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Lie symmetries and conserved quantities of constrained mechanical systems

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Summary. The Lie symmetries and conserved quantities of constrained mechanical systems are studied. Using the invariance of the ordinary differential equations under the infinitesimal transformations, the determining equations and the restriction equations of the Lie symmetries of the systems are established. The structure equation and the form of conserved quantities are obtained. We find the corresponding conserved quantity from a known Lie symmetry, that is a direct problem of the Lie symmetries. And then, the inverse problem of the Lie symmetries – finding the corresponding Lie symmetry from a known conserved quantity – is studied. Finally, the relation between the Lie symmetry and the Noether symmetry is given.

1 Introduction

The researches on the conserved quantities of mechanical systems not only have important mathematical significance, but also have profound physical background. There are two modern methods to find the conserved quantities, that is the Noether symmetry method and Lie symmetry method. The Noether method is making good progress [1]–[4]. Since the late seventies, the study on the Lie symmetries of mechanical systems had some results [5]–[6]. In this work, we study the Lie symmetries and conserved quantities of constrained mechanical systems, including the holonomic systems with remainder coordinates, the non-holonomic systems of Chetaev type and non-Chetaev type.

2 Holonomic systems with remainder coordinates

2.1 *Direct problem*

Let the position of a mechanical system be determined by n generalized coordinates $q^s (s = 1, \dots, n)$. For some needs, we introduce m remainder coordinates $q^{n+\gamma} (\gamma = 1, \dots, m)$ and have m ideal holonomic constraints

$$f^\beta(t, q^u) = 0 \qquad (\beta = 1, \dots, m; u = 1, \dots, n+m). \tag{2.1}$$

The restriction of constraints (2.1) on the virtual displacements is

$$\frac{\partial f^\beta}{\partial q^u}\, \delta q^u = 0. \tag{2.2}$$

From the d'Alembert-Lagrange principle and the formula (2.2), using the method of Lagrange multiplier, we can obtain the equations of motion of the system,

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{q}^u} - \frac{\partial T}{\partial q^u} = Q_u + \lambda_\beta \frac{\partial f^\beta}{\partial q^u} \tag{2.3}$$

or

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}^u} - \frac{\partial L}{\partial q^u} = Q_u'' + \lambda_\beta \frac{\partial f^\beta}{\partial q^u}, \tag{2.4}$$

where T is the kinetic energy, L is Lagrangian, Q_u the generalized forces, Q_u'' the generalized non-potential forces, λ_β the constraint multipliers. From Eqs. (2.1) and (2.4) we can determine λ_β as the functions of t, $\boldsymbol{q}$, $\dot{\boldsymbol{q}}$,

$$\lambda_\beta = \lambda_\beta(t, \boldsymbol{q}, \dot{\boldsymbol{q}}). \tag{2.5}$$

Substituting this into Eqs. (2.4) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}^u} - \frac{\partial L}{\partial q^u} = Q_u'' + \Lambda_u,$$

$$\Lambda_u = \Lambda_u(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) = \lambda_\beta \frac{\partial f^\beta}{\partial q^u}. \tag{2.6}$$

Let

$$\det\left(\frac{\partial^2 L}{\partial \dot{q}^u \partial \dot{q}^v}\right) \neq 0. \tag{2.7}$$

Expanding Eqs. (2.6), we can determine all accelerations as

$$\ddot{q}^u = \alpha^u(t, \boldsymbol{q}, \dot{\boldsymbol{q}}). \tag{2.8}$$

Equations (2.8) are called the equations of the holonomic system corresponding to the system (2.1), (2.4) with remainder coordinates. When the initial conditions satisfy Eqs. (2.1), the solution of Eqs. (2.8) gives the motion of the system.

Introducing the infinitesimal transformations

$$t^* = t + \Delta t, \qquad q^{u*}(t + \Delta t) = q^u(t) + \Delta q^u \tag{2.9}$$

or their expanded form

$$t^* = t + \varepsilon\xi^0(t, \boldsymbol{q}, \dot{\boldsymbol{q}}), \qquad q^{u*} = q^u + \varepsilon\xi^u(t, \boldsymbol{q}, \dot{\boldsymbol{q}}), \tag{2.10}$$

taking the infinitesimal generator

$$X^{(0)} = \xi^0 \frac{\partial}{\partial t} + \xi^u \frac{\partial}{\partial q^u} \tag{2.11}$$

and its first extended vector

$$X^{(1)} = X^{(0)} + (\dot{\xi}^u - \dot{q}^u \dot{\xi}^0)\frac{\partial}{\partial \dot{q}^u} \tag{2.12}$$

by using the invariance of the ordinary differential equations under the infinitesimal transformations [7], the invariance of Eqs. (2.8) under the infinitesimal transformations (2.10) leads to the satisfaction of the following determining equations:

$$\ddot{\xi}^u - \dot{q}^u \ddot{\xi}^0 - 2\dot{\xi}^0 \alpha^u = X^{(1)}(\alpha^u)\,, \tag{2.13}$$

and the invariance of the constraint equations (2.1) under the infinitesimal transformations (2.10) leads to the satisfaction of the following restriction equations:

$$X^{(0)}\big(f^\beta(t,\boldsymbol{q})\big) = 0\,. \tag{2.14}$$

If the generator ξ^0, ξ^u of the infinitesimal transformations satisfies the determining equations (2.13) and the restriction equations (2.14), then the corresponding transformations are called the Lie symmetrical transformations of the system (2.1), (2.4) with remainder coordinates; if the determining equations (2.13) are only satisfied, then the transformations are called the Lie symmetrical transformations of the corresponding holonomic system (2.8).

The Lie symmetry can lead to a conserved quantity under certain conditions. We have:
Proposition 1. For the infinitesimal generator ξ^0, ξ^u satisfying the determining equations (2.13) and the restriction equations (2.14), if there exists a gauge function $G = G(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ satisfying the structure equation

$$L\dot{\xi}^0 + X^{(1)}(L) + (Q_u{}'' + \Lambda_u)\,(\xi^u - \dot{q}^u \xi^0) + \dot{G} = 0\,, \tag{2.15}$$

then the holonomic system with remainder coordinates has the following conserved quantity:

$$I = L\xi^0 + \frac{\partial L}{\partial \dot{q}^u}(\xi^u - \dot{q}^u \xi^0) + G = \text{const.} \tag{2.16}$$

Proof.

$$\begin{aligned}\frac{\mathrm{d}I}{\mathrm{d}t} &= \dot{L}\xi^0 + L\dot{\xi}^0 + \frac{\partial L}{\partial \dot{q}^u}(\dot{\xi}^u - \ddot{q}^u \xi^0 - \dot{q}^u \dot{\xi}^0) + (\xi^u - \dot{q}^u \xi^0)\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}^u} \\ &\quad - L\dot{\xi}_0 - X^{(1)}(L) - (Q_u{}'' + \Lambda_u)\,(\xi^u - \dot{q}^u \xi^0) \\ &= (\xi^u - \dot{q}^u \xi^0)\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}^u} - \frac{\partial L}{\partial q^u} - Q_u{}'' - \Lambda_u\right) = 0\,.\end{aligned}$$

If there are no remainder coordinates, Proposition 1 becomes a result in [6]; and then, if there are no generalized non-potential forces, Proposition 1 becomes a result in [5].

The method of solution of the direct problem of the Lie symmetries is the following: firstly, establish the determining equations (2.13) and restriction equations (2.14) of the Lie symmetries for a given holonomic system with remainder coordinates and seek the generator ξ^0, ξ^u from these equations; secondly, substitute the generator obtained into the structure equation (2.15) to determine G; finally, substitute ξ^0, ξ^u and G into the formula (2.16) to obtain the conserved quantities of the Lie symmetries.

2.2 *Inverse problem*

Suppose that the system has an integral

$$I = I(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) = \text{const.} \tag{2.17}$$

Let us seek the corresponding Lie symmetry. Differentiating this with respect to t, we have

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \frac{\partial I}{\partial t} + \frac{\partial I}{\partial q^u}\dot{q}^u + \frac{\partial I}{\partial \dot{q}^u}\ddot{q}^u = 0\,. \tag{2.18}$$

Multiplying Eqs. (2.6) by

$$\bar{\xi}^u = \xi^u - \dot{q}^u \xi^0 \tag{2.19}$$

and taking the summation for u, we obtain

$$\bar{\xi}^u \left(\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{q}^u} - \frac{\partial L}{\partial q^u} - Q_u{}'' - \Lambda_u \right) = 0 . \tag{2.20}$$

Adding the formulae (2.18) and (2.20), separating the terms containing $\ddot{q}^v$ and taking their coefficients as zeros, we obtain

$$-\frac{\partial I}{\partial \dot{q}^v} + \frac{\partial^2 L}{\partial \dot{q}^u \partial \dot{q}^v} \bar{\xi}^u = 0 \qquad (v = 1, \ldots, n+m) . \tag{2.21}$$

From this we can determine $\bar{\xi}^u$ as

$$\bar{\xi}^u = \tilde{\omega}^{uv} \frac{\partial I}{\partial \dot{q}^v} , \tag{2.22}$$

where

$$\tilde{\omega}^{uv} \omega_{vw} = \delta^u_w , \qquad \omega_{uv} = \frac{\partial^2 L}{\partial \dot{q}^u \partial \dot{q}^v} . \tag{2.23}$$

Let the integral (2.17) be equal to the conserved quantity (2.16), i.e.,

$$L\xi^0 + \frac{\partial L}{\partial \dot{q}^u} \bar{\xi}^u + G = I . \tag{2.24}$$

Thus we can obtain the infinitesimal generator ξ^0, ξ^u from Eqs. (2.22) and (2.24) when the gauge function G is given. We have:

Proposition 2. If the generator ξ^0, ξ^u obtained from Eqs. (2.22) and (2.24) satisfies the determining equations (2.13) and the restriction equations (2.14), then the transformations (2.10) are Lie symmetrical transformations of the system (2.1), (2.4). If Eqs. (2.13) are only satisfied, then the transformations are Lie symmetrical transformations of the system (2.8).

2.3 Lie symmetry and Noether symmetry

If the transformations (2.9) satisfy Noether's identity

$$\frac{\partial L}{\partial t} \Delta t + \frac{\partial L}{\partial q^u} \Delta q^u + \frac{\partial L}{\partial \dot{q}^u} \Delta \dot{q}^u + L \frac{\mathrm{d}}{\mathrm{d}t}(\Delta t) + (Q_u{}'' + \Lambda_u)(\Delta q^u - \dot{q}^u \Delta t) = -\frac{\mathrm{d}}{\mathrm{d}t}(\Delta G) \tag{2.25}$$

and the restriction conditions

$$\frac{\partial f^\beta}{\partial q^u} (\xi^u - \dot{q}^u \xi^0) = 0 , \tag{2.26}$$

then the transformations (2.9) are called the Noether symmetrical transformations of the holonomic system with remainder coordinates (2.1), (2.4). We have:

Proposition 3. The structure equation (2.15) of the Lie symmetry is equivalent to Noether's identity (2.25).

Proposition 4. The restriction equations (2.14) of the Lie symmetry are equivalent to the restriction conditions (2.26).

Proposition 5. For the holonomic system (2.1), (2.4) with remainder coordinates, when the transformations (2.10) are Lie symmetrical, if and only if $\dot{G}$ is zero or the exact derivation of a function, then the transformations are Noether symmetrical.

Proposition 6. For the holonomic system (2.1), (2.4) with remainder coordinates, when the transformations (2.10) are Noether symmetrical, if and only if the generator satisfies the determining equations (2.13), then the transformations are Lie symmetrical.

2.4 Illustrative example

Let us consider a system whose Lagrangian is

$$L = \frac{1}{2}\{(\dot{q}^1)^2 + (\dot{q}^2)^2 + \omega^2(q^1)^2\} - mgq^2, \tag{2.27}$$

the generalized non-potential forces are

$$Q_1'' = -m\dot{q}^2, \qquad Q_2'' = m\dot{q}^1 \tag{2.28}$$

and the constraint equation is

$$f = q^2 - a(q^1)^2. \tag{2.29}$$

Equations (2.8) give

$$\begin{aligned} \ddot{q}^1 &= \frac{q^1}{1+4a^2(q^1)^2}[\omega^2 - 4a^2(\dot{q}^1)^2 - 2ag] = \alpha^1, \\ \ddot{q}^2 &= \frac{2a}{1+4a^2(q^1)^2}[\omega^2(q^1)^2 + (\dot{q}^1)^2 - 2ag(q^1)^2] = \alpha^2. \end{aligned} \tag{2.30}$$

Equations (2.13) and (2.14) give

$$\ddot{\xi}^1 - \dot{q}^1\ddot{\xi}^0 - 2\dot{\xi}^0\alpha^1 = X^{(1)}(\alpha^1), \qquad \ddot{\xi}^2 - \dot{q}^2\ddot{\xi}^0 - 2\dot{\xi}^0\alpha^2 = X^{(1)}(\alpha^2), \tag{2.31}$$

$$\xi^2 - 2aq^1\xi^1 = 0. \tag{2.32}$$

Equations (2.31) and (2.32) have the following solution:

$$\xi^0 = -1, \qquad \xi^1 = \dot{q}^1, \qquad \xi^2 = \dot{q}^2. \tag{2.33}$$

Substituting the generator (2.33) into the structure equation (2.15), we obtain the gauge function

$$G = -\frac{1}{2}m[(\dot{q}^1)^2 + (\dot{q}^2)^2] - \frac{1}{2}m\omega^2(q^1)^2 + mgq^2. \tag{2.34}$$

Substituting (2.33) and (2.34) into the formula (2.16), we obtain the conserved quantity

$$I = m[(\dot{q}^1)^2 + (\dot{q}^2)^2 - \omega^2(q^1)^2] + 2mgq^2 = \text{const.} \tag{2.35}$$

From the Proposition 5, we know that the transformations are also Noether symmetrical.

3 Nonholonomic systems of Chetaev type

3.1 Direct problem

Let the position of a mechanical system be determined by n generalized coordinates $q^s (s = 1, \ldots, n)$. Its motion is subjected to the ideal non-holonomic constraints of Chetaev type

$$f^\beta(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) = 0 \qquad (\beta = 1, \ldots, g). \tag{3.1}$$

The restriction of condition (3.1) on the virtual displacements is

$$\frac{\partial f^\beta}{\partial \dot{q}^s} \delta q^s = 0. \tag{3.2}$$

The equations of motion can be written in the form:

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{q}^s} - \frac{\partial L}{\partial q^s} = Q_s'' + \lambda_\beta \frac{\partial f^\beta}{\partial \dot{q}^s}. \tag{3.3}$$

Before integrating the equations of motion, we can determine λ_β as functions of $t, \boldsymbol{q}, \dot{\boldsymbol{q}}$ [9]

$$\lambda_\beta = \lambda_\beta(t, \boldsymbol{q}, \dot{\boldsymbol{q}}). \tag{3.4}$$

Substituting this into Eqs. (3.3), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{q}^s} - \frac{\partial L}{\partial q^s} = Q_s'' + \Lambda_s',$$

$$\Lambda_s' = \Lambda_s'(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) = \lambda_\beta \frac{\partial f^\beta}{\partial \dot{q}^s}. \tag{3.5}$$

Let

$$\det\left(\frac{\partial^2 L}{\partial \dot{q}^s \partial \dot{q}^k}\right) \neq 0. \tag{3.6}$$

Expanding Eqs. (3.4), we can obtain all accelerations as

$$\ddot{q}^s = \beta^s(t, \boldsymbol{q}, \dot{\boldsymbol{q}}). \tag{3.7}$$

Equations (3.7) are called the equations of motion of holonomic system corresponding to the non-holonomic system (3.1), (3.3). If the initial conditions satisfy Eqs. (3.1), the solution of Eqs. (3.7) gives the motion of non-holonomic system.

The invariance of Eqs. (3.7) and (3.1) under the infinitesimal transformations (2.10) $(u = 1, \ldots, n)$ leads to the satisfaction of the following determining equations:

$$\ddot{\xi}^s - \dot{q}^s \ddot{\xi}^0 - 2\dot{\xi}^0 \beta^s = X^{(1)}(\beta^s) \tag{3.8}$$

and the restriction equations

$$X^{(1)}\big(f^\beta(t, \boldsymbol{q}, \dot{\boldsymbol{q}})\big) = 0. \tag{3.9}$$

If the generator ξ^0, ξ^s satisfies Eqs. (3.8) and (3.9), then the corresponding transformations are called the Lie symmetrical transformations of the non-holonomic system (3.1), (3.3); if the determining equations (3.8) are only satisfied, then the transformations are called the Lie symmetrical transformations of the corresponding holonomic system (3.7).

The Lie symmetry can lead to a conserved quantity under certain conditions. We have:

Proposition 7. For the generator ξ^0, ξ^s satisfying the determining equations (3.8) and the restriction equations (3.9), if there exists a gauge function $G = G(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ satisfying the structure equation

$$L\dot{\xi}^0 + X^{(1)}(L) + (Q_s'' + \Lambda_s')\,(\xi^s - \dot{q}^s\xi^0) + \dot{G} = 0\,, \tag{3.10}$$

then the non-holonomic system of Chetaev type has the following conserved quantity:

$$I = L\xi^0 + \frac{\partial L}{\partial \dot{q}^s}(\xi^s - \dot{q}^s\xi^0) + G = \text{const.} \tag{3.11}$$

The proof of Proposition 7 is similar to that of Proposition 1.

If there are no non-holonomic constraints, Proposition 7 gives a result in [6].

The method of solution of the direct problem of the Lie symmetries is the following: Firstly, establish the determining equations (3.8) and restriction equations (3.9) of the Lie symmetries for a given non-holonomic system and determine the generator ξ^0, ξ^s from these equations; Secondly, substitute the generator into the structure equation (3.10) to determine G; Finally, substitute ξ^0, ξ^s and G into the formula (3.11) to obtain the conserved quantities of the Lie symmetries.

3.2 *Inverse problem*

Using a similar method in 2.2, we have

$$\bar{\xi}^s = \bar{\omega}^{sk}\,\frac{\partial I}{\partial \dot{q}^k}\,, \tag{3.12}$$

$$L\xi^0 + \frac{\partial L}{\partial \dot{q}^s}\,\bar{\xi}^s + G = I\,. \tag{3.13}$$

Proposition 8. If the generator ξ^0, ξ^s obtained from Eqs. (3.12) and (3.13) satisfies the determining equations (3.8) and the restriction equations (3.9), then the transformations are Lie symmetrical transformations of the non-holonomic system (3.1), (3.3). If Eqs. (3.8) are only satisfied, then the transformations are Lie symmetrical transformations of the corresponding holonomic system (3.7).

3.3 *Lie symmetry and Noether symmetry*

If the transformations satisfy Noether's identity

$$\frac{\partial L}{\partial t}\Delta t + \frac{\partial L}{\partial q^s}\Delta q^s + \frac{\partial L}{\partial \dot{q}^s}\Delta\dot{q}^s + L\,\frac{\mathrm{d}}{\mathrm{d}t}(\Delta t) + (Q_s'' + \Lambda_s')\,(\Delta q^s - \dot{q}^s\Delta t) = \frac{\mathrm{d}}{\mathrm{d}t}(\Delta G) \tag{3.14}$$

and the restriction conditions

$$\frac{\partial f^\beta}{\partial \dot{q}^s}(\xi^s - \dot{q}^s\xi^0) = 0\,, \tag{3.15}$$

then the transformations are called the generalized Noether quasi-symmetrical transformations of the non-holonomic system (3.1), (3.3) [4]. We have:

Proposition 9. The structure equation (3.10) is equivalent to Noether's identity (3.14).

Proposition 10. For the non-holonomic system (3.1), (3.3), when the transformations are symmetrical, if and only if $\dot{G}$ is zero or the exact derivation of a function, and the transformations satisfy the restriction conditions (3.15), then the transformations are generalized Noether quasi-symmetrical.

Proposition 11. For the non-holonomic system (3.1), (3.3), when the transformations are generalized Noether quasi-symmetrical, if and only if the generator ξ^0, ξ^s satisfies the determining equations (3.8) and the restriction equations (3.9), then the transformations are Lie symmetrical.

3.4 Illustrative example

In the Appell-Hamel's example [8], [9], the Lagrangian and the constraint equation are, respectively,

$$L = \frac{1}{2}m[(\dot{q}^1)^2 + (\dot{q}^2)^2 + (\dot{q}^3)^2] - mgq^3, \tag{3.16}$$

$$f = (\dot{q}^1) + (\dot{q}^2)^2 - (\dot{q}^3)^2 = 0. \tag{3.17}$$

Firstly, we study the direct problem. Eqs. (3.7) give

$$m\ddot{q}^1 = -\frac{1}{2}\frac{mg\dot{q}^1}{\dot{q}^3}, \qquad m\ddot{q}^2 = -\frac{1}{2}\frac{mg\dot{q}^2}{\dot{q}^3}, \qquad m\ddot{q}^3 + mg = \frac{1}{2}mg. \tag{3.18}$$

Equations (3.8) and Eqs. (3.9) give, respectively,

$$\begin{aligned} &\ddot{\xi}^1 - \dot{q}^1\ddot{\xi}^0 - 2\dot{\xi}^0\left(-\frac{1}{2}\,g\,\frac{\dot{q}^1}{\dot{q}^3}\right) = (\dot{\xi}^1 - \dot{q}^1\dot{\xi}^0)\left(-\frac{1}{2}\,g\,\frac{1}{\dot{q}^3}\right) + (\dot{\xi}^3 - \dot{q}^3\dot{\xi}^0)\left(\frac{1}{2}\,g\,\frac{\dot{q}^1}{(\dot{q}^3)^2}\right), \\ &\ddot{\xi}^2 - \dot{q}^2\ddot{\xi}^0 - 2\dot{\xi}^0\left(-\frac{1}{2}\,g\,\frac{\dot{q}^2}{\dot{q}^3}\right) = (\dot{\xi}^2 - \dot{q}^2\dot{\xi}^0)\left(-\frac{1}{2}\,g\,\frac{1}{\dot{q}^3}\right) + (\dot{\xi}^3 - \dot{q}^3\dot{\xi}^0)\left(\frac{1}{2}\,g\,\frac{\dot{q}^2}{(\dot{q}^3)^2}\right), \end{aligned} \tag{3.19}$$

$$\ddot{\xi}^3 - \dot{q}^3\ddot{\xi}^0 - 2\dot{\xi}^0\left(-\frac{1}{2}\,g\right) = 0, \qquad (\dot{\xi}^1 - \dot{q}^1\dot{\xi}^0)\,\dot{q}^1 + (\dot{\xi}^2 - \dot{q}^2\dot{\xi}^0)\,\dot{q}^2 - (\dot{\xi}^3 - \dot{q}^3\dot{\xi}^0)\,\dot{q}^3 = 0. \tag{3.20}$$

We can obtain the following solutions of Eqs. (3.19) and Eqs. (3.20):

$$\xi^0 = -1, \qquad \xi^s = 0 \qquad (s = 1,2,3), \tag{3.21}$$

$$\xi^0 = 0, \qquad \xi^1 = 0, \qquad \xi^2 = 0, \qquad \xi^3 = 1. \tag{3.22}$$

Substituting these generators into the structure equation (3.10), we obtain, respectively,

$$G = 0, \tag{3.23}$$

$$G = \frac{1}{2}\,mgt. \tag{3.24}$$

Substituting the generators (3.21), (3.22) and the gauge functions (3.23), (3.24) into the formula (3.11), we obtain the following conserved quantities:

$$I = \frac{1}{2}\,m[(\dot{q}^1)^2 + (\dot{q}^2)^2 + (\dot{q}^3)^2] + mgq^3 = \text{const.}, \tag{3.25}$$

$$I = m\dot{q}^3 + \frac{1}{2}\,mgt = \text{const.}. \tag{3.26}$$

And then we study the inverse problem. Suppose that the system possesses an integral (3.26), let us seek the corresponding Lie symmetry. Eqs. (3.12) and (3.13) give

$$\bar{\xi}^1 = 0, \qquad \bar{\xi}^2 = 0, \qquad \bar{\xi}^3 = 1,$$
$$L\xi^0 + G = \frac{1}{2}\, gt. \tag{3.27}$$

It is easy to verify that the generator determined by (3.27) satisfies Eqs. (3.19) and (3.20), therefore, the transformations are the Lie symmetry of the system (3.16), (3.17).

Finally, we study the relation between the Lie symmetry and the Noether symmetry.

The generator (3.21) satisfies the restriction (3.15); in view of Proposition 10, we know that the transformations are generalized Noether quasi-symmetrical transformations. The generator (3.22) does not satisfy the restriction (3.15); in view of Proposition 10, we know that the transformations are not generalized Noether quasi-symmetrical transformations. For the generator

$$\xi^0 = 0, \qquad \xi^1 = -\frac{1}{\dot{q}^2}, \qquad \xi^2 = \frac{\dot{q}^1}{(\dot{q}^2)^2}, \qquad \xi^3 = 0, \tag{3.28}$$

the identity (3.14) gives

$$G = -m\,\frac{\dot{q}^1}{\dot{q}^2}, \tag{3.29}$$

and the restriction conditions (3.15) are satisfied. Therefore, the transformations corresponding to the generator (3.28) are generalized Noether quasi-symmetrical transformations. It is easy to verify that the generator (3.28) does not satisfy the determining equations (3.19). From Proposition 11, we know that the transformations are not Lie symmetrical.

4 Non-holonomic systems of non-Chetaev type

4.1 Direct problem

Let the position of a mechanical system be determined by n generalized coordinates $q^s (s = 1, \ldots, n)$. Its motion is subjected to the ideal non-holonomic constraints of non-Cheataev type

$$F^\beta(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) = 0 \qquad (\beta = 1, \ldots, g). \tag{4.1}$$

Suppose that the restriction of constraints (4.1) on the virtual displacements is

$$F^\beta_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}})\, \delta q^s = 0. \tag{4.2}$$

In the general case

$$F^\beta_s \neq \frac{\partial F^\beta}{\partial \dot{q}^s}. \tag{4.3}$$

The equations of motion can be written in the form [10]

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}^s} - \frac{\partial L}{\partial q^s} = Q_s'' + \lambda_\beta F_s{}^\beta. \tag{4.4}$$

Let

$$\det\left(\frac{\partial^2 L}{\partial \dot{q}^s \partial \dot{q}^k}\right) \neq 0. \tag{4.5}$$

Before integrating the equations of motion, we can determine λ_β as function of $t, \boldsymbol{q}, \dot{\boldsymbol{q}}$,

$$\lambda_\beta = \lambda_\beta(t, \boldsymbol{q}, \dot{\boldsymbol{q}}). \tag{4.6}$$

Substituting this into Eqs. (4.4), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}^s} - \frac{\partial L}{\partial q^s} = Q_s'' + \Lambda_s'', \qquad \Lambda_s'' = \Lambda_s''(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) = \lambda_\beta F_s{}^\beta. \tag{4.7}$$

Expanding Eqs. (4.7), we can seek all accelerations as

$$\ddot{q}^s = \gamma^s(t, \boldsymbol{q}, \dot{\boldsymbol{q}}). \tag{4.8}$$

Equations (4.8) are called the equations of motion of the holonomic system corresponding to the non-holonomic system (4.1), (4.4). If the initial conditions satisfy Eqs. (4.1), the solution of Eqs. (4.8) gives the motion of the non-holonomic system.

Introducing the infinitesimal transformations (2.10) ($u = 1, \ldots, n$), the infinitesimal generator (2.11) and its first extended vector (2.12), the invariance of Eqs. (4.8) and Eqs. (4.1) under the infinitesimal transformations leads to the satisfaction of the following determining equations:

$$\ddot{\xi}^s - \dot{q}^s\ddot{\xi}^0 - 2\dot{\xi}^0\gamma^s = X^{(1)}(\gamma^s) \tag{4.9}$$

and the restriction equations

$$X^{(1)}\left(F^\beta(t, \boldsymbol{q}, \dot{\boldsymbol{q}})\right) = 0. \tag{4.10}$$

If the generator ξ^0, ξ^s satisfies Eqs. (4.9) and (4.10), corresponding transformations are called the Lie symmetrical transformations of a non-holonomic system of non-Chetaev type (4.1), (4.4); if the determining equations (4.9) are only satisfied, then the transformations are called the Lie symmetrical transformations of the corresponding holonomic systems (4.8).

The Lie symmetry can lead to a conserved quantity under certain conditions. We have:

Proposition 12. For the generator ξ^0, ξ^s satisfying the determining equations (4.9) and the restriction equations (4.10), if there exists a gauge function $G = G(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ satisfying the structure equation

$$L\dot{\xi}^0 + X^{(1)}(L) + (Q_s'' + \Lambda_s'')(\xi^s - \dot{q}^s\xi^0) + \dot{G} = 0, \tag{4.11}$$

the non-holonomic system of non-Chetaev type has the following conserved quantity:

$$I = L\xi^0 + \frac{\partial L}{\partial \dot{q}^s}(\xi^s - \dot{q}^s\xi^0) + G = \text{const.} \tag{4.12}$$

The proof of Proposition 12 is similar to that of Proposition 1. If the constraints are Chetaev type, Proposition 12 becomes Proposition 7.

The method of solution of the direct problem of the Lie symmetries for the non-holonomic system of non-Chetaev type is the following: firstly, establish the determining equations (4.9) and the restriction equations (4.10) of the Lie symmetries for a given non-holonomic system and determine the generator ξ^0, ξ^s from these equations; secondly, substitute the generator into the structure equation (4.11) to determine G; finally, substitute ξ^0, ξ^s and G into the formula (4.12) to obtain the conserved quantities of the Lie symmetries.

4.2 *Inverse problem*

Using a similar method in 2.2, we have

$$\bar{\xi}^s = \bar{\omega}^{sk} \frac{\partial I}{\partial \dot{q}^k}, \tag{4.13}$$

$$L\xi^0 + \frac{\partial L}{\partial \dot{q}^s} \bar{\xi}^s + G = I. \tag{4.14}$$

Proposition 13. If the generator ξ^0, ξ^s obtained from Eqs. (4.13) and (4.14) satisfies the determining equations (4.9) and the restriction equations (4.10), then the transformations are Lie symmetrical transformations of the non-holonomic system (4.1), (4.4) of non-Chetaev type. If Eqs. (4.9) are only satisfied, then the transformations are Lie symmetrical transformations of the corresponding holonomic system (4.8).

4.3 *Lie symmetry and Noether symmetry*

If the transformations satisfy Noether's identity

$$\frac{\partial L}{\partial t} \Delta t + \frac{\partial L}{\partial q^s} \Delta q^s + \frac{\partial L}{\partial \dot{q}^s} \Delta q^s + L \frac{\mathrm{d}}{\mathrm{d}t} (\Delta t) + (Q_s'' + \Lambda_s'')(\Delta q^s - \dot{q}^s \Delta t) = -\frac{\mathrm{d}}{\mathrm{d}t} (\Delta G) \tag{4.15}$$

and the restriction conditions

$$F_s^\beta(\xi^s - \dot{q}^s \xi^0) = 0, \tag{4.16}$$

then the transformations are called the generalized Noether quasi-symmetrical transformations of the non-holonomic system (4.1), (4.4).

Proposition 14. The structure equation (4.11) is equivalent to Noether's identity (4.15).

Proposition 15. For the non-holonomic system (4.1), (4.4), when the transformations are Lie symmetrical, if and only if $\dot{G}$ is zero or the exact derivation of a function, and the transformations satisfy the restriction condition (4.16), then the transformations are generalized Noether quasi-symmetrical.

Proposition 16. For the non-holonomic system (4.1), (4.4), when the transformations are generalized Noether symmetrical, if and only if the generator ξ^0, ξ^s satisfies the determining equations (4.9) and the restriction equations (4.10), then the transformations are Lie symmetrical.

4.4 *Illustrative example*

Suppose that the Lagrangian of a system is

$$L = \frac{1}{2}[(\dot{q}^1)^2 + (\dot{q}^2)^2] - q^2, \tag{4.17}$$

the constraint equation is

$$F = \dot{q}^2 - t\dot{q}^1 = 0, \tag{4.18}$$

and the equation of virtual displacements is

$$\delta q^1 - \delta q^2 = 0 . \tag{4.19}$$

Firstly, we study the direct problem of the Lie symmetries of the system. The equations of motion of the corresponding holonomic system are

$$\ddot{q}^1 = -\frac{\dot{q}^1 + 1}{t+1}, \qquad \ddot{q}^2 = \frac{\dot{q}^1 - t}{t+1} . \tag{4.20}$$

The determining equations (4.9) and the restriction equations (4.10) give, respectively,

$$\begin{aligned} &\ddot{\xi}^1 - \dot{q}^1\ddot{\xi}^0 - 2\dot{\xi}^0\left(-\frac{\dot{q}^1+1}{t+1}\right) = X^{(1)}\left(-\frac{\dot{q}^1+1}{t+1}\right), \\ &\ddot{\xi}^2 - \dot{q}^2\ddot{\xi}^0 - 2\dot{\xi}^0\left(\frac{\dot{q}^1-1}{t+1}\right) = X^{(1)}\left(\frac{\dot{q}^1-t}{t+1}\right), \end{aligned} \tag{4.21}$$

$$\dot{\xi}^2 - \dot{q}^2\dot{\xi}^0 - t(\dot{\xi}^1 - \dot{q}^1\dot{\xi}^0) = 0 . \tag{4.22}$$

We can get the following solutions of Eqs. (4.21) and (4.22):

$$\xi^0 = 0, \qquad \xi^1 = 1, \qquad \xi^2 = 0, \tag{4.23}$$

$$\xi^0 = 0, \qquad \xi^1 = 0, \qquad \xi^2 = 1, \tag{4.24}$$

$$\xi^0 = 0, \qquad \xi^1 = \xi^2 = 1, \tag{4.25}$$

$$\xi^0 = 0, \qquad \xi^1 = \ln(1+t), \qquad \xi^2 = t - \ln(1+t) . \tag{4.26}$$

Substituting the generator (4.25) into the structure equation (4.11), we obtain

$$G = t . \tag{4.27}$$

Substituting the formulae (4.25), (4.27) into the formula (4.12), we obtain the corresponding conserved quantity

$$I = \dot{q}^1 + \dot{q}^2 + t = \text{const.} \tag{4.28}$$

For the generator (4.23), (4.24), (4.26), there are no corresponding gauge functions G, therefore, there are no corresponding conserved quantities.

Secondly, we study the inverse problem of the Lie symmetries of the system. Suppose that the system possesses an integral (4.28) we now, seek for the corresponding Lie symmetries. Equations (4.13) and (4.14) give, respectively,

$$\bar{\xi}^1 = 1, \qquad \bar{\xi}^2 = 1, \tag{4.29}$$

$$L\xi_0 + G = t . \tag{4.30}$$

It is easy to verify that the determining equations (4.21) are satisfied for any G. Substituting the formulae (4.29), (4.30) into the restriction equations (4.10), we get

$$X^{(1)}(F) = \dot{q}^1\xi^0 . \tag{4.31}$$

Therefore, if and only if $\xi^0 = 0$, the restriction equations (4.10) are verified, and we obtain the generator of the Lie symmetrical transformations corresponding to the integral (4.28),

$$\xi^0 = 0 \qquad \xi^1 = \xi^2 = 1 . \tag{4.32}$$

Finally, we study the relation between the Lie symmetry and the Noether symmetry. The generators (4.23), (4.24), (4.26) correspond to the Lie symmetrical transformations. But they do not satisfy the restriction conditions (4.16). According to Proposition 15, we know that they are not generalized Noether quasi-symmetrical. Choosing the generator

$$\xi^0 = 0, \qquad \xi^1 = \xi^2 = t, \tag{4.33}$$

it satisfies the restriction conditions (4.16). From the identity (4.15), we can obtain

$$G = \frac{1}{2}\,t^2 - q^1 - q^2. \tag{4.34}$$

Therefore, the generator (4.33) corresponds to the generalized Noether quasi-symmetrical transformations. For the generator (4.33), the determining equations (4.21) and the restriction equations (4.22) are not satisfied. According to Proposition 16, we know that the transformations are not Lie symmetrical.

5 Conclusions

The Lie symmetry is an invariance of the ordinary differential equations under the infinitesimal transformations. It is different from the Noether symmetry. In this work we have studied the Lie symmetry of constrained mechanical systems, including the holonomic systems with remainder coordinates, the non-holonomic systems of Chetaev type and non-Chetaev type. For these systems, the invariance of the equations of motion leads to the satisfaction of the determining equations, and the invariance of the constraint equations leads to the satisfaction of the restriction equations.

We have studied two problems of the Lie symmetry, i.e., the direct problem: find the corresponding conserved quantity from a known Lie symmetry, and the inverse problem: find the corresponding Lie symmetry from a known integral.

A Lie symmetry does not always imply a conserved quantity. We have obtained the condition under which a Lie symmetry can lead to a conserved quantity. The condition is the satisfaction of the structure equation. A conserved quantity does not always correspond to a Lie symmetry.

We have given the relation between the Lie symmetry and the Noether symmetry of the systems.

In this work, we have given some examples to illustrate the application of the theoretical results.

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