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Strongly Balanced Cooperative Games[1]

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Abstract: Kaneko/Wooders (1982) derived a list of necessary and sufficient conditions for a partitioning game to have a nonempty core regardless of the payoff functions of its effective coalitions. The main purpose of our paper is to provide a graph-theoretical characterization of this family of games whose associated hypergraphs we call *strongly balanced:* we show that the strong balancedness condition is equivalent to the *normality* of the hypergraph, which is a type of *coloring property* (Lovasz (1972)). We also study interesting economic examples of *communication* and *assignment* games and provide direct proofs that their associated hypergraphs are strongly balanced.

1 Introduction

In many economic situations it may not be easy or even possible to form every coalition. Some of the games associated with such environments are described by exogenously given set W of *effective coalitions* and the payoff function defined over the set W and where the power of an "ineffective" coalition C is determined by the best possible partition of C into its effective subcoalitions. Kaneko/Wooders (1982) provided a list of necessary and sufficient conditions for such "partitioning" games to have a nonempty core regardless of a payoff function of effective coalitions. (Since the games satisfying this property are *balanced* for any choice of payoff function of W, we call the hypergraphs associated with these games *strongly balanced*.)

The main purpose of this paper is to provide an intuitive graph-theoretic interpretation of strong balancedness. We show that the strong balancedness property is equivalent to the *normality* condition, which is a type of "coloring property" of hypergraphs (see Lovasz (1972)). We believe that one can obtain further results by relating issues in the cooperative game theory and ideas from the hypergraph theory presented in Berge (1987).

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Kaneko/Wooders (1982) motivated their analysis by referring to economic environments where the effective coalitions are relatively small (e.g., "marriage", "assignment", "bridge" games). In this paper we however describe interesting economic environments (e.g., "communication" and "consecutive" games) whose associated hypergraphs are strongly balanced and where the effective coalitions are not necessarily small. Furthermore, we offer a direct proof that the well-known assignment games are strongly balanced.

The paper is organized as follows: In the next section we introduce necessary notations and definitions. In Sections 3–5 some applications, which give rise to strongly balanced hypergraphs, are examined. In Section 6 we prove the equivalence of normality and strong balancedness conditions.

2 Notations and Definitions

Let (N, W) be *a hypergraph*, where $N = \{1, 2, \ldots, n\}$ is a finite set and W is a (nonempty) family of subsets of N with $\{i\} \in W$ for each $i \in N$. The standard interpretation is that N is the set of players and W is the set of *effective* coalitions. That is, if a coalition C does not belong to W its power is represented by the power of effective subsets of C.[5] To introduce the central notion of this paper, of a *strongly balanced hypergraph*, we first recall that a collection of coalitions $\mathscr{S} = \{C_1, \ldots, C_H\}$ is called *balanced* if there exist positive numbers $\{\gamma_1, \ldots, \gamma_H\}$ (called *balancing weights*) such that

$$\sum_{C_h \in \mathscr{S}(i)} \gamma_h = 1 \quad \textit{for all} \quad i \in N,$$

where $\mathscr{S}(i) = \{C \in \mathscr{S} \mid i \in C\}$. Then

Definition 2.1: The hypergraph (N, W) is *strongly balanced* if every balanced collection $\mathscr{S}$, that consists of elements of W, contains a partition of N.

This notion has been used by Kaneko/Wooders (1982) in order to demonstrate that the core of the cooperative game associated with a hypergraph (N, W) is nonempty regardless of the payoff function of coalitions in W, if and only if the hypergraph (N, W) is strongly balanced.

In the next three sections we provide examples of economic models which give rise to strongly balanced hypergraphs.

[5] For formal details see Kaneko/Wooders (1982).

3 Communication Games

Consider a communication structure between players as suggested by Myerson (1977) (see also Owen (1986)). Let N be a set of players and E be an ordinary graph defined on N. There is a natural interpretation of the graph E: if players i and j are connected, that is, the link (i, j) belongs to E, then we say that i and j can communicate (without cost), whereas if i and j are not connected in E then they cannot communicate, which is equivalent to existence of infinite communication cost between these players.[6] The graph E induces a "communication" hypergraph (N, W_E) as follows: a coalition C belongs to W_E if and only if C is connected in E, i.e., for each two players i and j in S there is a path, consisting of links in E, which goes from i to j and stays within E. To state our first result, recall that a graph containing no cycles is called *a tree* and a graph is a *forest* if each of its connected components is a tree.

Proposition 3.1: The communication hypergraph (N, W_E) is strongly balanced if and only if the graph (N, E) is a forest.

Proof: Sufficiency.[7] Let (N, E) be a forest and let $N^1, \ldots, N^M$ be its connected components. Consider a balanced collection $\mathscr{S}$ which consists of elements of W. For each $m=1, \ldots, M$, denote $\mathscr{S}^k = \{C \in \mathscr{S} \mid C \subset N^m\}$. The definition of the set W_E implies that for each $C \in \mathscr{S}$ there exists m, $1 \leq m \leq M$ such that $C \in \mathscr{S}^m$. Thus, in order to show that $\mathscr{S}$ contains a partition, it suffices to demonstrate that each $\mathscr{S}^m$ contains a partition. Assume, therefore, without loss of generality, that (N, E) itself is a tree.

Consider 1 as the root of the tree and order elements of N by defining *j precedes k* (denoted $j \prec k$) if the path from 1 to k passes through j. We call *j an immediate predecessor* of k, denoted by $p(k)$, if $j \prec k$ and $(j, k) \in E$. Since (N, E) is a tree, it is easy to see that $\prec$ is a partial order and for all $k > 1$, $p(k)$ is unique.

We shall use the following two claims, the proof of which is left to the reader:

Claim 3.2: Let $C \in \mathscr{S}$ with $i \in C$ and $j \notin C$ for some $i, j \in N$. Then there exists a coalition $\hat{C}$ in $\mathscr{S}$ such that $j \in \hat{C}$ and $i \notin \hat{C}$.

Claim 3.3: Let $\{i, j\} \in E$ and let C and $\bar{C}$ be connected subsets of N with $i \in C \setminus \bar{C}$ and $j \in \bar{C} \setminus C$. Then $C \cup \bar{C}$ is connected and $C \cap \bar{C} = \emptyset$.

Let $C_1 \in \mathscr{S}$ be such that $1 \in C_1$. If $C_1 = N$, the conclusion follows. If $C_1 \neq N$, choose k_1 to be a minimal element in $N \setminus C_1$ with respect to the order $\prec$, that is, $j_1 = p(k_1)$, the immediate predecessor of k_1, does not belong to $N \setminus C_1$. By Claim 3.2, there exists $C_2 \in \mathscr{S}$ with $j_1 \notin C_2$ and $k_1 \in C_2$. Moreover, Claim 3.3 yields $C_1 \cap C_2 = \emptyset$

[6] A model of an exchange economy endowed with such a communication structure has been studied by Kirman/Oddou/Weber (1986).

[7] After our paper has been submitted for a publication, we found out that Demange (1990) has independently proved the strong balancedness of the hypergraph (N, W_E) in the case where the graph (N, E) is a tree.

and $\overline{C_2} = C_1 \cup C_2$ is connected. If $\overline{C_2} = N$ then $\{C_1, C_2\}$ is a desired partition of N. Otherwise, choose k_2 to be a minimal element in $N \setminus \overline{C_2}$ with respect to the order $\prec$ and proceed as before. Thus, we obtain a sequence of elements of $\mathscr{S}$, $C_1, C_2, \ldots, C_r, \ldots$, where for each r the set $\overline{C_r} = C_1 \cup C_2 \cup \ldots \cup C_r$ is connected and has an empty intersection with the set C_{r+1}. Since the sequence $\overline{C_r}$ is strictly increasing with respect to inclusion and N is finite, there exists R such that $\overline{C_R} = N$, yielding $\{C_1, C_2, \ldots, C_R\}$ as the desired partition of N.

Necessity. Let (N, E) be a graph which contains a cycle. Let the length of the cycle be $k \geq 3$. Without loss of generality, assume that the links $\{1, 2\}$, $\{2, 3\}$, ..., $\{k-1, k\}$, $\{k, 1\}$ belong to E. Denote $T = \{1, \ldots, k\}$ and consider the following family $\mathscr{S} = \{T_1, \ldots, T_n\}$ where for each $i = 1, \ldots, n$

$$T_i = \begin{cases} T \setminus \{i\} & \text{if } \; i \leq k \\ \{i\} & \text{if } \; i > k. \end{cases}$$

It is easy to see that $\mathscr{S}$ is a balanced collection of elements of W_E. Indeed, define the balancing weights by:

$$\gamma_i = \begin{cases} \dfrac{1}{k-1} & \text{if } \; i \leq k \\ 1 & \text{if } \; i > k. \end{cases}$$

However, $\mathscr{S}$ does not contain a partition of N. □

4 Consecutive Games

In their study of a Tiebout equilibrium in economies with local public goods, Greenberg/Weber (1986) introduced what they called *consecutive games*. To recall, a coalition C is *consecutive* if for every two players $i, j \in C$, $i < j$, every "intermediate" player k, $i \leq k \leq j$, also belongs to C. In our terminology, a game is consecutive if its hypergraph is consecutive, or, equivalently, all effective coalitions are consecutive. There are several types of models which naturally give rise to consecutive games[8]: local public goods, location, product differentiation[9], hierarchical and political games[10]. Consecutive games naturally give rise to the notion of *consecutive hypergraphs:* the consecutive hypergraph is the communication hypergraph induced by the graph E where the link $(i, j) \in E$ if and only if $|i - j| = 1$. Since such E is, obviously, a tree, the following result of Greenberg/Weber (1986), stated in graph-theoretical terms, constitutes a corollary of Proposition 3.1:

[8] See Greenberg/Weber (1991) for a more detailed discussion.
[9] See the recent paper by Demange/Henriet (1991) on the oligopoly theory.
[10] E.g., Axelrod (1970) examines configurations of multi-party coalitions in parliamentary democracies where parties, ranked from the left to the right, choose their location in a one-dimensional policy space.

Proposition 4.1: Every consecutive hypergraph is strongly balanced.

5 Assignment Games

Gale/Shapley (1962) introduced a class of games, called "marriage games", which has been later extended to "assignment games" (Shapley/Shubik (1972)) which can be described as follows: Consider the set of players $N=\{1, \ldots, n\}$ which consists of m men, numbered $1, \ldots, m$, and $n-m$ women, numbered $m+1, \ldots, n$, and the set of effective coalitions W which consists of all one-person coalitions and all "man-woman" coalitions, $\{i,j\}$ where $1 \leq i \leq m < j \leq n$. Then

Proposition 5.1: The corresponding "assignment" hypergraph (N, W) is strongly balanced.

Proof:[11] Let $\mathscr{S}=\{C_1, \ldots, C_H\}$ be a balanced collection of effective coalitions with balancing weights $\{\gamma_h\}_{h=1,\ldots,H}$. Form the $n \times n$ matrix A whose entries are defined by:

$$a_{ij}=a_{ji}=\gamma(\{i,j\}) \quad \textit{if} \quad \{i,j\} \in \mathscr{S}$$

$$a_{ij}=0 \quad \textit{for all other pairs} \quad i,j.$$

The matrix A can be represented as:

$$A=\begin{pmatrix} E & D \\ D^t & F \end{pmatrix}$$

where E is the $m \times m$ matrix and F is the $(n-m)\times(n-m)$ matrix. Since both E and F have non-zero entries only on the main diagonal, the matrix A is symmetric. Moreover, balancedness of $\mathscr{S}$ implies that the sums of entries in any one row or column are 1. Since all entries are non-negative, A is a doubly stochastic matrix. By the Birkhoff Theorem (see, e.g., Marshall/Olkin (1979)), any doubly stochastic matrix is a convex linear combination of permutation matrices. Thus,

$$A=\sum_{\ell=1}^{L} \lambda_\ell \; B_\ell$$

where all numbers $\lambda_1, \ldots, \lambda_L$ are positive, $\sum_{\ell=1}^{L} \lambda_\ell = 1$ and each B_ℓ is a permutation matrix. Take one of B_ℓ, say, $B=B_1$ and observe that its entry b_{ij} equals to 1

[11] An indirect proof of this proposition could be obtained by combining the results of Kaneko (1982) and Kaneko/Wooders (1982).

only if the corresponding entry of matrix A, a_{ij}, is positive. We can represent B by using its submatrices:

$$B = \begin{pmatrix} X & Y \\ Z & U \end{pmatrix}$$

where again X and Y occupy the first m rows, whereas X and Z occupy the first m columns. (Note that, in general, $Z \neq Y^t$.)

If we now disregard the submatrix Z, one can observe that the non-zero entries in the remaining submatrices, X, Y and U, give rise, as desired, to the set of marriages (matrix Y), of single men (matrix X) and of single women (matrix U), which constitutes a partition of N, consisting only of sets in $\mathscr{S}$. □

6 Strong Balancedness and Hypergraphs

In previous section we studied several classes of games, arising in different economic environments, whose associated hypergraphs are strongly balanced. We shall now provide necessary and sufficient conditions for hypergraphs to be strongly balanced. Kaneko/Wooders (1982) observed that unimodular hypergraphs are strongly balanced. We shall demonstrate that in our framework a weaker condition, called *normality*, is equivalent to the strong balancedness property. (The reader is referred to Berge (1987) for a proof of the fact that every unimodular hypergraph is normal.)

Let us introduce some notation and definitions. Let $H=(N, W)$ be a hypergraph, where $N=\{1, \ldots, n\}$ is a finite set and $W=\{C_1, \ldots, C_m\}$ is a (nonempty) family of subsets of N. For each $i \in N$ denote $W_i=\{C \in W \mid i \in C\}$ and let

$$\Delta(H) \equiv \max_{i \in N} |W_i|,$$

where $|W_i|$ stands for cardinality of the set W_i.

The *chromatic index* of H, denoted by $q(H)$, is the minimal number of colors required to color all the elements of W in such a way every two elements of W with a nonempty intersection have different colors. Let $j \in N$ be such that $\Delta(H)=|W_j|$. Then one would have use $|W_j|$ colors to color only coalitions in W_j so that any two of them have different colors, yielding

$$q(H) \geq \Delta(H).$$

The chromatic index is used in order to introduce a notion of "normal" hypergraph:

Definition 6.1: Let W' be a subset W. The hypergraph $H'=(N', W')$ is called a *partial hypergraph (induced by W')*, where $N'=\{i \in N \mid \exists C \in W' \ s.t. \ i \in C\}$. The hypergraph $H=(N, W)$ is called *normal* if $q(H')=\Delta(H')$ for each partial hypergraph H' of H. (This is a *perfect coloring property* (Lovasz (1972)).

We are now in position to state our next result which provides a graph-theoretical characterization of the strong balancedness property:

Proposition 6.2: If $\{i\}\in W$ for all $i\in N$ then a hypergraph $H=(N, W)$ is strongly balanced if and only if it is normal[12].

Proof of Proposition 6.2: Let a hypergraph $H=(N, W)$, where $\{i\}\in W$ for all $i\in N$, be given. Define the *matching polyhedron of H, P(H)*, by

$$P(H)\equiv\{q=(q_1, \ldots, q_M)\in\Re^M_+ \mid \sum_{\{m \mid C_m\in W_i\}} q_m\leq 1 \ \forall i\in N\}.$$

Denote by $\bar{P}(H)$ the subset of $P(H)$ where all the inequalities are replaced by equalities, i.e.,

$$\bar{P}(H)\equiv\{q=(q_1, \ldots, q_M)\in\Re^M_+ \mid \sum_{\{m \mid C_m\in W_i\}} q_m=1 \ \forall i\in N\}.$$

Observe now that Proposition 6.2 is a corollary of the following lemmata:

Lemma 6.3: H is strongly balanced if and only if the set of the extreme points of $\bar{P}(H)$ consists of integers (i.e., vectors whose coordinates are integer numbers).

Lemma 6.4: The set of the extreme points of $\bar{P}(H)$ consists of integers if and only if the set of the extreme points of $P(H)$ consists of integers.

Lemma 6.5: H is normal if and only if the set of the extreme points of $P(H)$ consists of integers.

Lemma 6.3 was proved by Kaneko/Wooders (1982) whereas Lemma 6.5 follows from Lovasz (1972). It remains, therefore, to prove Lemma 6.4:

Proof of Lemma 6.4: We shall show that if the set of the extreme points of $\bar{P}(H)$ consists of integers then the set of the extreme points of $P(H)$ also consists of integers. (The other direction is straightforward.)

Let $W=\{C_1, \ldots, C_M\}$ be the family of the effective coalitions, where, without loss of generality, $C_i=\{i\}$ for $i=1, \ldots, n$. Let q be an extreme point of the polyhedron $P(H)$ for which there exists $i\in N$ such that

$$\sum_{\{m \mid C_m\in W_i\}} q_m<1. \tag{1}$$

[12] Note that Definition 6.1 implies that all partial hypergraphs of a normal hypergraph are also normal. Proposition 6.2 implies, therefore, that if a hypergraph (N, W) is strongly balanced then for each subset W' of W the hypergraph (N, W') is also strongly balanced.

(If such a point does not exist the proof of the lemma is completed.) Assume, without loss of generality, that (1) holds for $i=1, \ldots, k$. Put $\delta_i \equiv \sum_{\{m \mid C_m \in W_i\}} q_m$ and note that $\delta_i > 0$ for $i=1, \ldots, k$. Denote

$$\bar{q} = q + \alpha, \tag{2}$$

where $\alpha \in \Re_+^M$ is defined by

$$\alpha^i = \begin{cases} \delta_i & \textit{if} \quad 1 \leq i \leq k \\ 0 & \textit{otherwise.} \end{cases}$$

We claim that $\bar{q}$ is an extreme point of $\bar{P}(H)$. Otherwise, there exist two different vectors q_1 and q_2 in $\bar{P}(H)$ and $0<\lambda<1$ such that $\bar{q}=\lambda q_1+(1-\lambda)q_2$. Observe that since the first k coordinates of the vector $\bar{q}$ are equal to 1, so are the first k coordinates of the vectors q_1 and q_2. Since $q=\lambda(q_1-\alpha)+(1-\lambda_1)(q_2-\alpha)$, it follows that q is a convex combination of two different vectors in $\bar{P}(H)$, a contradiction to q being an extreme point of $P(H)$.

Lemma 6.3 implies that $\bar{q}$, being an extreme point of $\bar{P}(H)$, consists of integers. Since $\bar{q}^i = q^i$ for all $i>k$, it remains to show that each of the numbers $q^1, \ldots, q^k$, is an integer. Assume, without loss of generality, that the positive numbers $\delta_1, \ldots, \delta^k$, are ordered in such a way that $\delta_1 \geq \delta_2 \ldots \geq \delta_k$. By (2), the vector q could be represented as a convex combination of vectors $v_1, \ldots, v_k, v_{k+1}$:

$$q = \sum_{j=1}^{k+1} (\delta_{j-1} - \delta_j) v_j, \tag{3}$$

where $\delta_0 \equiv 1$, $\delta_{k+1} \equiv 0$ and the vectors $v_j \in P(H)$, $j=1, \ldots, k+1$, are given by

$$v_j^i = \begin{cases} 0 & \textit{if} \quad 1 \leq i < j \\ 1 & \textit{if} \quad j \leq i \leq k \\ q^i & \textit{if} \quad k < i \leq M. \end{cases}$$

Since $\delta_1 > 0$ and q is an extreme point of $P(H)$, it follows that $\delta_1 = 1$. As $\delta_2 > 0$ the same argument yields $\delta_2 = 1$. By repeating this argument $k-2$ times, if necessary, we obtain $\delta_i = 1$ for $i=1, \ldots, k$. Thus, (3) implies that $q = v_{k+1}$, i.e., $q^1 = \ldots = q^k = 0$. □

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