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Regular Simple Games

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Abstract: Using Kelley's intersection number (and a variant of it) we define two classes of simple games, the regular and the strongly regular games. We show that the strongly regular games are those in which the set of winning coalitions and the set of losing coalitions can be strictly separated by a finitely additive probability measure. This, in particular, provides a combinatorial characterization for the class of finite weighted majority games within the finite simple games. We also prove that regular games have some nice properties and show that the finite regular games are exactly those simple games which are uniquely determined by their counting vector. This, in particular, generalizes a result of Chow and Lapidot.

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1 Introduction

In this work we use Kelley's intersection number and a variant of it to define, combinatorially, two classes of simple games, the regular and strongly regular games. In the case where the set of players is finite the class of strongly regular games coincides with the class of weighted majority games. This, in particular, gives us a combinatorial characterization for the finite weighted majority games within the finite simple games. In the general case the class of strongly regular games is a proper subset of the weighted majority games. The class of regular games have some nice properties which are possessed by the weighted majority games.

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In Section 2 we define the notions which are relevant to our work. In Section 3 we show that the strongly regular games are those simple games in which the set of winning coalitions and the set of losing coalitions can be strictly separated by a finitely additive probability measure. We also derive some corollaries from this result and discuss some examples. In Section 4 we show that regular games possess some properties of weighted majority games. In Section 5 we show that the finite regular games are exactly those games which are uniquely determined by their counting vector. This, in particular, generalizes a result of Chow and Lapidot.

2 Preliminaries

In this section we define the notions which are relevant to our work.

Let I be a set and $\mathcal{C}$ be an algebra of subsets of I. A simple game on $(I, \mathcal{C})$ is a function $v : \mathcal{C} \to \{0, 1\}$ such taht $v(\emptyset) = 0$ and $v(I) = 1$. The members of I are called *players*, the members of $\mathcal{C}$ coalitions. If the set I of players is finite, the game v is called *finite* game. A simple game v on $(I, \mathcal{C})$ is *monotonic* if $v(S) \geqslant v(T)$ for each $S, T \in \mathcal{C}$ such that $S \supset T$. The *dual* of a simple game v is the simple game v^*, where $v^*(S) = v(I) - v(I - S)$, for each $S \in \mathcal{C}$. v is *constant* sum if $v^* = v$. Let v be a simple game on $(I, \mathcal{C})$ and let $z \notin I$. Denote by $\mathcal{C}_0$ the smallest algebra which contains $\mathcal{C} \cup \{\{z\}\}$. The *constant sum extension* of v is the simple game v_0 on $(I \cup \{z\}, \mathcal{C}_0)$, where

$$v_0(S) = \begin{cases} v(S) & z \notin S \\ v^*(S - \{z\}) & z \in S. \end{cases}$$

Note that v_0 is monotonic iff v is monotonic and $v \leqslant v^*$.

Let σ be a finite sequence (or set) of coalitions in $\mathcal{C}$. We denote by $|\sigma|$ the number of members in σ and by $m(\sigma)$ the maximum number of members in σ with a non-empty intersection. Note that if $\sigma = (S_1, \ldots, S_n)$, then $m(\sigma) = \| \sum_{i=1}^{n} \chi_{S_i} \|_\infty$. Let $\mathcal{B} \subset \mathcal{C}$. Define $i(\mathcal{B}) = \inf \left\{ \frac{m(\sigma)}{|\sigma|} \mid \sigma \text{ in a finite subset of } \mathcal{B} \right\}$; $i^*(\mathcal{B}) = \inf \left\{ \frac{m(\sigma)}{|\sigma|} \mid \sigma \text{ is a finite sequence of members in } \mathcal{B} \right\}$. Note that $i^*(\mathcal{B}) = \inf \{ \|f\|_\infty \mid f \in \operatorname{conv} \{\chi_S \mid S \in \mathcal{B}\}\}$.

It is clear that $i^*(\mathcal{B}) \leqslant i(\mathcal{B})$ for each $\mathcal{B} \subset \mathcal{C}$. The number $i^*(\mathcal{B})$ is called the *intersection number of* $\mathcal{B}$, it was introduced in Kelley (1959) in order to provide a necessary and sufficient condition for the existence of a strictly positive measure on a Boolean algebra (for extensions of Kelley results see Wilhelm).

Let v be a simple game on $(I, \mathcal{C})$ and let $\mathcal{W} = \{S \in \mathcal{C} \mid v(S) = 1\}$ (i.e., $\mathcal{W}$ is the set of winning coalitions in v). Define

$$i(v) = i(\mathcal{W}) \quad \text{and} \quad i^*(v) = i^*(\mathcal{W}).$$

The game v is called *regular* if $\max\,(i(v_0), i(v_0^*)) > \frac{1}{2}$ (recall that if w is a simple game, w_0 denotes the constant sum extension of w). It is *strongly regular* if $\max\,(i^*(v_0), i^*(v_0^*)) > \frac{1}{1}$. It is clear that a constant sum simple game v is regular (strongly regular) iff $i(v) > \frac{1}{2}$ $\left(i^*(v) > \frac{1}{2}\right)$.

It is not difficult to construct an example of an infinite simple game which is regular but not strongly regular. Indeed, let I be the set of natural numbers and $\mathcal{C} = 2^I$, the class of all subsets of I. Define a simple game v on $\mathcal{C}$ by

$$v(S) = 1 \Leftrightarrow [S = I - \{1\}, \text{ or } 1 \in S \text{ and } |S| \geqslant 2].$$

It is easy to see that v is a constant sum game and $i(v) = \frac{2}{3}$ and therefore v is regular. For each $n \in I$ we consider the $2n$-term sequence $\sigma_n = (\{1, 2\}, \ldots, \{1, n\}, I - \{1\}, \ldots, I - \{1\})$. It is clear that $\dfrac{m(\sigma_n)}{|\sigma_n|} = \dfrac{n+1}{2n}$. Therefore $i^*(v) \leqslant \dfrac{n+1}{2n}$ for each $n \in I$; thus $i^*(v) \leqslant \frac{1}{2}$.

We note that it is possible to construct a regular finite simple game which is not strongly regular. Indeed, Gabelman's example (see section D of Winder 1971) has this property.

Finally a simple game v on $(I, \mathcal{C})$ is a *weighted majority* game if there exist a finitely additive probability measure μ on $\mathcal{C}$ and $0 < q < 1$ such that $v(S) = 1$ iff $\mu(S) \geqslant q$. The pair (μ, q) is called a *representation* of v and we write $v = (\mu, q)$.

3 Strongly Regular Games and Weighted Majority Games

In this section we characterize the class of strongly regular games (within the simple games) and study the relationships between this class and the class of weighted majority games. We start with the main result of this section.

Theorem A: Let v be a simple game on (I, C). Then v is strongly regular iff there exists a finitely additive probability measure μ on (I, C) such that

$$\sup \{\mu(S) | S \in C, v(S) = 0\} < \inf \{\mu(S) | S \in C, v(S) = 1\}. \tag{3.1}$$

Proof: We first prove the sufficiency part of the theorem. Let μ be a probability measure on C which satisfies (3.1). Let $q = \inf \{\mu(S) | S \in C, v(S) = 1\}$ and $r = \sup \{\mu(S) | S \in C, v(S) = 0\}$. Choose $0 < \epsilon < q - r$. We will show that $q > \frac{1}{2}$, implies $i^*(v_0) > \frac{1}{2}$ and $q < \frac{1}{2}$, implies $i^*(v_0^*) > \frac{1}{2}$. Assume that $q > \frac{1}{2}$. We define $\mu_0 : C_0 \to \mathbb{R}_+$ by $\mu_0(S) = \mu(S)$ if $z \notin S$ and $\mu_0(S) = \mu(S - \{z\}) + 2q - \epsilon - 1$ if $z \in S$. It is easy to see that μ_0 is a measure on C_0 and $\mu_0(S) \geqslant q$ if $v_0(S) = 1$. Let $0 < \delta < \frac{q}{2q - \epsilon} - \frac{1}{2}$. By the definition of $i^*(v_0)$ there exist coalitions $S_1, \ldots, S_n$ in C_0 such that $v_0(S_i) = 1$ for each $1 \leqslant i \leqslant n$ and $\frac{1}{n} \| \sum_{i=1}^{n} \chi_{S_i} \|_\infty < i^*(v_0) + \delta$. Therefore $\frac{1}{n} \sum_{i=1}^{n} \chi_{S_i} \leqslant (i^*(v_0) + \delta) \chi_{I \cup \{z\}}$. Hence, $\frac{1}{n} \sum_{i=1}^{n} \int_{I \cup \{z\}} \chi_{S_i} d\mu_0 \leqslant (i^*(v_0) + \delta) \mu_0(I \cup \{z\})$. This implies that $\frac{1}{n} \sum_{i=1}^{n} \mu_0(S_i) \leqslant (i^*(v_0) + \delta)(2q - \epsilon)$. Therefore $i^*(v_0) \geqslant \frac{q}{2q - \epsilon} - \delta > \frac{1}{2}$. If $q \leqslant \frac{1}{2}$ we define a measure $\mu_0 : C_0 \to \mathbb{R}_+$ by $\mu_0(S)$ if $z \notin S$ and $\mu_0(S) = \mu(S - \{z\}) + 1 - 2q + \epsilon$ if $z \in S$. Then $\mu_0(S) \geqslant 1 - q + \epsilon$ if $v_0^*(S) = 1$. Let $0 < \delta < \frac{1 - q + \epsilon}{2 - 2q + \epsilon} - \frac{1}{2}$. By the same argument which was used above we get $i^*(v_0^*) \geqslant \frac{(1 - q + \epsilon)}{(2 - 2q + \epsilon)} - \delta > \frac{1}{2}$.

We now prove the necessity part of the theorem. We first assume, that v is a constant sum game. Let $B(I, C)$ be the Banach space of all real valued, bounded, measurable functins on (I, C) with the supremum norm. Let $\alpha = i^*(v) > \frac{1}{2}$. Define $K = \operatorname{conv}(\{\alpha \chi_I\} \cup \{\chi_S | S \in C, v(S) = 1\})$. The definition of $i^*(v)$ implies that if

$r_1, \ldots, r_n$ are non-negative rational numbers with $\sum_{i=1}^{n} r_i = 1$, and $S_1, \ldots, S_n$ are coalitions in C such that $v(S_i) = 1$ for each $1 \leqslant i \leqslant n$, then $\| \sum_{i=1}^{n} r_i \chi_{S_i} \|_\infty \geqslant i^*(v)$. Therefore $\|f\| \geqslant \alpha$ for each $f \in K$. Let $B_\alpha = \{f \in B(I,\ C) \mid \|f\| < \alpha\}$. Then K and B_α are nonempty disjoint convex subsets of $B(I,\ C)$ and B_α is open. Therefore by the separation theorem (see Theorem 8 page 417 in Dunford/Schwartz), there exist $q \in \mathbb{R}$ and a nonzero continuous linear functional $\phi : B(I,\ C) \to \mathbb{R}$ such that $\phi(f) \geqslant q$ for each $f \in K$ and $\phi(f) < q$ for each $f \in B_\alpha$. Since $0 \in B_\alpha, q > 0$ and $\phi(\chi_I) > 0$. Therefore we may assume, without loss of generality, that $\phi(\chi_I) = 1$. Since $\alpha \chi_I \in K$, $q \leqslant \alpha$. On the other hand $\alpha \chi_I \in \mathrm{cl}(B_\alpha)$. Therefore $q = \alpha$. We now show that $\phi(\chi_S) \geqslant 0$ for each $S \in C$. Assume, on the contrary, that there is $S \in C$ such that $\phi(\chi_S) < 0$. Let $0 < \epsilon < \dfrac{-\alpha\phi(\chi_S)}{1 - \phi(\chi_S)}$. Then $\phi((\alpha - \epsilon)\chi_{I-S}) = \alpha - \epsilon - (\alpha - \epsilon)\phi(\chi_S) = \alpha - \alpha\phi(\chi_S) - \epsilon(1 - \phi(\chi_S)) > \alpha$, which is impossible because $(\alpha - \epsilon)\chi_{I-S} \in B_\alpha$. We now use the fact that the dual of $B(I,\ C)$ is the space of all bounded and finitely additive measures on $(I,\ C)$ (see Theorem IV.5.1. page 258 in Dunford/Schwartz). This yields the existence of a finitely additive measure μ on C such taht $\phi(f) = \int_I f d\mu$ for each $f \in B(I,\ C)$. Since $\mu(S) = \phi(\chi_S)$ for each $S \in C$, what we have just shown above implies that μ is a probability measure. Now if $S \in C$ and $v(S) = 0$, then $\chi_{I-S} \in K$. Therefore $\mu(I - S) \geqslant \alpha$. Thus $\mu(S) \leqslant 1 - \alpha < \frac{1}{2}$. Hence $\sup \{\mu(S) | S \in C, v(S) = 0\} \leqslant \frac{1}{2} < \alpha \leqslant \inf \{\mu(S) | S \in C, v(S) = 1\}$.

Now if v is not a constant sum game. Then v_0 and v_0^* are constant sum games. Since v is strongly regular, $i^*(v_0) > \frac{1}{2}$ or $i^*(v_0^*) > \frac{1}{2}$. If $i^*(v_0) > \frac{1}{2}$ then by what we have shown above there exists a finitely additive probability measure μ_0 on $(I \cup \{z\},\ C_0)$ such that $\sup \{\mu_0(S) | S \in C_0 \text{ and } v_0(S) = 0\} < \inf \{\mu_0(S) | S \in C_0, v_0(S) = 1\}$. Let λ be the restriction of μ_0 to C. Now, v_0 coincides with v on C. Therefore $\mu = (1/\lambda(I))\lambda$ is a probability measure on $(I,\ C)$ which satisfies (3.1). Assume now that $i^*(v_0^*) > \frac{1}{2}$. Then there is a finitely additive probability measure μ_0 on $(I \cup \{z\},\ C_0)$ such that $r = \sup \{\mu_0(S) | S \in C_0, v_0^*(S) = 0\} < \inf \{\mu_0(S) | S \in C_0,\ v_0^*(S) = 1\} = q$. Choose $0 < \epsilon < q - r$. Let λ be the restriction of μ_0 to C. Then for each $S \in C$, $v(S) = 1$ implies $\lambda(S) > 1 - q + \epsilon$, and $v(S) = 0$ implies $\lambda(S) \leqslant 1 - q$. Let $\mu = (1/\lambda(I))\lambda$. Then μ is a probability measure on $(I,\ C)$ which satisfies (3.1).

Corollary 3.1: A strongly regular game on $(I,\ C)$ is a weighted majority game.

Corollary 3.2: A finite simple game on (I, C) (i.e. the set I of players is finite) is a weighted majority game iff it is strongly regular.

Note that Corollary 3.2 gives a combinatorial characterization of finite weighted majority games.

Remark 3.3: Let v be a constant sum simple game on (I, C). It is clear that if v is strongly regular then $\frac{1}{2}\chi_I \notin \mathrm{conv}\,\{\chi_S \mid v(S)=1\}$. If I is finite and v is monotonic, then the converse is also true. Indeed, since $K = \mathrm{conv}\,\{\chi_S \mid v(S)=1\}$ is convex and compact, there is $0 \neq p \in \mathbb{R}^I$ and $\alpha \in \mathbb{R}$ such that $p \cdot x > \alpha \geqslant p \cdot \frac{1}{2}\chi_I$ for each $x \in K$. As v is monotonic, we may assume that $p \in \mathbb{R}^I_+$ and $\sum_{i \in I} p_i = 1$. Then $p \cdot x > \frac{1}{2}$ for each $x \in K$. Therefore $\|x\|_\infty > \frac{1}{2}$ for each $x \in K$. As K is compact, $i^*(v) = \inf\{\|x\|_\infty \mid x \in K\} > \frac{1}{2}$.

If μ is a measure on (I, C) we denote by $R(\mu)$ the range of μ.

Proposition 3.4: Let v be a constant sum weighted majority game on (I, C). If v has a representation (μ, q) such that $q > \frac{1}{2}$, or $R(\mu)$ is closed subset of $[0, 1]$ (in particular if (I, C) is a σ-algebra and μ is countably additive), then v is strongly regular.

Proof: Assume first that v has a representation (μ, q) such that $q > \frac{1}{2}$. Let $0 < \epsilon < q - \frac{1}{2}$. There exist coalitions $S_1, \ldots, S_n$ in C such that $v(S_i) = 1$ for each $1 \leqslant i \leqslant n$ and $\frac{1}{n}\sum_{i=1}^{n} \chi_{S_i} < (i^*(v) + \epsilon)\chi_I$, therefore $\frac{1}{n}\sum_{i=1}^{n} \int_I \chi_{S_i} d\mu \leqslant i^*(v) + \epsilon$. Thus $\frac{1}{n}\sum_{i=1}^{n} \mu(S_i) \leqslant i^*(v) + \epsilon$. Since $q \leqslant \frac{1}{n}\sum_{i=1}^{n} \mu(S_i)$, $i^*(v) \geqslant q - \epsilon > \frac{1}{2}$.

Assume now that v has a representation (μ, q) such that $R(\mu)$ is closed subset of $[0, 1]$. We will show that $q > \frac{1}{2}$. Let $\alpha = \inf\{\mu(S) \mid S \in C, v(S) = 1\}$. Then $\alpha \leqslant q$. Since $R(\mu)$ is closed, there is $S \in C$ such that $\mu(S) = \alpha$. As v is a constant sum, $\alpha \geqslant \frac{1}{2}$. Now, if $\alpha = \frac{1}{2}$ then $\mu(S) = \frac{1}{2}$ and $\mu(I - S) = \frac{1}{2}$, which is impossible because v is a constant sum. Therefore $\alpha > \frac{1}{2}$.

The following example shows that the assumption that $q \geqslant \frac{1}{2}$ or $R(\mu)$ is closed in Proposition 3.4 cannot be removed.

Example 3.5: Let $I = \{1, 2, 3, \ldots\}$ and C the algebra of finite subsets of I and their complements. Define a measure λ on C by $\lambda(S) = \sum_{n \in S} 2^{-n}$ for each $S \in C$. Note that $\frac{1}{3} \notin R(\lambda)$. For otherwise $\frac{1}{3}$ can be represented in the form $\frac{p}{2^l}$ where p and l are positive integers, which is of course impossible. Define a finitely additive measure ξ on C by $\xi(S) = 1$ if S is infinite and $\xi(S) = 0$ if S is finite. Let $\mu = \frac{1}{4}\xi + \frac{3}{4}\lambda$. Then μ is a finitely additive probability measure on C. Since $\frac{1}{3} \notin R(\lambda)$, we have $\frac{1}{2} \notin R(\mu)$. Define a simple game v on (I, C) by $v(S) = 1$ if $\mu(S) \geqslant \frac{1}{2}$, and $v(S) = 0$ otherwise. Since $\frac{1}{2} \notin R(\mu)$, v is a constant sum weighted majority game. We will show that v is not regular. Assume, on the contrary that $i(v) > \frac{1}{2}$. Let n be a natural number such that $\frac{1}{n} < i(v) - \frac{1}{2}$. For each natural number k let $S_k = \{1, 3, 5, \ldots, 2k-1\}$. Then for each $k \in I$ we have $\mu(S_k) = \frac{1}{2}\left(1 - \left(\frac{1}{4}\right)^k\right)$. Therefore $\mu(S_k \cup \{2k\}) > \frac{1}{2}$ and $\mu(I - S_k) > \frac{1}{2}$. Let $\sigma = \{S_k \cup \{2k\}\}_{k=1}^{n} \cup \{I - S_k\}_{k=1}^{n}$. It is easy to see that $i(\sigma) \leqslant \frac{n+1}{2n}$. Since $i(v) \leqslant i(\sigma)$, this contradicts the choice of n. Thus $i(v) \leqslant \frac{1}{2}$ and v is not regular.

The following example shows that the assumption that v is a constant sum in Proposition 3.4 is also essential.

Example 3.6: Let I be the set of natural numbers and $C = 2^I$, the class of all subsets of I. Let μ on C be the measure defined by $\mu(S) = \sum_{n \in S} 2^{-n}$, for each $S \subset I$. Define a weighted majority game v on C by $v(S) = 1$ iff $\mu(S) = 1$. It is easy to see that $i(v_0^*) = 0$. We will show that $i^*(v_0) \leqslant \frac{1}{2}$. Assume, on the contrary, that $i^*(v_0) > \frac{1}{2}$. Let n be a natural number such that $\frac{1}{n} + \frac{1}{2} < i^*(v_0)$. Consider the $2n$-term sequence $\sigma = (\{1, z\}, \ldots, \{n, z\}, I, \ldots, I)$. It is clear that $\frac{m(\sigma)}{|\sigma|} = \frac{n+1}{2n}$. Since $i^*(v_0) \leqslant \frac{m(\sigma)}{|\sigma|}$, we get a contradiction. Therefore v is not strongly regular.

4 Properties of Regular Games

In this section we prove some properties of regular games. We start with the following simple proposition.

Propositon 4.1: Let v be a regular game on $(I, \mathcal{C})$. Then it is monotonic.

Proof: Let $S \in \mathcal{C}$ such that $v(S) = 1$, and let $T \in \mathcal{C}$, $T \supset S$. Assume, on the contrary, that $v(T) = 0$. Then $v_0(T) = 0$ and $v^*(I - T) = 1$. Therefore $v_0(I \cup \{z\} - T) = 1$ and $v^*(I - T) = 1$. Since $v(S) = 1$, $v^*(I - S) = 0$ and $v_0^*(S \cup \{z\}) = 1$. As $S \cap (I - T) = \emptyset$, we have $i(v_0) \leqslant \frac{1}{2}$ and $i(v_0^*) \leqslant \frac{1}{2}$, which contradicts the fact that v is regular.

Let v be a simple game on $(I, \mathcal{C})$. A coalition $S \in \mathcal{C}$ is at *least as desirable as* a coalition T, written $S \succsim T$, if for each $U \in \mathcal{C}$ such that $U \cap (S \cup T) = \emptyset$ we have $v(T \cup U) = 1$ implies $v(S \cup U) = 1$. The relation $\succsim$ was introduced in Lapidot (1968). It generalizes the relation of desirability for players (see Definition 9.1 in Maschler/Peleg 1966). It is also studied in Einy (1985), Einy/Neyman (1988). It is clear that in weighted majority game $\succsim$ is complete. The following proposition shows that in regular games $\succsim$ is also complete.

Propositon 4.2: Let v be a regular game on $(I, \mathcal{C})$. Then the desirability relation of v is complete.

Proof: We assume first that v is a constant sum. Assume, on the contrary, that there exist $S_1, S_2 \in \mathcal{C}$, which are incomparable with respect to $\succsim$. Then there exist T_1, $T_2 \in \mathcal{C}$ such that $(S_1 \cup S_2) \cap (T_1 \cup T_2) = \emptyset$ and $v(S_1 \cup T_1) = 1$, $v(S_1 \cup T_2) = 0$, $v(S_2 \cup T_2) = 1$, $v(S_2 \cup T_1) = 0$. Since v is a constant sum, $v(I - (S_1 \cup T_2)) = 1$ and $v(I - (S_2 \cup T_1)) = 1$. Let $\sigma = \{S_1 \cup T_1, S_2 \cup T_2, I - (S_1 \cup T_2), I - (S_2 \cup T_1)\}$. Then $\frac{1}{|\sigma|} \sum_{S \in \sigma} X_S = \frac{1}{2} X_I$. But this contradicts the fact that v is regular. Assume now that v is an arbitrary regular game on $(I, \mathcal{C})$. Then by what we have just shown v_0 or v_0^* has a complete desirability relation. By Theorem 4.2 and Theorem 5.1 of Einy (1985), v has a desirability relation. (The results in Einy 1985 are formulated only for finite simple games, but the same proofs work also in the general case).

5 Finite Regular Games

Along this section we assume that $I = \{1, \ldots, n\}$ and $\mathcal{C} = 2^I$, the class of all subsets of I.

Let v be a simple game on $(I, \mathcal{C})$ and $i \in I$. Denote

$$\mathcal{W}(v) = \{S \mid S \subset I,\ v(S) = 1\}, \qquad w(v) = |\mathcal{W}(v)| \tag{5.1}$$

$$f_v(i) = |\{S \mid S \in \mathcal{W}(v), i \in S\}| = \sum_{\substack{S \in \mathcal{W}(v) \\ i \in S}} v(S) \tag{5.2}$$

$$\beta_v(i) = \sum_{S \subset I} (v(S) - v(S - \{i\})). \tag{5.3}$$

Note that $\beta_v(i)$ is the non-normalized Banzhaf index of v. The counting vector of v is the $(n + 1)$ dimensional vector. $f_v = (f_v(1), \ldots, f_v(n),\ w(v))$ (see Chow 1961 and Lapidot 1972). The *characteristic vector* of v is the $(n + 1)$-dimensional vector $\beta_v = (\beta_v(1), \ldots, \beta_v(n), w(v))$. Note that for each $i \in I$ we have

$$\begin{aligned} w(v) - f_v(i) &= \sum_{\substack{S \in \mathcal{W}(v) \\ i \notin S}} v(S) = \sum_{S \in \mathcal{W}(v)} v(S - \{i\}) + \sum_{S \notin \mathcal{W}(v)} v(S - \{i\}) \\ &= \sum_{\substack{S \in \mathcal{W}(v) \\ i \in S}} v(S) - \sum_{\substack{S \in \mathcal{W}(v) \\ i \in S}} (v(S) - v(S - \{i\})) \\ &\quad + \sum_{S \notin \mathcal{W}(v)} (v(S - \{i\}) - v(S)) \\ &= f_v(i) - \beta_v(i). \end{aligned}$$

Therefore

$$f_v(i) = (\beta_v(i) + w(v))/2, \qquad \text{for each } i \in I. \tag{5.4}$$

We also have

$$\begin{aligned} \beta_{v^*}(i) &= \sum_{S \subset I} (v^*(S) - v^*(S - \{i\})) = \sum_{S \subset I} (v(I - S) \cup \{i\}) - v(I - S)) \\ &= \sum_{T \subset I} (v(T \cup \{i\}) - v(T)) = \beta_v(i). \end{aligned}$$

Thus we have

$$\beta_v(i) = \beta_{v^*}(i), \quad \text{for each } i \in I. \tag{5.5}$$

(For monotonic games, (5.5) is Theorem 5 in Dubey/Shapley).

Chow (1961) and Lapidot (1972) showed that a weighted majority games is uniquely determined by its counting vector. The following theorem shows that the monotonic simple games which are uniquely determined by their counting vector are exactly the regular games (in particular, it generalizes the Chow/Lapidot result).

Theorem B: Let v be a monotonic simple game on (I, C). Then v is regular iff it is uniquely determined by its counting vector.

Proof: We first assume that v is regular and show that it is uniquely determined by its counting vector. Let u be a simple game on (I, C) such that $f_u = f_v$. We show that $u = v$. Assume, on the contrary, that $u \neq v$. Let $i \in I$. Then by (5.4) and (5.5) we have

$$\begin{aligned} f_{v^*}(i) &= (\beta_{v^*}(i) + \omega(v^*))/2 = (\beta_v(i) + \omega(v^*))/2 \\ &= (\beta_u(i) + \omega(u^*))/2 = (\beta_{u^*}(i) + \omega(u^*))/2 = f_{u^*}(i). \end{aligned}$$

By the definition of v_0 and u_0, we have $f_{v_0}(i) = f_v(i) + f_{v^*}(i) = f_u(i) + f_{u^*}(i) = f_{u_0}(i)$. By duality, $f_{v_0^*}(i) = f_{u_0^*}(i)$. Also we have $f_{v_0}(z) = \omega(v^*) = \omega(u^*) = f_{u_0}(z)$ and by duality, $f_{v_0^*}(z) = f_{u_0^*}(z)$. Thus for each $k \in I \cup \{z\}$, $f_{v_0}(k) = f_{u_0}(k)$ and $f_{v_0^*}(k) = f_{u_0^*}(k)$. Now, since $v \neq u$, $v_0 \neq u_0$ and $v_0^* \neq u_0^*$. Let $\sigma = \mathcal{W}(v_0) - \mathcal{W}(u_0)$, $\sigma' = \mathcal{W}(u_0) - \mathcal{W}(v_0)$, $\tau = \mathcal{W}(v_0^*) - \mathcal{W}(u_0^*)$, $\tau' = \mathcal{W}(u_0^*) - \mathcal{W}(v_0^*)$. Then $\sigma' = \{(I \cup \{z\}) - S \mid S \in \sigma\}$, $\tau' = \{(I \cup \{z\}) - S \mid S \in \tau\}$. Since $u_0 \neq v_0$ and $u_0^* \neq v_0^*$, $\sigma, \sigma' \neq \emptyset$ and $\tau, \tau' \neq \emptyset$. Let $I_0 = I \cup \{z\}$. Since $f_{v_0}(k) = f_{u_0}(k)$ and $f_{v_0^*}(k) = f_{u_0^*}(k)$ for each $k \in I_0$, we have $\sum_{S \in \mathcal{W}(v_0)} X_S = \sum_{S \in \mathcal{W}(u_0)} X_S$ and $\sum_{S \in \mathcal{W}(v_0^*)} X_S = \sum_{S \in \mathcal{W}(u_0^*)} X_S$. Therefore $\sum_{S \in \sigma} X_S = \sum_{S \in \sigma'} X_S$ and $\sum_{S \in \tau} X_S = \sum_{S \in \tau'} X_S$. This yields that $\sum_{S \in \sigma} X_S = \sum_{S \in \sigma} X_{I_0 - S}$ and $\sum_{S \in \tau} X_S = \sum_{S \in \tau} X_{I_0 - S}$, which imply that $\frac{1}{|\sigma|} \sum_{S \in \sigma} X_S = \frac{1}{2} X_{I_0}$ and $\frac{1}{|\tau|} \sum_{S \in \tau} X_S = \frac{1}{2} X_{I_0}$. Therefore $i(v_0) \leqslant \frac{1}{2}$ and $i(v_0^*) \leqslant \frac{1}{2}$, which is impossible because v is regular.

Assume now that v is uniquely determined by f_v and show that it is regular. Assume, on the contrary, that v is not regular. Then there exists a set $\sigma \subset \mathcal{W}(v_0)$ such that

$\frac{1}{|\sigma|} \sum_{S\in\sigma} \chi_S \leqslant \frac{1}{2} \chi_{I_0}$, where $I_0 = I \cup \{z\}$. Without loss of generality we may assume that $|\sigma|$ is an even positive integer. For otherwise we can omit one member from σ and the inequality $\frac{1}{|\sigma|} \sum_{S\in\sigma} \chi_S \leqslant \frac{1}{2} \chi_{I_0}$ is still valid. Let $\sigma' = \{I_0 - S | S \in \sigma\}$. Then $\sum_{S\in\sigma} \chi_S \leqslant \sum_{T\in\sigma'} \chi_T$. Let Γ be the set of all constant sum simple games on (I_0, C_0) such that if $u_0 \in \Gamma$, then

(1) $\beta_{u_0}(i) \geqslant \beta_{v_0}(i)$ for each $i \in I_0$,

(2) there is $i \in I_0$ such that $\beta_{u_0}(i) \geqslant 4 + \beta_{v_0}(i)$,

(3) $|W(u_0) - W(v_0)|$ is an even integer.

We will show that $\Gamma \neq \emptyset$. Indeed, define a game u_0 on (I_0, C_0) by $u_0(S) = v_0(S)$ if $S \notin \sigma' \cup \sigma$, and $u_0(S) = v_0(I_0 - S)$ if $S \in \sigma' \cup \sigma$. Since $\sum_{S\in\sigma} \chi_S \leqslant \sum_{S\in\sigma'} \chi_S$, $f_{u_0}(i) \geqslant f_{v_0}(i)$ for each $i \in I_0$. By (5.4), $\beta_{u_0}(i) \geqslant \beta_{v_0}(i)$ for each $i \in I_0$. We show that $\beta_{u_0} \neq \beta_{v_0}$. Let u be the game on (I, C) which satisfies $u(S) = u_0(S)$ for each $S \in C$. It is clear that u_0 is the constant sum extension of u. We show now that $f_{u_0} \neq f_{v_0}$. Assume, on the contrary, that $f_{u_0} = f_{v_0}$. Since $f_{v_0}(z) = \omega(v^*)$, $f_{u_0}(z) = \omega(u^*)$, $\omega(v_0) = \omega(v) + \omega(v^*)$, and $\omega(u_0) = \omega(u) + \omega(u^*)$, we have $\omega(u) = \omega(v)$ and $\omega(u^*) = \omega(v^*)$. Let $i \in I$. Then $f_{v_0}(i) = f_v(i) + f_{v^*}(i)$ and $f_{u_0}(i) = f_u(i) + f_{u^*}(i)$. Now, (5.4) and (5.5) imply that $f_{v^*}(i) = f_v(i) + \frac{\omega(v^*) - \omega(v)}{2}$ and $f_{u^*}(i) = f_u(i) + \frac{\omega(u^*) - \omega(u)}{2}$. Therefore $f_v(i) = f_u(i)$. As $\omega(v) = \omega(u)$, $f_v = f_u$, which is impossible because v is uniquely determined by f_v and $u \neq v$. Therefore $f_{u_0} \neq f_{v_0}$. Since $\omega(u_0) = \omega(v_0) = 2^n$, there is $i \in I_0$ such that $f_{u_0}(i) \neq f_{v_0}(i)$. By (5.4), $\beta_{u_0}(i) \neq \beta_{v_0}(i)$. We will show that $\beta_{u_0}(i) \geqslant 4 + \beta_{v_0}(i)$. As $f_{u_0}(i) > f_{v_0}(i)$, $\sum_{S\in\sigma} \chi_S(i) < \sum_{S\in\sigma'} \chi_S(i)$, and thus since $|\sigma|$ is even $\sum_{S\in\sigma} \chi_S(i) \leqslant \frac{|\sigma|}{2} - 1$. Therefore $\sum_{s\in\sigma'} \chi_S(i) - \sum_{S\in\sigma} \chi_S(i) = |\sigma| - 2 \sum_{S\in\sigma} \chi_S(i) \geqslant 2$. Hence $f_{u_0}(i) \geqslant 2 + f_{v_0}(i)$. As $\omega(u_0) = \omega(v_0)$, $\beta_{u_0}(i) \geqslant 4 + \beta_{v_0}(i)$. Since $|W(u_0) - W(v_0)| = |\sigma|$ is even integer, $u_0 \in \Gamma$.

Let Γ^* be the set of all constant sum simple games on (I_0, C_0) such that if $u \in \Gamma^*$ then

(1) $\beta_u(i) \geqslant \beta_{v_0^*}(i)$, for each $i \in I_0$,

(2) there is $i \in I_0$ such that $\beta_u(i) \geqslant 4 + \beta_{v_0^*}(i)$,

(3) $|W(u) - W(v_0^*)|$ is an even integer.

As v_0^* is not regular, by the same arguments which were used above, it can be shown that $\Gamma^* \neq \emptyset$. We now need the following claim.

Claim 5.1: At least one of the games v_0 or v_0^* is monotonic.

Assume, on the contrary, that v_0 and v_0^* are not monotonic. Then there exist $S_1 \subset S_2 \subset I_0$ and $T_1 \subset T_2 \subset I_0$ such that $v_0(S_1) = 1$, $v_0(S_2) = 0$ and $v_0^*(T_1) = 1$, $v_0^*(T_2) = 0$. Since v and v^* are monotonic, $z \in S_2 - S_1$ and $z \in T_2 - T_1$. Now $v(S_1) = 1$, $v(I - S_1) \geqslant v(I - S_2) = 1$ and $v(T_1) \leqslant v(T_2 - \{z\}) = 0$, $v(I - T_1) = 0$. Let u be a simple game on (I, C) such that $\mathcal{W}(u) = (\mathcal{W}(v) \cup \{T_1, I - T_1\}) - \{S_1, I - S_1\}$. Then $\sum_{S \in \mathcal{W}(u)} \chi_S = \sum_{S \in \mathcal{W}(v)} \chi_S$ and $\omega(u) = \omega(v)$, which means that $f_u = f_v$. But this is impossible because $u \neq v$ and v is uniquely determined by f_v.

We are now ready to complete the proof of Theorem B. By Claim 5.1. at least one of the games v_0 or v_0^* is monotonic. Assume, without loss of generality, that v_0 is monotonic. Let $u_0 \in \Gamma$ such that $\sum_{i \in I_0} \beta_{u_0}(i) \leqslant \sum_{i \in I_0} \beta_u(i)$, for each $u \in \Gamma$. Let $i \in I_0$ such that $\beta_{u_0}(i) \geqslant \beta_{v_0}(i) + 4 \geqslant 4$. ($\beta_{v_0}(i) \geqslant 0$ because v_0 is monotonic). Let $S \subset I_0$ such that $u_0(S) = 1$ and $u_0(S - \{i\}) = 0$. Let w_0 be a simple game on (I_0, C_0) such that $\mathcal{W}(w_0) = (\mathcal{W}(u_0) \cup \{S - \{i\}, I_0 - S\}) - \{S, (I_0 - S) \cup \{i\}\}$. Then w_0 is a constant sum, $\beta_{w_0}(j) = \beta_{u_0}(j)$ for $j \in I_0 - \{i\}$, and $\beta_{w_0}(i) = \beta_{u_0}(i) - 4 \geqslant \beta_{v_0}(i)$. Since v_0 is monotonic and w_0 is not, $w_0 \neq v_0$. We will get a contradiction to the minimality of $\sum_{i \in I_0} \beta_{u_0}(i)$ by showing that $w_0 \in \Gamma$. It is easy to check that $|\mathcal{W}(w_0) - \mathcal{W}(v_0)|$ is even. We will show that $\beta_{w_0} \neq \beta_{v_0}$. Assume not. Let w be the restriction of w_0 to (I, C). By (5.4) $f_{v_0}(z) = (\beta_{v_0}(z) + 2^n)/2$ and $f_{w_0}(z) = (\beta_{w_0}(z) + 2^n)/2$. As $f_{v_0}(z) = \omega(v^*)$ and $f_{w_0}(z) = \omega(w^*)$, we have $\omega(v^*) = \omega(w^*)$. Therefore $\omega(v) = \omega(w)$. Let $i \in I$. Then $\beta_{v_0}(i) = 2\beta_v(i)$, $\beta_{w_0}(i) = 2\beta_w(i)$. Therefore $\beta_v(i) = \beta_w(i)$ and by (5.4) we obtain $f_v(i) = (\beta_v(i) + \omega(v))/2 = (\beta_w(i) + \omega(w))/2 = f_w(i)$. Thus $f_v = f_w$, which implies that $v = w$. But this is impossible because $v_0 \neq w_0$. Now, since $\beta_{w_0} \neq \beta_{v_0}$ there is $j \in I$ such that $\beta_{w_0}(j) > \beta_{v_0}(j)$. We will show that $\beta_{w_0}(j) \geqslant \beta_{v_0}(j) + 4$ and then $w_0 \in \Gamma$. Let $\sigma = \mathcal{W}(w_0) - \mathcal{W}(v_0)$, $\sigma' = \mathcal{W}(v_0) - \mathcal{W}(w_0)$. As $f_{w_0}(j) > f_{v_0}(j)$ and $|\sigma|$ is even, we have $\sum_{S \in \sigma} \chi_S(j) - \sum_{S \in \sigma'} \chi_S(j) \geqslant 2$, which means that $f_{w_0}(j) \geqslant f_{v_0}(j) + 2$. Therefore $\beta_{w_0}(j) \geqslant \beta_{v_0}(j) + 4$.

Note that a regular finite simple game may not be a weighted majority game. Indeed Gabelman discovered a montonic simple game with 15 players which is uniquely determined by its counting vector and is not a weighted majority game. (See Section D in Winder 1971.)

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