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Unbounded families and the cofinality of the infinite symmetric group

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Abstract. In this paper, we study the relationship between the cofinality $c(\mathrm{Sym}(\omega))$ of the infinite symmetric group and the minimal cardinality $\underset{\sim}{b}$ of an unbounded family F of ${}^{\omega}\omega$.

1 Introduction

Suppose that G is a group that is not finitely generated. Then G can be expressed as the union of a chain of proper subgroups. The cofinality of G, written $c(G)$, is defined to be the least cardinal λ such that G can be expressed as the union of a chain of λ proper subgroups. If κ is an infinite cardinal, then $\mathrm{Sym}(\kappa)$ denotes the group of all permutations of the set $\kappa = \{\alpha | \alpha < \kappa\}$. In [9], Macpherson and Neumann proved that $c(\mathrm{Sym}(\kappa)) > \kappa$. In [10], we studied the possibilities for the value of $c(\mathrm{Sym}(\omega))$. In particular, we proved that it is consistent that $c(\mathrm{Sym}(\omega))$ and 2^{ω} can be any two prescribed regular uncountable cardinals, subject only to the obvious requirement that $c(\mathrm{Sym}(\omega)) \leq 2^{\omega}$. In this paper, we shall consider the relationship between $c(\mathrm{Sym}(\omega))$ and two wellknown cardinal invariants of the continuum.

Definition 1.1. (a) ${}^{\omega}\omega$ is the set of functions from ω to ω.
(b) If $f, g \in {}^{\omega}\omega$, then $f \leq^* g$ iff there exists $n_0 \in \omega$ such that $f(n) \leq g(n)$ for all $n \geq n_0$.

Definition 1.2. (a) A family $F \subseteq {}^{\omega}\omega$ is *dominating* if for every $g \in {}^{\omega}\omega$, there exists $f \in F$ such that $g \leq^* f$.
(b) $\underset{\sim}{d}$ is the minimal cardinality of a dominating family F of ${}^{\omega}\omega$.

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Definition 1.3. (a) A family $F \subseteq {}^{\omega}\omega$ is *unbounded* if there does not exist $g \in {}^{\omega}\omega$ such that $f \leq^* g$ for all $f \in F$.
(b) $\underset{\sim}{b}$ is the minimal cardinality of an unbounded family F of ${}^{\omega}\omega$.

Proposition 1.4. $c(\mathrm{Sym}(\omega)) \leq \underset{\sim}{d}$.

Proof. Let $\underset{\sim}{d} = \lambda$ and let $F = \{f_i | i < \lambda\}$ be a dominating family. We may assume that each f_i is strictly increasing. For each $\theta < \lambda$, define

$$G_\theta = < g | \text{There exist } i, j < \theta \text{ such that } g \leq^* f_i \text{ and } g^{-1} \leq^* f_j > .$$

Then $\mathrm{Sym}(\omega) = \bigcup_{\theta<\lambda} G_\theta$. So it is enough to prove that each G_θ is a proper subgroup. Suppose that there exists $\theta < \lambda$ such that $G_\theta = \mathrm{Sym}(\omega)$. Let C be the closure of $\{f_i | i < \theta\}$ under the taking of compositions. Since $\lambda \geq \omega_1$, it follows that $|C| < \lambda$. Notice that each $f \in C$ is also strictly increasing.

Claim 1.5 *For each $g \in G_\theta = \mathrm{Sym}(\omega)$, there exists $f \in C$ such that $g \leq^* f$.*

Proof of Claim 1.5 It suffices to prove that if $g, h \in G_\theta$ satisfy $g \leq^* f_1$ and $h \leq^* f_2$ for some $f_1, f_2 \in C$, then $g \circ h \leq^* f_1 \circ f_2$. Choose n_0 so that $g(n) \leq f_1(n)$ and $h(n) \leq f_2(n)$ for all $n \geq n_0$. Since $h \in \mathrm{Sym}(\omega)$, there exists $n_1 > n_0$ such that $h(n) \geq n_0$ for all $n \geq n_1$. Hence $n \geq n_1$ implies that $g(h(n)) \leq f_1(h(n)) \leq f_1(f_2(n))$. □

Expanding C if necessary, we can suppose that $d \in C$ where $d(n) = 2n$. Since $|C| < \underset{\sim}{d}$, there exists $\varphi \in {}^{\omega}\omega$ such that $\varphi \not\leq^* f$ for all $f \in C$. We can assume that φ is strictly increasing and that $|\omega - \operatorname{ran}\varphi| = \omega$. Hence there exists $g \in \mathrm{Sym}(\omega)$ such that $g(2n) = \varphi(n)$ for all n. By Claim 1.5, there exists $f \in C$ such that $g \leq^* f$. But then $\varphi(n) = g(2n) \leq f \circ d(n)$ for all sufficiently large n, which is a contradiction. □

So the order relationships between the cardinals mentioned above are given by the following diagram, where $\kappa \longrightarrow \lambda$ means that $\kappa \leq \lambda$ is provable in ZFC.

$$\begin{array}{ccccc} c(Sym(\omega)) & \longrightarrow & \underset{\sim}{d} & \longrightarrow & 2^{\omega} \\ \uparrow & & \uparrow & & \\ \omega_1 & \longrightarrow & \underset{\sim}{b} & & \end{array}$$

The results in [4] and [10] show that no further theorems are provable in ZFC, except possibly for one which relates $\underset{\sim}{b}$ and $c(\mathrm{Sym}(\omega))$. The next two results rule out this possibility.

Theorem 1.6. *$\omega_1 = c(\mathrm{Sym}(\omega)) < \underset{\sim}{b} = \underset{\sim}{d} = 2^{\omega}$ is consistent with ZFC.*

Proof. Let $M \vDash MA + \neg CH$. Then $M \vDash \underset{\sim}{b} = \underset{\sim}{d} = 2^{\omega}$. Let B be the measure algebra corresponding to the product measure space ${}^{\omega_1}\{0,1\}$. (See Jech [6].) For each $\alpha < \omega_1$, let B_α be the measure algebra of ${}^{\alpha}\{0,1\}$. Let H be an M-generic ultrafilter on B. It is well known that $M[H] \vDash \omega_1 < \underset{\sim}{b} = \underset{\sim}{d} = 2^{\omega}$. For each $\alpha < \omega_1$, let $H_\alpha = H \cap B_\alpha$ and let $G_\alpha = \mathrm{Sym}(\omega) \cap M[H_\alpha]$. Then $\mathrm{Sym}(\omega) \cap M[H] = \bigcup_{\alpha<\omega_1} G_\alpha$, and each G_α is a proper subgroup of $\mathrm{Sym}(\omega) \cap M[H]$. Thus $M[H] \vDash c(\mathrm{Sym}(\omega)) = \omega_1$. □

Theorem 1.7. $\omega_1 = \underset{\sim}{b} < c(\mathrm{Sym}(\omega)) = \omega_2 = 2^\omega$ *is consistent with ZFC.*

In Sect. 4, we will use a countable support iteration of proper forcings to prove Theorem 1.7. The forcing notions, which are the individual steps in our iteration, will be discussed in Sects. 2 and 3.

Our notation follows that of Kunen [8]. Thus if $\mathbb{P}$ is a notion of forcing and $p, q \in \mathbb{P}$, then $q \leq p$ means that q is a strengthening of p. If V is the ground model, then we often denote the generic extension by $V^{\mathbb{P}}$ if we do not want to specify a particular generic filter $G \subseteq \mathbb{P}$. If $\tilde{\tau}$ is a $\mathbb{P}$-name, then $(\tilde{\tau})_G$ denotes the element of $V[G]$ which $\tilde{\tau}$ names.

A condition $p \in \mathbb{P}$ is an atom if there do not exist incompatible conditions q and r such that $q, r \leq p$. $\mathbb{P}$ is atomic if for all $p \in \mathbb{P}$, there exists an atom q with $q \leq p$. In this case, $V[G] = V$ for every generic filter $G \subseteq \mathbb{P}$.

$\mathbb{P}$ is said to be ${}^\omega\omega$-bounding if for each $g \in ({}^\omega\omega \cap V^{\mathbb{P}})$ there exists $f \in ({}^\omega\omega \cap V)$ such that $g \leq^* f$.

If A and B are sets, then ${}^AB = \{f | f : A \to B\}$. We use the convention that if $g \in {}^\omega\omega$, then $g(-1) = 0$. This is used in expressions such as $\prod_{n<\omega} \mathrm{Sym}(g(n) \smallsetminus g(n-1))$. A subset A of ω is called a moiety if $|A| = |\omega \smallsetminus A| = \omega$.

Let U be a nonprincipal ultrafilter on ω and let $\mathscr{I}$ be the dual ideal. U is a p-point if for any partition $\{I_n | n < \omega\}$ of ω satisfying $I_n \in \mathscr{I}$ for all $n < \omega$, there exists $X \in U$ such that $|I_n \cap X| < \omega$ for all $n < \omega$. If there exists $X \in U$ such that $|I_n \cap X| \leq 1$ for all $n < \omega$, then U is said to be selective for the partition $\{I_n | n < \omega\}$.

Let G be a subgroup of $\mathrm{Sym}(\omega)$, and let $\{a_n | n < \omega\}$ be the increasing enumeration of $A \in [\omega]^\omega$. $G_{\{A\}}$ denotes the setwise stabiliser of A in G, and $G(A)$ denotes the subgroup of $\mathrm{Sym}(A)$ which is induced on A by $G_{\{A\}}$. If $\pi \in \mathrm{Sym}(\omega)$, then $\pi^A \in \mathrm{Sym}(A)$ is defined by $\pi^A(a_n) = a_{\pi(n)}$ for all $n < \omega$. If Γ is a subgroup of $\mathrm{Sym}(\omega)$, then $\Gamma^A = \{\pi^A | \pi \in \Gamma\}$. If H and K are subsets of $\mathrm{Sym}(\omega)$, then $\langle H, K\rangle$ denotes the subgroup generated by the subset $H \cup K$. Similarly, if Φ is a property, then $\langle g \in \mathrm{Sym}(\omega) | g \text{ has property } \Phi\rangle$ denotes the subgroup generated by the subset $\{g \in \mathrm{Sym}(\omega) | g \text{ has property } \Phi\}$.

2 A variant of Grigorieff forcing

In this section, we shall introduce a modified form of Grigorieff forcing, which is designed to adjoin a generic permutation $g : \omega \to \omega$ rather than a generic subset $S \subseteq \omega$. In order to motivate the definition, we shall first point out two unsuccessful attempts at making the modification.

Definition 2.1. Let U be a p-point and let $\mathscr{I}$ be the dual ideal. Then $\mathbb{P}(U)$ consists of the set

$$\{p : \omega \to \omega | \mathrm{dom}(p) \in \mathscr{I} \text{ and } p \in \mathrm{Sym}(\mathrm{dom}\, p)\}$$

partially ordered by reverse inclusion.

Proposition 2.2. $\mathbb{P}(U)$ *collapses all cardinals* λ *such that* $\omega_1 \leq \lambda \leq 2^\omega$.

Proof. Let $T = \{\pi_\alpha | \alpha < 2^\omega\}$ be an enumeration of the infinite cycles $\pi \in \mathrm{Sym}(\omega)$ which act transitively on ω. For each $\alpha < 2^\omega$, the set

$$D_\alpha = \{p \in \mathbb{P}(U) | \text{There exists } A \in [\mathrm{dom}\, p]^\omega \text{ such that } p \upharpoonright A = \pi_\alpha^A\}$$

is dense in $\mathbb{P}(U)$. Let $G \subseteq \mathbb{P}(U)$ be a generic filter, and let $g = \bigcup G \in \mathrm{Sym}(\omega)$. Then for each $\alpha < 2^\omega$, there exists $A_\alpha \in [\omega]^\omega$ such that $g \restriction A_\alpha = \pi_\alpha^{A_\alpha}$. If $\alpha < \beta < 2^\omega$, then $A_\alpha \neq^* A_\beta$ and so $A_\alpha \cap A_\beta = \emptyset$. The result follows. □

In order to avoid the above problem, it seems necessary to restrict our partial order to those conditions p which are bounded by some fixed function $f \in {}^\omega\omega$.

Definition 2.3. If $f \in {}^\omega\omega$ is strictly increasing, then

$$\mathbb{P}_0(U, f) = \{p \in \mathbb{P}(U) | p(n) \leq f(n) \text{for all } n < \omega\}.$$

Unfortunately there is now a serious possibility that our notion of forcing has become trivial. Define $h \in {}^\omega\omega$ recursively by

$$\begin{aligned} h(0) &= f(1) \\ h(n+1) &= f(h(n)), \quad n < \omega. \end{aligned}$$

Proposition 2.4. *If U is selective for the partition $\{h(n) \setminus h(n-1) | n < \omega\}$, then $\mathbb{P}_0(U, f)$ is atomic.*

Proof. Let $p \in \mathbb{P}_0(U, f)$ be arbitrary. Then there exists $B \subseteq \omega \setminus \mathrm{dom}\, p$ such that $B \in U$ and $|B \cap (h(n) \setminus h(n-1))| \leq 1$ for all $n < \omega$. Let $\{b_n | n < \omega\}$ be the increasing enumeration of B. Replacing B by $\{b_{2n} | n < \omega\}$ or $\{b_{2n+1} | n < \omega\}$ if necessary, we can suppose that $b_{n+1} > f(b_n)$ for all $n < \omega$. Extend p to a condition q such that $\mathrm{dom}\, q = \omega \setminus B$ by defining $q(m) = m$ for all $m \in \mathrm{dom}\, q \setminus \mathrm{dom}\, p$. Suppose now that $r \leq q$ and that $\{c_n | n < \omega\}$ is the increasing enumeration of $C = \mathrm{dom}\, r \setminus \mathrm{dom}\, q$. Then $c_{n+1} > f(c_n)$ for all $n < \omega$, and so $r(m) = m$ for all $m \in C$. Thus q is an atom. □

In particular, $\mathbb{P}_0(U, f)$ is always atomic if U is a Ramsey ultrafilter. We are now ready to give the correct definition of modified Grigorieff forcing.

Definition 2.5. Suppose that $f \in {}^\omega\omega$ is strictly increasing, and that U is a p-point which is *not* selective for the partition $\{f(n) \setminus f(n-1) | n < \omega\}$. Then

$$\mathbb{P}(U, f) = \{p \in \mathbb{P}(U) | p(n) \leq f(n) \textit{ for all } n < \omega\}.$$

Theorem 2.6. *$\mathbb{P}(U, f)$ is proper and ${}^\omega\omega$-bounding.*

In the proof of Theorem 2.6, we need the following material from Grigorieff [5].

Definition 2.7. Let U be a p-point.

(a) A is a U-tree if A is a nonempty subset of $Seq([\omega]^{<\omega})$ closed under taking initial subsequences.
(b) If $s \in A$, then the ramification of A at s, denoted by $R_A(s)$, is the set of all $a \in [\omega]^{<\omega}$ such that $s^\frown < a > \in A$.
(c) A is a strong U-tree if for each $s \in A$, there exists $X \in U$ such that $[X]^{<\omega} \subseteq R_A(s)$.
(d The branch $H = \langle H(n) | n < \omega \rangle$ of A is a U-branch if $\bigcup\{H(n) | n < \omega\} \in U$.

Theorem 2.8. *[5] If U is a p-point, then every strong U-tree has a U-branch.*

In order to prove that $\mathbb{P}(U, f)$ is proper, it is enough to show that player II has a winning strategy in the game $\mathscr{G}_\omega$. (See Jech [7].)

Definition 2.9. The *countable choice* game $\mathcal{G}_\omega$ for $\mathbb{P}(U, f)$ is defined as follows. First player I selects a condition $p \in \mathbb{P}(U, f)$, and then player I and II begin to play alternatively. At the n^{th} stage of the game for each $n < \omega$, player I plays an ordinal name $\tilde{\alpha}_n$ and then player II plays a countable set B_n. Player II wins the game iff there exists $q \leq p$ such that for all $n < \omega$, $q \Vdash \tilde{\alpha}_n \in B_n$.

Proof of Theorem 2.6 We shall show that player II has a winning strategy in the game $\mathcal{G}_\omega$ for $\mathbb{P}(U, f)$. Suppose that player I initially selects $p \in \mathbb{P}(U, f)$. As the game proceeds, player II will construct a U-tree A and a decreasing function $Q : A \to \mathbb{P}(U, f)$ such that the following conditions are satisfied.

(2.10)
(a) $Q(\phi) = p$.
(b) If $a \in [\omega \setminus \operatorname{dom} p]^{<\omega}$, then $\langle a \rangle \in A$.
(c) For every $s \in A, \operatorname{dom} Q(s) \cap (\bigcup R_A(s) \bigcup \cup s) = \phi$.
(d) After the $(n-1)^{\text{st}}$ move of player I, player II decides which sequences $s = \langle a_0, \dots, a_n \rangle$ of length $n + 1$ lie in A, and defines the conditions $Q(\langle a_0, \dots, a_{n-1} \rangle)$ for each sequence $\langle a_0, \dots, a_{n-1} \rangle \in A$ of length n.

Suppose at the n^{th} stage that player I plays $\tilde{\alpha}_n$. Then player II proceeds as follows. Let $s = \langle a_0, \dots, a_n \rangle \in A$ be arbitrary, and let $m_s = \max(\bigcup s)$. Since $Q(\langle a_0, \dots, a_{n-1} \rangle) \cap \bigcup s = \phi$, there exists $q' \leq Q(\langle a_0, \dots, a_{n-1} \rangle)$ such that $(\omega \setminus \operatorname{dom} q') \cap (f(m_s) + 1) = \bigcup s$.

Thus if $r \leq q'$ satisfies $\bigcup s \subseteq \operatorname{dom} r$, then $r \restriction \bigcup s \in \operatorname{Sym}(\bigcup s)$. Let $\{\pi_i | 0 \leq i \leq t\}$ enumerate those elements $\pi \in \operatorname{Sym}(\bigcup s)$ such that $\pi \in \mathbb{P}(U, f)$. Choose inductively conditions $q' \geq q_0 \geq \dots \geq q_t$ and ordinals $\beta_0, \dots, \beta_t$ such that for each $0 \leq i \leq t$

(2.11)
(a) $\operatorname{dom}(q_i) \cap \bigcup s = \phi$;
(b) $q_i \cup \pi_i \Vdash \tilde{\alpha}_n = \beta_i$.

Now define $R_A(s) = [\omega \setminus (\bigcup s \cup \operatorname{dom} q_t)]^{<\omega}$, $Q(s) = q_t$ and $B_s = \{\beta_i | 0 \leq i \leq t\}$. Notice that $Q(s) \Vdash \tilde{\alpha}_n \in B_s$. Also $\omega \setminus (\bigcup s \cup \operatorname{dom} q_t) \in U$, and so A will actually be a strong U-tree. Finally player II plays

$$B_n = \{\beta | \text{ There exists } s = \langle a_0, \dots, a_n \rangle \in A \text{ such that } \beta \in B_s\}.$$

We shall now show that player II wins the above play of the game $\mathcal{G}_\omega$. By Theorem 2.8, A has a branch $H = \langle a_n | n < \omega \rangle$ such that $\bigcup_{n \in \omega} a_n \in U$. Let $q = \bigcup_{n \in \omega} Q(\langle a_0, \dots, a_n \rangle)$. Then $\operatorname{dom} q \cap \bigcup_{n \in \omega} a_n = \phi$, and it follows easily that $q \in \mathbb{P}(U, f)$. Also for each $n < \omega$,

$$q \leq Q(\langle a_0, \dots, a_n \rangle) \Vdash \tilde{\alpha}_n \in B_{\langle a_0, \dots, a_n \rangle} \subseteq B_n.$$

This completes the proof that $\mathbb{P}(U, f)$ is proper.

Finally we prove that $\mathbb{P}(U, f)$ is ${}^\omega\omega$-bounding. So suppose that $p \Vdash \tilde{h} \in {}^\omega\omega$. Then we can consider a play of the game $\mathcal{G}_\omega$ in which player I plays p and $\tilde{\alpha}_n = \tilde{h}(n)$ for $n < \omega$. The above argument yields a condition $q \leq p$ and a sequence of finite sets $B_{\langle a_0, \dots, a_n \rangle} \subseteq \omega$ for $n < \omega$ such that $q \Vdash$ For each $n < \omega$, $\tilde{h}(n) \in B_{\langle a_0, \dots, a_n \rangle}$. The result follows. □

Definition 2.12. If $g \in {}^{\omega}\omega$ is strictly increasing, then

$$P_g = \Pi_{n<\omega}\mathrm{Sym}(F_n), \text{ where } F_n = g(n) \setminus g(n-1).$$

$\mathbb{P}(U, f)$ was designed so that the following density condition would hold.

Lemma 2.13. *Suppose that $g \in {}^{\omega}\omega$ is strictly increasing and that $p \in \mathbb{P}(U, f)$. Then there exists $J \in [\omega \setminus \mathrm{dom}\, p]^{\omega}$ such that*

$$p \cup \pi \in \mathbb{P}(U, f) \text{ for all } \pi \in P_g^J. \tag{2.14}$$

Proof. We shall make use of the following result.

Claim 2.15. *Suppose that $\{a_n | n < \omega\}$ is the increasing enumeration of $A \in U$. Then for all t, $j < \omega$ there exists $m < \omega$ such that $j < m$ and $a_{m+t} \leq f(a_m)$.*

Proof of Claim 2.15. Let $A \in U$, and suppose that the result fails for some t, $j < \omega$. Then we clearly have that $|A \cap (f(n) \setminus f(n-1))| \leq t$ for all sufficiently large $n < \omega$. But this implies that U is selective for the partition $\{f(n) \setminus f(n-1) | n < \omega\}$, which is a contradiction. □

Let $p \in \mathbb{P}(U, f)$ and let $\{a_n | n < \omega\}$ be the increasing enumeration of $A = \omega \setminus \mathrm{dom}\, p$. Using Claim 2.15, we can construct a sequence of finite subsets of A by induction

$$B_0, C_0, B_1, C_1, \ldots, B_n, C_n, \ldots \qquad n < \omega$$

which satisfies the following conditions.

(2.16)

(a) $\max(B_n) < \min(C_n) \leq \max(C_n) < \min(B_{n+1})$.

(b) $|B_n| = |C_n| = g(n) - g(n-1)$.

(c) $\max(B_n) \leq f(\min(B_n))$.

(d) $\max(C_n) \leq f(\min(C_n))$.

If $\bigcup_{n<\omega} B_n \notin U$, then we can take $J = \bigcup_{n<\omega} B_n$. Otherwise, we can take $J = \bigcup_{n<\omega} C_n$. □

Next we shall explain the manner in which modified Grigorieff forcing will be used in the proof of Theorem 1.7. Assume that the continuum hypothesis holds in the ground model V. Let $S_0 \subseteq \{\alpha < \omega_2 | cf(\alpha) = \omega_1\}$ be a stationary subset of ω_2. Let $d \in {}^{\omega}\omega$ be defined by $d(n) = 2n$. We shall define by induction a countable support iteration of proper forcings $\langle \mathbb{P}_\alpha, \tilde{\mathbb{Q}}_\alpha | \alpha < \omega_2 \rangle$ such that for each $\alpha < \omega_2$, $\mathbb{P}_\alpha \Vdash |\tilde{\mathbb{Q}}_\alpha| = 2^{\omega}$. Then Shelah III 4.1 [11] implies that for each $\alpha < \omega_2$, $\mathbb{P}_\alpha \Vdash CH$. Thus, by Booth [2], we see that for each $\alpha < \omega_2$, $\mathbb{P}_\alpha \Vdash$ There exists a p-point $\tilde{U}_\alpha$ which is not selective for the partition $\{d(n) \setminus d(n-1) | n \in \omega\}$. So we can define $\tilde{\mathbb{Q}}_\alpha = \mathbb{P}(\tilde{U}_\alpha, d)$ for each $\alpha \in S_0$. (We shall specify $\tilde{\mathbb{Q}}_\alpha$ for $\alpha \in \omega_2 \setminus S_0$ in the next section.) Let $H \subseteq \mathbb{P}_{\omega_2}$ be a V-generic filter.

Theorem 2.17. *With the above assumptions, suppose that $V[H] \vDash c(\mathrm{Sym}(\omega)) = \omega_1$. Then the following statement is true in $V[H]$.*

(2.18) It is possible to express $\mathrm{Sym}(\omega) = \bigcup_{i<\omega_1} G_i$ as the union of an increasing chain of proper subgroups such that for each strictly increasing $g \in {}^{\omega}\omega$, there exists $i < \omega_1$ with $P_g \leq G_i$.

Proof. We shall work inside $V[H]$. Suppose that $c(\mathrm{Sym}(\omega)) = \omega_1$. Express $\mathrm{Sym}(\omega) = \bigcup_{i<\omega_1} \Gamma_i$ as the union of an increasing chain of proper subgroups.

Claim 2.19. *For each strictly increasing $g \in {}^\omega\omega$ and moiety A, there exists $i < \omega_1$ such that $P_g^A \leq \Gamma_i(A)$.*

Proof of Claim 2.19. Suppose that the result fails for $g \in {}^\omega\omega$ and A. For each $i < \omega_1$, choose $\pi_i \in P_g$ such that $\pi_i^A \notin \Gamma_i(A)$. Let $\Pi = \langle \pi_i | i < \omega_1 \rangle$.

Then $\Pi^A \not\leq \Gamma_i(A)$ for all $i < \omega_1$. If B is a second moiety, then there exists $\varphi \in \mathrm{Sym}(\omega)$ such that $\varphi \restriction A$ is an order-preserving bijection between A and B. It follows that for all moieties B and all $i < \omega_1$, $\Pi^B \not\leq \Gamma_i(B)$.

For each $\alpha < \omega_2$, let $H_\alpha = H \cap \mathbb{P}_\alpha$. Using Lemma 5.10 [1], we see that there exists an ω_1-closed unbounded set $C \subseteq \omega_2$ such that for all $\alpha \in C$

(2.20) (a) $\Pi, \langle \Gamma_i \cap V[H_\alpha] | i < \omega_1 \rangle \in V[H_\alpha]$;

(b) for each moiety $B \in V[H_\alpha]$ and $i < \omega_1$,
$\Gamma_i(B) \cap V[H_\alpha] = (\Gamma_i \cap V[H_\alpha])(B)$.

Hence there exists $\alpha \in C \cap S_0$. Let $U_\alpha \in V[H_\alpha]$ be the p-point such that $(\tilde{\mathbb{Q}}_\alpha)_{H_\alpha} = \mathbb{P}(U_\alpha, d)$. Since $\Pi^B \not\leq \Gamma_i(B)$ for all moieties B and all $i < \omega_1$, Lemma 2.13 implies that for all $i < \omega_1$, the set

$$D_i = \{p \in \mathbb{P}(U_\alpha, d) | \text{ There exists a moiety } B \text{ and an element } \pi \in \Pi \text{ such that } p \restriction B = \pi^B \notin (\Gamma_i \cap V[H_\alpha])(B)\}$$

is dense in $\mathbb{P}(U_\alpha, d)$. Thus $\mathbb{P}(U_\alpha, d)$ adjoins a permutation ψ such that for all $i < \omega_1$, there exists a moiety $B \in V[H_\alpha]$ and a $\pi \in \Pi$ such that $\psi \restriction B = \pi^B \notin (\Gamma_i \cap V[H_\alpha])(B)$. But this contradicts (2.20)(b). □

Fix a moiety A, and consider $\mathrm{Sym}(A) = \bigcup_{i<\omega_1} \Gamma_i(A)$. Using Lemma 2.4 [9], we see that each $\Gamma_i(A)$ must be a proper subgroup of $\mathrm{Sym}(A)$. So the result follows easily from Claim 2.19. □

We end this section with a group-theoretic result which is also needed in the proof of Theorem 1.7.

Definition 2.21. If $\varphi \in {}^\omega\omega$ is strictly increasing, then

$$S_\varphi = \langle \pi \in \mathrm{Sym}(\omega) | \pi, \pi^{-1} \leq^* \varphi \rangle.$$

Proposition 2.22. *Suppose that* $\mathrm{Sym}(\omega) = \bigcup_{i<\omega_1} G_i$ *is an increasing chain of subgroups such that for each strictly increasing $g \in {}^\omega\omega$, there exists $i < \omega_1$ such that $P_g \leq G_i$. Then for each strictly increasing $\varphi \in {}^\omega\omega$, there exists $i < \omega_1$ such that $S_\varphi \leq G_i$.*

We shall make use of the following result.

Lemma 2.23 (10). *Suppose that $f \in {}^\omega\omega$ is strictly increasing and that $\pi \in$* $\mathrm{Sym}(\omega)$ *satisfies:*

(2.24) *for all $n < \omega$, if $\ell \in f(n)$ then $\pi(\ell)$, $\pi^{-1}(\ell) \in f(n+1)$.*

Then $\pi \in \langle P_{h_0}, P_{h_1} \rangle$, where $h_0, h_1 \in {}^\omega\omega$ are defined by $h_0(n) = f(2n)$ and $h_1(n) = f(2n+1)$. □

Proof of Proposition 2.22. For each $t < \omega$, define $\varphi_t \in {}^\omega\omega$ by $\varphi_t(n) = \varphi(n+t)$, and define $f_t \in {}^\omega\omega$ recursively by

$$\begin{aligned} f_t(0) &= 0 \\ f_t(n+1) &= \text{the least } m > f_t(n) \text{ such that} \\ & \quad\ \varphi_t(\ell) \in m \text{ for all } \ell \in f_t(n). \end{aligned}$$

If π, $\pi^{-1} \leq^* \varphi$, then there exists $t < \omega$ such that $\pi(\ell)$, $\pi^{-1}(\ell) \leq \varphi_t(\ell)$ for all $\ell < \omega$. This implies that if $n < \omega$ and $\ell \in f_t(n)$, then $\pi(\ell)$, $\pi^{-1}(\ell) \leq \varphi_t(\ell) \in f_t(n+1)$. So the result follows from Lemma 2.23. □

3 Products of Mathias forcing

In [3], Canjar proved that if $\underset{\sim}{d} = 2^\omega$, then there exists an ultrafilter U on ω such that its associated Mathias forcing $\mathbb{Q}_U$ does not adjoin a dominating real. We shall use a carefully chosen sequence of such Mathias forcings to eliminate chains of length ω_1 which satisfy the conclusion of Proposition 2.22.

Definition 3.1. Let U be an ultrafilter on ω. The associated Mathias forcing $\mathbb{Q}_U$ consists of all conditions $p = (s, A)$ such that

(i) $s : n \longrightarrow \omega$ is *a* strictly increasing function for some $n < \omega$;
(ii) $A \in U$.
We define $(t, B) \leq (s, A)$ iff
(iii) $t \supseteq s$ and $B \subseteq A$;
(iv) $\operatorname{ran} t \setminus \operatorname{ran} s \subseteq A$.

Suppose that $\tilde{G} \subseteq \mathbb{Q}_U$ is a generic filter, and let $\tilde{g} = \bigcup\{s|$ There exist $A \in U$ such that $(s, A) \in \tilde{G}\}$ be the corresponding Mathias real. For each $A \in [\omega]^\omega$, let $e_A \in {}^\omega\omega$ be the function such that $\{e_A(n)|n < \omega\}$ is the increasing enumeration of A. Then it is easily checked that $e_A \leq^* \tilde{g}$ for all $A \in U$. This suggests that we consider the following subgroups of $\mathrm{Sym}(\omega)$ in the ground model.

Definition 3.2. If $\mathscr{F}$ is a nonprincipal filter on ω, then

$$S_{\mathscr{F}} = \langle \pi \in \mathrm{Sym}(\omega) | \pi, \pi^{-1} \leq^* e_A \textit{ for some } A \in \mathscr{F} \rangle.$$

Example 3.3. $\mathscr{F}$ is said to be *rapid* if for each $f \in {}^\omega\omega$, there exists $A \in \mathscr{F}$ such that $f \leq^* e_A$. Clearly if $\mathscr{F}$ is rapid, then $S_{\mathscr{F}} = \mathrm{Sym}(\omega)$.

Example 3.4. Let V be the ground model. Suppose that $\mathbb{Q}_U$ does not adjoin a dominating real. Thus for all $\tilde{h} \in {}^\omega\omega \cap V^{\mathbb{Q}_U}$, there exists $f \in {}^\omega\omega \cap V$ such that $f \not\leq^* \tilde{h}$. Let $\tilde{g} \in V^{\mathbb{Q}_U} = V[\tilde{G}]$ be the Mathias real which is adjoined by $\mathbb{Q}_U$. For each $1 \leq n < \omega$, let $\tilde{g}_n = \tilde{g} \circ \ldots \circ \tilde{g}$ be the n-fold composition of $\tilde{g}$. Choose $\tilde{h}$ so that $\tilde{g}_n \leq^* \tilde{h}$ for all $n < \omega$. Arguing as in the proof of Claim 1.5, we see that if $\pi \in S_U$ then $\pi \leq^* \tilde{g}_n$ for some $n < \omega$. By assumption, there exists $f \in {}^\omega\omega \cap V$ such that $f \not\leq^* \tilde{h}$. It follows easily that $S_U \neq \mathrm{Sym}(\omega)$ in the ground model V.

The remainder of this section is devoted to the proof of the following result.

Theorem 3.5. (CH) *Suppose that $F \subseteq {}^{\omega}\omega$ is an unbounded family. Then there exist ultrafilters U_0, U_1 on ω such that the following conditions are satisfied.*

(3.6) *(a) For each $i \in \{0,1\}$, $\mathbb{Q}_{U_i} \Vdash F$ remains unbounded.*
(b) $\langle S_{U_0}, S_{U_1}\rangle = \mathrm{Sym}(\omega)$.

Our proof will follow that of Canjar [3] quite closely, but with a few extra ingredients. The following definitions are taken from [3].

Definition 3.7. P denotes the set of finite strictly increasing functions $s : n \to \omega$ for some $n < \omega$. If $A \subseteq \omega$ and $s, t \in P$, then we will write $t < (s, A)$ to indicate that $s \subseteq t$ and $\operatorname{ran} t \setminus \operatorname{ran} s \subseteq A$.

Throughout this section, a name τ in the $\mathbb{Q}_U$-forcing language will be called a *term* iff τ is a $\mathbb{Q}_U$-name for an element of ${}^{\omega}\omega$.

Definition 3.8. A *preterm* is a partial map from $P \times \omega$ into ω. If U is an ultrafilter and τ is a term in the $\mathbb{Q}_U$-forcing language, then the preterm $\hat{\tau}$ for τ is the partial map given by $\hat{\tau}(s, n) = k$ iff there exists $A \in U$ such that $(s, A) \Vdash \tau(n) = k$.

Definition 3.9. A preterm $\hat{\tau}$ is said to be *complete* over a set $A \subseteq \omega$ if whenever $A = \bigcup_{i<k} A_i$ is written as a finite union of sets, then there exists an $i < k$ satisfying the following condition.

(3.10) For all $s \in P$ and $n < \omega$, there exist $t < (s, A_i)$ and $m < \omega$ such that $\hat{\tau}(t, n) = m$.

A preterm $\hat{\tau}$ is complete over a family of sets iff it is complete over every set in the family.

Lemma 3.10 (3). *If U is an ultrafilter and $\hat{\tau}$ is the preterm for some term τ in the $\mathbb{Q}_U$-forcing language, then $\hat{\tau}$ is complete over U, and hence also over every subfamily of U.*

In the rest of this section, we regard $\mathscr{P}(\omega)$ as a topological space by identifying $\mathscr{P}(\omega)$ with the space ${}^{\omega}2$ equipped with the product topology. The ultrafilters U_0 and U_1 will be constructed simultaneously as the unions of ω_1-length chains of filters, each of which is σ-compact. (A subset E of a topological space is σ-compact if E is a countable union of compact sets.) Notice that a filter is σ-compact iff it has a σ-compact generating set.

The next result is a slight variation of Lemma 16 [3].

Lemma 3.11. *Suppose that $F \subseteq {}^{\omega}\omega$ is an unbounded family, $\mathscr{F}$ is a σ-compact filter and $\hat{\tau}$ is a preterm which is complete over $\mathscr{F}$. Then there exists a σ-compact filter $\mathscr{F}' \supseteq \mathscr{F}$ and an $f \in F$ with the property that if $U \supseteq \mathscr{F}'$ is any ultrafilter and τ is any term in the $\mathbb{Q}_U$-forcing language whose preterm is $\hat{\tau}$, then $\mathbb{Q}_U \Vdash f \not\leq^* \tau$.*

Proof. Let $P = \{s_i \mid i < \omega\}$ be a fixed enumeration. For each j, n, $k < \omega$ and $s \in P$, define

$$\begin{aligned}
Thick(s,n,k) &= \{A \subseteq \omega \mid \text{There exists } t < (s,A) \text{ and } \\
&\quad\ i < k \text{ such that } \hat{\tau}(t,n) = i\} \\
Thin(s,n,k) &= \mathscr{P}(\omega) \setminus Thick(s,n,k) \\
Bad(j,k) &= \{B \subseteq \omega \mid \text{There exist } m \leq j \text{ and } i_0,\dots,i_{m-1} < j \\
&\quad\ \text{such that } B = \bigcup_{\ell<m} B_\ell \text{ for some} \\
&\quad\ B_\ell \in Thin(s_{i_\ell}, j, k)\}
\end{aligned}$$

Members of $Thick(s,n,k)$, $Thin(s,n,k)$ and $Bad(j,k)$ will be called $(s,n,k) - Thick$, $(s,n,k) - Thin$ and $(j,k) - Bad$ sets respectively. Notice that $Thick(s,n,k)$ is always open, and hence $Thin(s,n,k)$ and $Bad(j,k)$ are always closed. Also $Thick(s,n,k)$ is closed under superset, and so $Thin(s,n,k)$ is closed under subset.

By Lemma 18 [3], given $j < \omega$ and a compact family $K \subseteq \mathscr{P}(\omega)$ such that $\hat{\tau}$ is complete over K, there exists $k < \omega$ such that no $A \in K$ is $(j,k) - Bad$. Let $\mathscr{F} = \bigcup_{n<\omega} K_n$, where each K_n is compact. For each $n < \omega$, there exists a function $h_n \in {}^\omega\omega$ such that

(3.12) $\qquad$ for all $A \in K_n$ and $j < \omega$, A is not $(j, h_n(j))$-Bad.

Choose $h \in {}^\omega\omega$ such that $h_n \leq^* h$ for all $n < \omega$. Since F is an unbounded family, there exists $f \in F$ and $S \in [\omega]^\omega$ such that $h(n) < f(n)$ for all $n \in S$. Define

$$\begin{aligned}
I[f] &= \{B \subseteq \omega \mid \text{There exists } s \in P \text{ and } m < \omega \text{ such that for all} \\
&\quad\ m < n \in S,\ B \text{ is } (s,n,f(n)) - Thin\}.
\end{aligned}$$

Then $I[f]$ is σ-compact and is closed under subset. We will let $\mathscr{F}'$ be the σ-compact filter generated by $\mathscr{F} \cup \{\omega \setminus B \mid B \in I[f]\}$. First we must check that this family has the Finite Intersection Property. If not, then there exists a set $A \in \mathscr{F}$ and a fin.te subfamily $\{B_k \mid k < n\} \subseteq I[f]$ such that $A = \bigcup_{k<n} B_k$. For each $k < n$, there exists $s_{i_k} \in P$ and $m_k < \omega$ such that B_k is $(s_{i_k}, m, f(m)) - Thin$ for all $m_k < m \in S$. Let $A \in K_r$. Choose $j \in S$ such that $h_r(j) \leq h(j) < f(j)$ and $j > \max(\{n\} \cup \{m_k \mid k < n\} \cup \{i_k \mid k < n\})$. Then each B_k is $(s_{i_k}, j, h_r(j)) - Thin$, and hence A is $(j, h_r(j)) - Bad$. This contradicts (3.12).

Finally suppose that $U \supseteq \mathscr{F}'$ is an ultrafilter and that τ is a term in the $\mathbb{Q}_U$-forcing language whose preterm is $\hat{\tau}$. We check that $\mathbb{Q}_U \Vdash f \not\leq^* \tau$. Suppose that $m < \omega$ and that $(s,E) \in \mathbb{Q}_U$. Then $E \notin I[f]$ and so there exists $m < n \in S$ such that that E is not $(s,n,f(n)) - Thin$. Thus there exists $t < (s,E)$ and $i < f(n)$ such that $\hat{\tau}(t,n) = i$. So for some $B \in U$, $(t,B) \Vdash \tau(n) = i$. Then $(t, B \cap E) \leq (s,E)$ and $(t, B \cap E) \Vdash \tau(n) < f(n)$. Thus $\mathscr{F}'$ and $f \in F$ satisfy our requirements. $\square$

Notice that if U is an ultrafilter on ω and $A \in U$, then $P_{e_A} \leq S_U$.

Lemma 3.13. *Let $\mathscr{F}_0$ and $\mathscr{F}_1$ be σ-compact filters, each of which includes the filter of cofinite sets. If $g \in {}^\omega\omega$ is strictly increasing, then there exist sets $A, B \in [\omega]^\omega$ such that the following conditions are satisfied.*

(3.14) $\qquad$ *(a)* $\quad P_g \leq \langle P_{e_A}, P_{e_B}\rangle$;

$\qquad\qquad$ *(b)* $\quad \{A\} \cup \mathscr{F}_0$ *and* $\{B\} \cup \mathscr{F}_1$ *have the Finite Intersection Property.*

Proof. For each $j \in \{0,1\}$, let $\mathscr{F}_j = \bigcup_{t<\omega} K_t^j$, where each K_t^j is compact. We shall inductively define integers a_n and b_n for $n < \omega$, so that $A = \{a_n | n < \omega\}$ and $B = \{b_n | n < \omega\}$ satisfy our requirements. At the 0^{th} stage of the construction, we set $\ell_0 = 0$ and $b_0 = 0 < a_0 = g(1)$. Suppose that we have just completed the n^{th} stage for some $n \geq 0$.

Case 1. $n = 2t$ is even.

Suppose inductively that we have defined a_i, b_i for $i \leq \ell_n$, and that $b_{\ell_n} < a_{\ell_n}$. If $C \in K_t^0$, then C is infinite and hence there exists $k < \omega$ such that $(k \setminus (a_{\ell_n}+1)) \cap C \neq \phi$. Since K_t^0 is compact, there is a fixed $k < \omega$ such that $(k \setminus (a_{\ell_n}+1)) \cap C \neq \phi$ for all $C \in K_t^0$. Let $k = a_{\ell_n} + r$. Then we set $\ell_{n+1} = \ell_n + r - 1$, and $a_{\ell_n+s} = a_{\ell_n} + s$ for $1 \leq s < r$. Let m be the least integer such that $g(m) > a_{\ell_{n+1}}$. Then we set $b_{\ell_n+s} = g(m+s-1)$ for $1 \leq s < r$. Note that $a_{\ell_{n+1}} < b_{\ell_{n+1}}$.

Case 2. $n = 2t+1$ is odd.

Suppose inductively that we have defined a_i, b_i for $i \leq \ell_n$, and that $a_{\ell_n} < b_{\ell_n}$. We proceed as in Case 1 with the roles of A and B reversed. At this stage, we ensure that $B \cap C \neq \phi$ for each $C \in K_t^1$.

This completes the construction of A and B. It is clear that both $\{A\} \cup \mathscr{F}_0$ and $\{B\} \cup \mathscr{F}_1$ have the Finite Intersection Property. Let m_n, $n < \omega$, be the strictly increasing sequence defined by $g(m_{2t}) = a_{\ell_{2t}}$ and $g(m_{2t+1}) = b_{\ell_{2t+1}}$ for $t < \omega$. Define

$$L = \{n | \text{ There exists } t < \omega \text{ such that } m_{2t} \leq n < m_{2t+1}\}$$

and

$$R = \{n | \text{ There exists } t < \omega \text{ such that } m_{2t-1} \leq n < m_{2t}\}.$$

(Here we set $m_{-1} = 0$.) Then we see that

$$\prod_{n \in L} \text{Sym}(g(n+1) \setminus g(n)) \leq P_{e_B}$$

and

$$\prod_{n \in R} \text{Sym}(g(n+1) \setminus g(n)) \leq P_{e_A}.$$

Thus $P_g \leq \langle P_{e_A}, P_{e_B} \rangle$, as required. □

Proof of Theorem 3.5. (*CH*) Let $\{\hat{\tau}_\alpha | \alpha < \omega_1\}$ be the set of all preterms, and let $\{g_\alpha | \alpha < \omega_1\}$ be the set of all strictly increasing $g \in {}^\omega\omega$. Using Lemmas 3.11 and 3.13, we can construct smooth increasing sequences of σ-compact filters $\langle \mathscr{F}_\alpha^j | \alpha < \omega_1 \rangle$ for $j \in \{0,1\}$ such that the following conditions are satisfied.

(a) $\mathscr{F}_0^0 = \mathscr{F}_0^1$ is the filter of cofinite sets.
(b) There exists $A_\alpha \in \mathscr{F}_{\alpha+1}^0$ and $B_\alpha \in \mathscr{F}_{\alpha+1}^1$ such that $P_{g_\alpha} \leq \langle P_{e_{A_\alpha}}, P_{e_{B_\alpha}} \rangle$.
(c) For each $j \in \{0,1\}$, if $\hat{\tau}_\alpha$ is complete over $\mathscr{F}_\alpha^j$, then there exists $f_\alpha^j \in F$ such that if $U \supseteq \mathscr{F}_{\alpha+1}^j$ is any ultrafilter and τ is any term in the $\mathbb{Q}_U$-forcing language whose preterm is $\hat{\tau}_\alpha$, then $\mathbb{Q}_U \Vdash f_\alpha^j \not\leq^* \tau$.

Finally for $j \in \{0,1\}$, let U_j be any ultrafilter which includes $\bigcup_{\alpha<\omega_1} \mathscr{F}_\alpha^j$. Using Lemma 2.23, we see that U_0 and U_1 satisfy our requirements. □

4 The proof of Theorem 1.7

Throughout this section, V denotes the ground model. We shall make use of the following result of Shelah, which is a special case of Lemma 1.13 [12].

Lemma 4.1. *Suppose that $\langle \mathbb{P}_\alpha, \tilde{\mathbb{Q}}_\alpha | \alpha < \lambda \rangle$ is a countable support iteration of proper forcings of limit length λ. If for each $\alpha < \lambda$,*

$$\mathbb{P}_\alpha \Vdash {}^\omega\omega \cap V \text{ is an unbounded family}$$

then also

$$\mathbb{P}_\lambda \Vdash {}^\omega\omega \cap V \text{ is an unbounded family.}$$

We suppose that the continuum hypothesis holds in V; and that $\{\alpha < \omega_2 | cf(\alpha) = \omega_1\} = S_0 \sqcup S_1$ is a partition into two stationary subsets of ω_2 such that $\diamondsuit_{\omega_2}(S_1)$ holds. We shall define by induction a countable support iteration of proper forcings $\langle \mathbb{P}_\alpha, \tilde{\mathbb{Q}}_\alpha | \alpha < \omega_2 \rangle$ such that for each $\alpha < \omega_2$, $\mathbb{P}_\alpha \Vdash |\tilde{\mathbb{Q}}_\alpha| = 2^\omega$. Remember that this implies that for each $\alpha < \omega_2$, $\mathbb{P}_\alpha \Vdash CH$. We shall also assume inductively that

(4.2) $\qquad \mathbb{P}_\alpha \Vdash {}^\omega\omega \cap V$ is an unbounded family.

Lemma 4.1 deals with requirement (4.2) at limit stages of the construction. Suppose then that $\mathbb{P}_\alpha$ has been defined.

Case 1. $\alpha \notin S_1$.

Then, as described in Sect. 2, we can choose $\tilde{\mathbb{Q}}_\alpha = \mathbb{P}(\tilde{U}_\alpha, d)$ for some p-point $\tilde{U}_\alpha \in V^{\mathbb{P}_\alpha}$. By Theorem 2.6, $\mathbb{P}_\alpha \Vdash \tilde{\mathbb{Q}}_\alpha$ is ${}^\omega\omega$-bounding. Hence $\alpha + 1$ also satisfies (4.2).

Case 2. $\alpha \in S_1$.

Using $\diamondsuit_{\omega_2}(S_1)$, we choose a chain of proper subgroups, $\mathrm{Sym}(\omega) \cap V^{\mathbb{P}_\alpha} = \bigcup_{i<\omega_1} G_i^\alpha$, inside $V^{\mathbb{P}_\alpha}$. By Theorem 3.5, there exists an ultrafilter $\tilde{U}_\alpha \in V^{\mathbb{P}_\alpha}$ such that

(4.3) $\qquad$ (a) $\mathbb{Q}_{\tilde{U}_\alpha} \Vdash {}^\omega\omega \cap V$ remains unbounded;

$\qquad\qquad$ (b) for all $i < \omega_1$, $S_{\tilde{U}_\alpha} \cap V^{\mathbb{P}_\alpha} \not\leq G_i^\alpha$.

We define $\tilde{\mathbb{Q}}_\alpha = \mathbb{Q}_{\tilde{U}_\alpha}$. Again $\alpha + 1$ satisfies (4.2).

This completes the construction of our notion of forcing. Let $H \subseteq \mathbb{P}_{\omega_2}$ be a V-generic filter. By Lemma 4.1, ${}^\omega\omega \cap V$ is an unbounded family in $V[H]$. Hence $V[H] \models \underset{\sim}{b} = \omega_1$. Also a standard argument shows that $V[H] \models 2^\omega = \omega_2$. Thus it only remains to prove that $V[H] \models c(\mathrm{Sym}(\omega)) = \omega_2$.

From now on, we shall work inside $V[H]$. Suppose that $c(\mathrm{Sym}(\omega)) = \omega_1$. By Theorem 2.17 and Proposition 2.22, it is possible to express $\mathrm{Sym}(\omega) = \bigcup_{i<\omega_1} G_i$ as the union of an increasing chain of proper subgroups such that for each strictly increasing $\varphi \in {}^\omega\omega$, there exists an $i < \omega_1$ such that $S_\varphi \leq G_i$. For each $\alpha < \omega_2$, let $H_\alpha = H \cap \mathbb{P}_\alpha$. Using Lemma 5.10 [1], we see that there exists an ω_1-closed unbounded set $C \subseteq \omega_2$ such that for all $\alpha \in C$, $\langle G_i \cap V[H_\alpha] | i < \omega_1 \rangle \in V[H_\alpha]$ is a chain of proper subgroups of $\mathrm{Sym}(\omega) \cap V[H_\alpha]$. By $\diamondsuit_{\omega_2}(S_1)$, we can assume that there exists $\alpha \in S_1 \cap C$ such that $\langle G_i \cap V[H_\alpha] | i < \omega_1 \rangle = \langle G_i^\alpha | i < \omega_1 \rangle$. Let $U_\alpha \in V[H_\alpha]$

be the ultrafilter such that $(\tilde{\mathbb{Q}}_\alpha)_{H_\alpha} = \mathbb{Q}_{U\alpha}$. Then $S_{U_\alpha} \cap V[H_\alpha] \not\leq G_i \cap V[H_\alpha]$ for all $i < \omega_1$. However, $\mathbb{Q}_{U_\alpha}$ adjoins a strictly increasing $\varphi \in {}^\omega\omega$ such that $S_{U_\alpha} \leq S_\varphi$. Now there exists $i < \omega_1$ such that $S_\varphi \leq G_i$, and hence $S_{U_\alpha} \cap V[H_\alpha] \leq G_i \cap V[H_\alpha]$. This is a contradiction, and so we must have that $c(\mathrm{Sym}(\omega)) = \omega_2$. This completes the proof of Theorem 1.7.

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