sin 6\beta,\\
128\omega_{04} &= -18 - [20 - 35\cos 6\beta]\delta;
\end{aligned} \tag{24}$$

where we have the relations

$$\gamma = 1 - 2\mu, \qquad \delta^2 = 1 + 3\gamma^2, \qquad \zeta^2 = 3 + \gamma^2$$

$$\cos 2\beta = \frac{1}{\delta}, \qquad \sin 2\beta = \frac{\gamma}{\delta}\sqrt{3}, \qquad \cos\alpha = \frac{\gamma}{\zeta}, \qquad \sin\alpha = \frac{\sqrt{3}}{\zeta}$$

which for $\mu = \mu_1$ give

$$\gamma = \tfrac{1}{9}\sqrt{69}, \qquad \delta = \tfrac{4}{3}\sqrt{2}, \qquad \zeta = \tfrac{2}{9}\sqrt{78},$$

$$\cos 2\beta = \tfrac{3}{8}\sqrt{2}, \qquad \sin 2\beta = \tfrac{1}{8}\sqrt{46}, \qquad \cos\alpha = \tfrac{1}{2}\sqrt{\frac{23}{26}},$$

$$\sin\alpha = \tfrac{9}{2}\sqrt{\frac{1}{26}}. \tag{25}$$

Thus we have

$$H_2(\xi, \eta, p_\xi, p_\eta) = \tfrac{1}{2}(p_\xi^2 + p_\eta^2) - \tfrac{1}{4}(1 - 2\sqrt{2})\xi^2 - \tfrac{1}{4}(1 + 2\sqrt{2})\eta^2 - (\xi p_\eta - \eta p_\xi).$$

The real symplectic co-ordinate change which brings $H_2$ into its real infinitesimal symplectic normal form is found by Burgoyne and Cushman (1974). It is given by

$$\xi = -\frac{1}{2\lambda}(x_1 - 2y_2), \qquad \eta = -\frac{\sqrt{2}}{2}\lambda(x_2 - 2y_1),$$

$$p_\xi = -\tfrac{1}{2}\lambda(3x_2 - 2y_1), \qquad p_\eta = -\frac{\sqrt{2}}{4\lambda}(3x_1 - 2y_2), \tag{26}$$

where

$$\lambda = \left(1 - \frac{1}{\sqrt{2}}\right)^{1/2}. \tag{27}$$

The new quadratic Hamiltonian becomes

$$H_2(x, y) = \tfrac{1}{2}\sqrt{2}\,(x_1 y_2 - x_2 y_1) + \tfrac{1}{2}(x_1^2 + x_2^2). \tag{28}$$

For the higher order terms we have, using complex conjugate co-ordinates $(z, \bar{z})$ as defined in Section 3,

$$\xi^p \eta^q = d_{pq}(\lambda)(z_1 + \bar{z}_1 + 2iz_2 - 2i\bar{z}_2)^p(-iz_1 + i\bar{z}_1 - 2z_2 - 2\bar{z}_2)^q$$

$$= d_{pq}(\lambda)\tilde{G}_{pq}(z_1, z_2, \bar{z}_1, \bar{z}_2) \quad \text{where} \quad d_{pq}(\lambda) = (-\tfrac{1}{2})^{p+q}\left(\frac{1}{2\lambda}\right)^p \left(\frac{\lambda}{\sqrt{2}}\right)^q$$

$$\xi^q \eta^p = d_{qp}(\lambda)\tilde{G}_{qp}(z_1, z_2, \bar{z}_1, \bar{z}_2).$$

It follows that

$$\tilde{G}_{qp}(z_1, z_2, \bar{z}_1, \bar{z}_2) = \tilde{G}_{pq}(i\bar{z}_1, i\bar{z}_2, -iz_1, -iz_2). \tag{29}$$

Thus if we write $d_{pq}$ instead of $d_{pq}(\lambda)$ we have

$$\tilde{H}_3(z, \bar{z}) = \Sigma_3 h_{lk}(l, k) = d_{30}\omega_{30}\tilde{G}_{30}(z, \bar{z}) + d_{21}\omega_{21}\tilde{G}_{21}(z, \bar{z}) +$$
$$+ d_{12}\omega_{12}\tilde{G}_{21}(i\bar{z}, -iz) + d_{03}\omega_{03}\tilde{G}_{30}(i\bar{z}, -iz).$$

We may now compute the $h_{lk}$, they are listed in Table II. For convenience define

$$a = \sqrt{2}\left(\frac{\lambda^3}{4}\omega_{03} - \frac{1}{8\lambda}\omega_{21}\right),$$

$$b = \frac{1}{8\lambda^3}\omega_{30} - \frac{\lambda}{4}\omega_{12},$$

$$c = \sqrt{2}\left(3\frac{\lambda^3}{4}\omega_{03} + \frac{1}{8\lambda}\omega_{21}\right), \tag{30}$$

$$d = \frac{3}{8\lambda^3}\omega_{30} + \frac{\lambda}{4}\omega_{12}$$

TABLE II
List of coefficient $h_{lk}$

| | |
|---|---|
| $h_{3000} = \bar{h}_{0030} = -\frac{1}{8}(b + ia)$ | $h_{2010} = \bar{h}_{1020} = -\frac{1}{8}(d - ic)$ |
| $h_{2100} = \bar{h}_{0021} = -\frac{1}{4}(c + id)$ | $h_{2001} = \bar{h}_{0120} = -\frac{3}{4}(a - ib)$ |
| $h_{1200} = \bar{h}_{0012} = \frac{1}{2}(d + ic)$ | $h_{1110} = \bar{h}_{1011} = \frac{1}{2}(c - id)$ |
| $h_{0300} = \bar{h}_{0003} = a + ib$ | $h_{1101} = \bar{h}_{0111} = -d + ic$ |
| | $h_{0210} = \bar{h}_{1002} = \frac{3}{2}(a - ib)$ |
| | $h_{0201} = \bar{h}_{0102} = c - id$ |

To find the coefficients of $\tilde{H}_4$ we proceed in the same way. In this case we only have to find the coefficient $h_{0202}$. We have

$$\tilde{H}_4(z, \bar{z}) = d_{40}\omega_{40}\tilde{G}_{40}(z, \bar{z}) + d_{31}\omega_{31}\tilde{G}_{31}(z, \bar{z}) + d_{22}\omega_{22}\tilde{G}_{22}(z, \bar{z}) + \\ + d_{13}\omega_{13}\tilde{G}_{31}(i\bar{z}, -iz) + d_{04}\omega_{04}\tilde{G}_{40}(i\bar{z}, -iz).$$

Thus the coefficient $h_{0202}$ in $\tilde{H}_4(z, \bar{z}) = \Sigma_4 h_{lk}(l, k)$ is

$$h_{0202} = \frac{3}{8\lambda^4}\omega_{40} + \tfrac{1}{4}\omega_{22} + \tfrac{3}{2}\lambda^4\,\omega_{04}. \tag{31}$$

Using (24), (25), and (27) we obtain

$$h_{0202} = -\tfrac{1}{128}\left[(\tfrac{9}{4} + \tfrac{3}{2}\sqrt{2})(18–20\delta–35\delta\cos 6\beta) + \right.\\ \left. + (\tfrac{9}{4} - \tfrac{3}{2}\sqrt{2})(18 + 20\delta - 35\delta\cos 6\beta) + 9 + \tfrac{105}{2}\delta\cos 6\beta\right] \\ = -\tfrac{1}{128}\left[90–105\left(\frac{4}{\delta^2} - 3\right) - 60\sqrt{2}\,\delta\right] = -2^{-10} \times 1015 = -\tfrac{1015}{1024}. \tag{32}$$

Knowing the coefficients of $\tilde{H}_3$ and using the results of Section 4 it is possible to compute $c_{0202}$ which is the coefficient of the monomial (0, 2, 0, 2) in $\frac{1}{2}\mathscr{L}_{\tilde{F}}\tilde{H}_3$. To do this use (21) and Table II, putting $\alpha = \frac{1}{2}\sqrt{2}$ and $\rho = -1$ in (21). We obtain

$$c_{0202} = 4(a^2 + 5c^2) + 4(b^2 + 5d^2) - 4\sqrt{2}(2c^2 - ac) + 4\sqrt{2}(2d^2 - bd). \tag{33}$$

Substituting (30) into (33) and collecting terms gives

$$c_{0202} = \tfrac{335}{4}(\omega_{30}^2 + \omega_{03}^2) + \tfrac{21}{2}(\omega_{30}\omega_{12} + \omega_{03}\omega_{21}) + \tfrac{3}{4}(\omega_{21}^2 + \omega_{12}^2) + \\ + 59\sqrt{2}(\omega_{30}^2 - \omega_{03}^2) + 7\sqrt{2}(\omega_{30}\omega_{12} - \omega_{03}\omega_{21}). \qquad (34)$$

Using (23), (25) and some trigonometric formulas an exact calculation of the terms of (34) gives

$$\begin{aligned}\omega_{30}^2 + \omega_{03}^2 &= 2^{-8}[(10\gamma\cos 3\beta - 3\zeta\cos(\alpha+\beta))^2 + \\ &\quad + (10\gamma\sin 3\beta + 3\zeta\sin(\alpha+\beta))^2] \\ &= 2^{-8}[109\,\gamma^2 + 27 - 60\,\gamma\zeta\cos\alpha\cos 4\beta + 60\,\gamma\zeta\sin\alpha\sin 4\beta] \\ &= 2^{-8}\left[\left(169 + \frac{240}{\delta^2}\right)\gamma^2 + 27\right] = 2^{-9}\times 3^{-3}\times 12337\end{aligned}$$

and similarly

$$\begin{aligned}\omega_{30}^2 - \omega_{03}^2 &= -2^{-12}\times 2577\sqrt{2}, \\ \omega_{21}^2 + \omega_{12}^2 &= 2^{-9}\times 951, \\ \omega_{21}^2 - \omega_{12}^2 &= 2^{-12}\times 5337\sqrt{2}, \\ \omega_{03}\omega_{21} + \omega_{30}\omega_{12} &= -2^{-9}\times 659, \\ \omega_{03}\omega_{21} - \omega_{30}\omega_{12} &= -2^{-12}\times\frac{11129}{3}\sqrt{2}.\end{aligned}$$

Substituting these values into (34) gives

$$c_{0202} = 2^{-10}\times 3^{-3}\times 29293.$$

Now our aim was to compute the coefficient $a_2$ of $(y_1^2 + y_2^2)^2$ in the 4th order normal form of $H$. This coefficient $a_2$ equals the coefficient $a_{0202}$ of the monomial $(0, 2, 0, 2) = z_2^2\bar{z}_2^2$. Thus for the planar restricted three body problem we have

$$a_2 = a_{0202} = c_{0202} + h_{0202} = \tfrac{59}{864} \sim 0.068\,287\ldots. \qquad (35)$$

## Appendix

In this appendix we will outline how to construct a versal deformation (Arnol'd, 1972) of the normalized Hamiltonian function

$$H = H_2 + H_4,$$

see (5), (11).

This enables us to study the system $X_H$ passing through resonance. In the case of the planar restricted three body problem at $L_4$ we will discuss the relation of the parameters of this versal deformation with the mass parameter which generates a 1-parameter family.

Consider the Hamiltonian function of Section 3. Normalizing up to order 4 and omitting higher order terms gives the truncated Hamiltonian

$$H^*(x, y) = \alpha(x_1 y_2 - x_2 y_1) - \tfrac{1}{2}\rho(x_1^2 + x_2^2) + a_1(x_1 y_2 - x_2 y_1)^2 + \\ + a_2(y_1^2 + y_2^2)^2 + a_3(x_1 y_2 - x_2 y_1)(y_1^2 + y_2^2); \qquad \rho = \pm 1. \qquad (36)$$

The coefficients are constants.

For the restricted three body problem we have $\alpha = \frac{1}{2}\sqrt{2}$, $a_2 = \frac{59}{864}$, $a_1$ and $a_3$ two fixed constants and $\rho = -1$.

It follows from the construction of the normal form that $X_{H^*}$ is an integrable system with integrals the energy $H^*$ and $M(x, y) = x_1 y_2 - x_2 y_1$.

The linearized vectorfield corresponding to $H^*$ is $X$. Consider the orbit of $X$ under the adjoint action of the real symplectic group $\mathrm{Sp}(\omega, \mathbb{R})$. Note that im $\mathrm{ad}_X$ is the tangent space to this orbit. As in Corollary 2.1 we may construct a space $Y$ transversal to the orbit at $X$. The infinitesimal symplectic matrices

$$e_1 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad e_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

form a basis of $Y$. Thus a two parameter versal deformation of $X$ is

$$X + v_1 e_1 + v_2 e_2 = X_v. \qquad (37)$$

Let $X_\mu \notin Y$ be some other deformation of $X$, depending smoothly on the parameter $\mu$ with $X_{\mu_1} = X$. As a direct consequence of the implicit function theorem we have the following:

PROPOSITION 5.1. *For $v$ near zero and $\mu$ near $\mu_1$ we have $X_\mu$ conjugate to $X_v$ by a smooth family of real symplectic mappings $P_\mu$. Moreover there are smooth real valued functions $\psi_1$ and $\psi_2$ with $v_1 = \psi_1(\mu)$, $v_2 = \psi_2(\mu)$.*

Consider now the planar restricted three body problem at $L_4$. The matrix for the linear vectorfield, depending smoothly on the mass ratio $\mu$, is found in Deprit (1966) and is given by

$$B_\mu = \begin{pmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ \frac{1}{2}(1 - \frac{3}{2}\delta) & 0 & 0 & 1 \\ 0 & \frac{1}{2}(1 + \frac{3}{2}\delta) & -1 & 0 \end{pmatrix}; \quad \delta^2 = 4(3\mu^2 - 3\mu + 1).$$

By a linear symplectic transformation $P$, see (26), we can bring $B_{\mu_1}$ into normal form $A = P^{-1}B_{\mu_1}P$. By (37) a versal deformation of $A$ is

$$X_\nu = \begin{pmatrix} 0 & -\frac{1}{2}\sqrt{2}-\nu_1 & \nu_2 & 0 \\ \frac{1}{2}\sqrt{2}+\nu_1 & 0 & 0 & \nu_2 \\ -1 & 0 & 0 & -\frac{1}{2}\sqrt{2}-\nu_1 \\ 0 & -1 & \frac{1}{2}\sqrt{2}+\nu_1 & 0 \end{pmatrix}.$$

Consider the family $\tilde{B}_\mu = P^{-1}B_\mu P$. By the above proposition $\tilde{B}_\mu$ and $X_\nu$ must have the same characteristic polynomials. Thus

$$\begin{aligned} \nu_1 &= -\tfrac{1}{2}\sqrt{2} + (\tfrac{1}{4} + \tfrac{1}{4}(27\mu - 27\mu^2)^{1/2})^{1/2}, \\ \nu_2 &= \tfrac{1}{4} - \tfrac{1}{4}(27\mu - 27\mu^2)^{1/2}; \end{aligned} \tag{38}$$

furthermore $(\frac{1}{2}\sqrt{2}+\nu_1)^2 = \frac{1}{2} - \nu_2$. Note that the eigenvalues of $X$ are

$$(\tfrac{1}{2}\sqrt{2}+\nu_1)\left[\pm i \pm \left(\frac{-\nu_2}{\frac{1}{2}-\nu_2}\right)^{1/2}\right]. \tag{39}$$

We see that the sign of $\nu_2$ determines whether we have purely imaginary eigenvalues or eigenvalues with non-zero real part. Furthermore we see that $\nu_2$ completely determines the frequency ratio.

We have

$$\left.\frac{d\nu_2}{d\mu}\right|_{\mu=\mu_1} = -\tfrac{3}{8}\sqrt{69} \neq 0$$

and thus the 1-parameter family generated by the mass-parameter is generic.

Of course the above discussion also holds if we consider the corresponding Hamiltonian functions, instead of the vectorfields. Thus a versal deformation of $H_2^*$ is

$$H_{2,\nu}^*(x,y) = H_2^*(x,y) + \nu_1(x_1y_2 - x_2y_1) + \tfrac{1}{2}\nu_2(y_1^2+y_2^2). \tag{40}$$

Here $(x_1y_2 - x_2y_1)$ and $(y_1^2+y_2^2)$ are the basis elements of $Y_2$ defined as in Section 2.

We now extend the above discussion to 4-jets of functions. Consider the Hamiltonian $H^*$, (36). The orbit $\mathcal{O}$ of the 4-jet of $H$ at zero under symplectic diffeomorphisms $\phi$ of $(\mathbb{R}^4, \omega)$ which leave the origin fixed is the set of all 4-jets of functions $H^*\circ\phi$. Note that $\mathcal{O}$ depends only on the 4-jet of $\phi$ at zero. The space transversal to $\mathcal{O}$ is $Y_2 + Y_4$. Thus we have a five parameter versal deformation of $H^*$ which we may write as

$$\begin{aligned} H_\nu^*(x,y) = {} & \alpha(x_1y_2 - x_2y_1) - \tfrac{1}{2}\rho(x_1^2+x_2^2) + \tfrac{1}{2}\nu(y_1^2+y_2^2) + \\ & + a_1(x_1y_2 - x_2y_1)^2 + a_2(y_1^2+y_2^2)^2 + \\ & + a_3(y_1^2+y_2^2)(x_1y_2 - x_2y_1); \end{aligned} \tag{41}$$

considering the coefficients as parameters in the neighborhood of some fixed value ($\nu$ near 0) except for $\rho$ which is $+1$ or $-1$.

Hence by an extension of Proposition 5.1 we deduce that the one parameter family of 4-jets of Hamiltonian functions of the planar restricted three body problem at $L_4$, generated by the mass parameter, can be brought into (41) by the 4-jet at 0 of a smooth family of symplectic diffeomorphisms which preserve the origin.

The coefficients $\alpha$, $\nu$, $a_1$, $a_2$, and $a_3$ depend smoothly on $\mu$ for $\mu$ near $\mu_1$. For $\alpha$ and $\nu$ the dependence on $\mu$ follows from (38) and (40). We have $\alpha = \frac{1}{2}\sqrt{2} + \nu_1$ and $\nu = \nu_2$. So $\nu$ is the detuning parameter.

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