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Stresses in an Orthotropic Semi-infinite Strip Due to Edge Loads

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With 5 Figures

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Summary

The stresses in an orthotropic elastic semi-infinite strip subject to plane strain are investigated. Symmetrical distributions of surface tractions are prescribed on the sides of the strip, while along the end the boundary conditions are arbitrary. By using an integral transform method the problem is reduced to a singular integral equation. The dependence of the stress singularity and the stress-intensity factors on the orthotropic properties of the strip is investigated. Stress distributions over the strip end are evaluated numerically.

1. Introduction

Problems involving isotropic elastic semi-infinite strips have received considerable attention in the literature. Specifically, in [1]—[4] the behavior of the stresses at the strip corners have been analyzed. On the other hand it appears that little consideration has been given to the case of an orthotropic half-strip, a problem of importance in composite material applications. In this paper we consider the elastic behavior of such a strip, with surface tractions prescribed along the sides and arbitrary conditions on the end. Particular attention is given to the case of a fixed end, for which stress singularities occur at the corners.

2. Formulation

Consider the orthotropic elastic semi-infinite strip shown in Fig. 1. Let the sides $x_2 = \pm h$ be subject to arbitrary symmetrical distributions of stresses given by

$$\begin{aligned} \sigma_{12}(x_1, h) &= P_1(x_1), \qquad \sigma_{12}(x_1, -h) = -P_1(x_1), \\ \sigma_{22}(x_1, h) &= P_2(x_1), \qquad \sigma_{22}(x_1, -h) = P_2(x_1). \end{aligned} \tag{1}$$

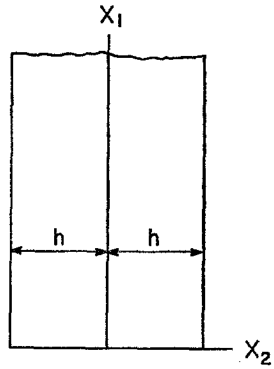


Fig. 1. Coordinate system

On the end $x_1 = 0$ any one of the following pairs of conditions may be specified:

$$u_1(0, x_2) = 0, \qquad u_2(0, x_2) = 0 \tag{2}$$

$$\sigma_{11}(0, x_2) = P(x_2), \qquad \sigma_{12}(0, x_2) = Q(x_2) \tag{3}$$

$$u_1(0, x_2) = 0, \qquad \sigma_{12}(0, x_2) = Q(x_2) \tag{4}$$

$$u_2(0, x_2) = 0, \qquad \sigma_{11}(0, x_2) = P(x_2) \tag{5}$$

where $P(x_2)$ and $Q(x_2)$ are symmetrical and antisymmetrical functions of x_2, respectively.

Assuming a state of plane strain, the stress-displacement relations are

$$\begin{aligned} \sigma_{11} &= A_{11} \frac{\partial u_1}{\partial x_1} + A_{12} \frac{\partial u_2}{\partial x_2} \\ \sigma_{22} &= A_{12} \frac{\partial u_1}{\partial x_1} + A_{22} \frac{\partial u_2}{\partial x_2} \\ \sigma_{12} &= A_{66} \left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \right) \end{aligned} \tag{6}$$

and the corresponding displacement equations of equilibrium become [5]

$$\begin{aligned} A_{11} \frac{\partial^2 u_1}{\partial {x_1}^2} + A_{66} \frac{\partial^2 u_1}{\partial {x_2}^2} + (A_{12} + A_{66}) \frac{\partial^2 u_2}{\partial x_1\, \partial x_2} &= 0 \\ (A_{12} + A_{66}) \frac{\partial^2 u_1}{\partial x_1\, \partial x_2} + A_{66} \frac{\partial^2 u_2}{\partial {x_1}^2} + A_{22} \frac{\partial^2 u_2}{\partial {x_2}^2} &= 0. \end{aligned} \tag{7}$$

Here A_{ij} denote the stiffness coefficients for the orthotropic material. Results for the corresponding problem of plane stress can be obtained from the present formulation by a simple rearrangement of the elastic constants [6].

In order to solve the differential Eq. (7) we introduce the Fourier transforms

$$\begin{aligned} \bar{u}_1(t, x_2) &= \int_0^\infty u_1(x_1, x_2) \sin tx_1 \, dx_1 \\ \bar{u}_2(t, x_2) &= \int_0^\infty u_2(x_1, x_2) \cos tx_1 \, dx_1 \end{aligned} \tag{8}$$

in which case $\bar{u}_1(t, x_2)$ and $\bar{u}_2(t, x_2)$ are governed by the ordinary differential equations

$$\begin{aligned} \bar{L}_1 &\equiv A_{66} \frac{d^2\bar{u}_1}{dx_2{}^2} - t(A_{12} + A_{66}) \frac{d\bar{u}_2}{dx_2} - t^2 A_{11}\bar{u} = -tA_{11}u_1(0, x_2) \\ \bar{L}_2 &\equiv A_{22} \frac{d^2\bar{u}_2}{dx_2{}^2} + t(A_{12} + A_{66}) \frac{d\bar{u}_1}{dx_2} - t^2 A_{66}\bar{u}_2 \\ &= (A_{12} + A_{66}) \frac{du_1(0, x_2)}{dx_2} + A_{66} \frac{du_2}{dx_1}(0, x_2). \end{aligned} \tag{9}$$

The solution to the system of equations (9) for boundary conditions (4) can easily be obtained, because in this case the functions on the right-hand sides of Eqs. (9) are known. Similarly for the boundary conditions (5) the solution is readily found if the Fourier sine transform is applied to u_2 and the cosine transform is applied to u_1. Attention is focused here on the more complicated situations which arise when boundary conditions (2) or (3) are prescribed.

We introduce the functions

$$\begin{aligned} U(x_2) &= \begin{Bmatrix} u_1(0, x_2), & |x_2| \leqq h \\ 0, & |x_2| > h \end{Bmatrix} = \int_0^\infty \bar{U}(p) \cos px_2 \, dp \\ \bar{q}(x_2) &= \begin{Bmatrix} \sigma_{12}(0, x_2), & |x_2| \leqq h \\ 0, & |x_2| > h \end{Bmatrix} = \int_0^\infty \bar{q}(p) \sin px_2 \, dp \end{aligned} \tag{10}$$

in which case Eqs. (9) can be written as

$$\begin{aligned} \bar{L}_1 &= -tA_{11} \int_0^\infty \bar{U}(p) \cos px_2 \, dp \\ \bar{L}_2 &= -\int_0^\infty [A_{12}p\bar{U}(p) - \bar{q}(p)] \, sin \, px_2 \, dp. \end{aligned} \tag{11}$$

It can be verified that a particular solution to (11) may be expressed as

$$\bar{u}_1^p = \int_0^\infty \frac{1}{D_{pt}} \{[(2\alpha_1^2 + \nu_1) p^2 + \alpha_2^4 t^2] t\bar{U} + (\nu_1/A_{66} + 1/A_{22}) tp\bar{q}\} \cos px_2 \, dp$$
$$\bar{u}_2^p = \int_0^\infty \frac{1}{D_{pt}} [(\nu_1 p^2 - \alpha_2^4 t^2) p\bar{U} - (p^2/A_{22} + \alpha_2^4 t^2/A_{66}) \bar{q}] \sin px_2 \, dp \tag{12}$$

where

$$D_{pt} = p^4 + 2\alpha_1^2 p^2 t^2 + \alpha_2^4 t^4, \qquad \nu_1 = \frac{A_{12}}{A_{22}}$$
$$\alpha_1^2 = \frac{A_{11}A_{22} - 2A_{12}A_{66} - A_{12}^2}{2A_{22}A_{66}}, \qquad \alpha_2^4 = \frac{A_{11}}{A_{22}}. \tag{13}$$

The form of the complementary solution to (11) depends upon the values of the coefficients in the characteristic equation

$$\varkappa^4 - 2\alpha_1^2 \varkappa^2 + \alpha_2^4 = 0. \tag{14}$$

If $D \equiv \alpha_1^2 - \alpha_2^2 > 0$ the roots of (14) are real (say $\pm\varkappa_i$ $(i = 1, 2)$) and the complementary solution is

$$\bar{u}_1^0 = \delta_1 B_1(t) \cosh \varkappa_1 t x_2 + \delta_2 B_2(t) \cosh \varkappa_2 t x_2$$
$$\bar{u}_2^0 = B_1(t) \sinh \varkappa_1 t x_2 + B_2(t) \sinh \varkappa_2 t x_2 \tag{15}$$

where

$$\delta_i = \frac{A_{66} - A_{22}\varkappa_i^2}{(A_{12} + A_{66}) \varkappa_i}, \quad \varkappa_1 = \alpha_4 + \alpha_5, \quad \varkappa_2 = \alpha_4 - \alpha_5 \quad (i = 1, 2)$$
$$\alpha_4 = \sqrt{(\alpha_1^2 + \alpha_2^2)/2}, \quad \alpha_5 = \sqrt{D/2}. \tag{16}$$

In this case the material will be classified as type I [7].

If $\alpha_1^4 < \alpha_2^4$ the roots of (14) are complex and the complementary solution is given by

$$\bar{u}_1^0 = [\varepsilon_2 B_1(t) - \varepsilon_1 B_2(t)] X_1(x_2, t) + [\varepsilon_1 B_1(t) + \varepsilon_2 B_2(t)] X_2(x_2, t)$$
$$\bar{u}_2^0 = B_1(t) X_3(x_2, t) + B_2(t) X_4(x_2, t) \tag{17}$$

in which

$$\begin{aligned} X_1(x_2, t) &= \cosh(\alpha_4 x_2 t)\cos(\bar{\alpha}_5 x_2 t) \\ X_2(x_2, t) &= \sinh(\alpha_4 x_2 t)\sin(\bar{\alpha}_5 x_2 t) \\ X_3(x_2, t) &= \cosh(\alpha_4 x_2 t)\sin(\bar{\alpha}_5 x_2 t) \\ X_4(x_2, t) &= \sinh(\alpha_4 x_2 t)\cos(\bar{\alpha}_5 x_2 t) \end{aligned} \tag{18}$$

$$\varepsilon_i = \frac{\alpha_{6-i}\left[A_{66} - (-1)^i\sqrt{A_{11}A_{22}}\right]}{\alpha_2^2(A_{12} + A_{66})} \quad (i = 1, 2), \qquad \bar{\alpha}_5 = \sqrt{-D/2}.$$

Materials of this type will be classified as type II.

For $|\alpha_1^2| > \alpha_2^2$ with $\alpha_1^2 < 0$ the roots of Eq. (14) are pure imaginary. Since this case generally does not occur in practice [7] we shall not consider it here.

Application of (8) to the boundary conditions (1) for $x_2 = \pm h$ yields

$$\begin{aligned} &\frac{d\bar{u}_1}{dx_2} - t\bar{u}_2(t, h) = \bar{P}_1(t) \\ &\frac{d\bar{u}_2}{dx_2} + \nu_1 t\bar{u}_1(t, h) = \nu_1 u_1(0, h) + \bar{P}_2(t) \end{aligned} \tag{19}$$

where

$$\bar{P}_1(t) = \frac{1}{A_{66}}\int_0^\infty P_1(x_1)\sin tx_1\, dx_1, \qquad \bar{P}_2(t) = \frac{1}{A_{22}}\int_0^\infty P_2(x_1)\cos tx_1\, dx_1. \tag{20}$$

Next substituting the complete solution $\bar{u}_i = \bar{u}_i^p + \bar{u}_i^0$ into (19), and using the inversion formulas

$$\bar{U}(p) = \frac{2}{\pi}\int_0^\infty U(x_2)\cos px_2\, dx_2, \qquad \bar{q}(p) = \frac{2}{\pi}\int_0^\infty q(x_2)\sin px_2\, dx_2 \tag{21}$$

together with the integral evaluations [8]

$$\begin{aligned} &\int_0^\infty \frac{p}{D_{pt}}\sin p(h - y)\, dp = \frac{\pi}{2\alpha_3 t^2}\, v_2(y, t) \\ &\int_0^\infty \frac{p^3}{D_{pt}}\sin p(h - y)\, dp = \frac{\pi}{2}\left[v_1(y, t) - v_2(y, t)\,\alpha_1^2/\alpha_3\right] \\ &\int_0^\infty \frac{1}{D_{pt}}\cos p(h - y)\, dp = \frac{\pi}{4\alpha_2^2 t^3}\left[v_1(y, t)/\alpha_4 + v_2(y, t)/\alpha_5\right] \\ &\int_0^\infty \frac{p^2}{D_{pt}}\cos p(h - y)\, dp = \frac{\pi}{4t}\left[v_1(y, t)/\alpha_4 - v_2(y, t)/\alpha_5\right] \end{aligned} \tag{22}$$

where

$$v_1(y, t) = e^{-\alpha_4(h-y)t} \cosh [\alpha_5(h - y) t]$$
$$v_2(y, t) = e^{-\alpha_4(h-y)t} \sinh [\alpha_5(h - y) t] \tag{23}$$
$$\alpha_3 = 2\alpha_4\alpha_5$$

we obtain the following equation for the determination of $B_1(t)$ and $B_2(t)$

$$\sum_{j=1,2} a_{ij}(t) B_j(t) = I_i(t) \qquad (i = 1, 2) \tag{24}$$

in which

$$a_{1j}(t) = \begin{Bmatrix} s_j \sinh (\varkappa_j th) \\ r_{5-j}X_3(h, t) - (-1)^j r_{2+j}X_4(h, t) \end{Bmatrix}$$
$$a_{2j}(t) = \begin{Bmatrix} s_{2+j} \cosh (\varkappa_j th) \\ r_jX_1(h, t) + (-1)^j r_{3-j}X_2(h, t) \end{Bmatrix}. \tag{25}$$

In (25) and hereafter the upper and lower expressions within braces { } relate to material types I and II, respectively; furthermore

$$s_j = \delta_j\varkappa_j - 1, \qquad s_{2+j} = \varkappa_j + \nu_1\delta_j$$
$$r_1 = \alpha_4 + \nu_1\varepsilon_2, \qquad r_2 = \bar{\alpha}_5 - \nu_1\varepsilon_1$$
$$r_3 = \alpha_4\varepsilon_2 + \bar{\alpha}_5\varepsilon_1 - 1, \qquad r_4 = \alpha_4\varepsilon_1 - \bar{\alpha}_5\varepsilon_2$$
$$I_i(t) = I_i^U(t) + I_i^q(t) + \bar{P}_i(t) \qquad (i = 1, 2)$$
$$I_1^U(t) = \frac{\lambda_1}{4} \int_{-h}^{h} U'(y) [v_1(y, t)/\alpha_4 - v_2(y, t)/\alpha_5] \, dy$$
$$I_2^U(t) = -\frac{\lambda_2}{2\alpha_3} \int_{-h}^{h} U'(y) \, v_2(y, t) \, dy, \qquad U'(y) = \partial U'/\partial y \tag{26}$$
$$I_1^q(t) = \frac{1}{2A_{22}} \int_{-h}^{h} q(y) [m_1v_1(y, t) - m_2v_2(y, t)] \, dy$$
$$I_2^q(t) = -\frac{1}{2A_{22}} \int_{-h}^{h} q(y) [v_1(y, t) - m_3v_2(y, t)] \, dy$$
$$\lambda_1 = \frac{A_{12}^2 - A_{11}A_{22}}{A_{22}A_{66}}, \qquad \lambda_2 = \frac{A_{66}}{A_{22}} \lambda_1, \qquad \lambda_3 = \frac{A_{12} - \sqrt{A_{11}A_{22}}}{A_{66}}$$
$$m_1 = \frac{\lambda_3}{2\alpha_4}, \qquad m_2 = \frac{A_{12} + \sqrt{A_{11}A_{22}}}{2\alpha_5A_{66}}, \qquad m_3 = \frac{\lambda_1}{2\alpha_3}.$$

Solving Eqs. (24) for $B_j(t)$, substituting the results into (17), and then taking the inverse transforms of (12) and (17) yields expressions for the displacement components $u_i(x_1, x_2)$. Using these expressions, together with the integration formulas

$$\begin{aligned} \int_0^\infty p\overline{U} \cos px_2 \, dp &= \frac{1}{\pi} \int_{-h}^{h} \frac{U'(y)}{x_2 - y} \, dy \\ \int_0^\infty \overline{q} \cos px_2 \, dp &= -\frac{1}{\pi} \int_{-h}^{h} \frac{q(y)}{x_2 - y} \, dy \end{aligned} \tag{27}$$

we arrive at the following equations for strain and stress needed in the subsequent analysis:

$$\begin{aligned} \frac{\partial u_2(0, x_2)}{\partial x_2} = \frac{\gamma_1}{2\pi} \int_{-h}^{h} \frac{U'(y)}{x_2 - y} \, dy + \frac{\gamma_2}{2\pi A_{22}} \int_{-h}^{h} \frac{q(y)}{x_2 - y} \, dy \\ + \frac{2}{\pi} \int_0^\infty \frac{1}{\Delta_t} [w_1(x_2, t) \, I_1(t) + w_2(x_2, t) \, I_2(t)] \, dt \end{aligned} \tag{28}$$

and

$$\begin{aligned} \sigma_{11}(0, x_2) = \frac{A_{11}\gamma_3}{\pi} \int_{-h}^{h} \frac{U'(y)}{x_2 - y} \, dy - \frac{\gamma_4}{\pi} \int_{-h}^{h} \frac{q(y)}{x_2 - y} \, dy \\ + \frac{2A_{11}}{\pi} \int_0^\infty \frac{1}{\Delta_t} [w_3(x_2, t) \, I_1(t) + w_4(x_2, t) \, I_2(t)] \, dt \end{aligned} \tag{29}$$

where

$$\gamma_1 = \frac{\lambda_3 A_{66}}{\alpha_4 A_{22}}, \quad \gamma_2 = \frac{A_{66} + \sqrt{A_{11} A_{22}}}{\alpha_4 A_{66}}, \quad \gamma_3 = \frac{\lambda_1 A_{66}}{2\alpha_4 A_{11}}, \quad \gamma_4 = -\frac{\lambda_1}{2}$$

$$\begin{aligned} w_{2i-1}(x_2, t) &= \begin{Bmatrix} C_{2i-1,1} X_5(x_2, t) - C_{2i-1,2} X_6(x_2, t) \\ w_{2i-1}^{(1)}(t) \, X_1(x_2, t) + w_{2i-1}^{(2)}(t) \, X_2(x_2, t) \end{Bmatrix} \\ w_{2i}(x_2, t) &= \begin{Bmatrix} C_{2i,2} X_7(t) \, X_6(x_2, t) - C_{2i,1} X_8(t) \, X_5(x_2, t) \\ w_{2i}^{(1)}(t) \, X_1(x_2, t) + w_{2i}^{(2)}(t) \, X_2(x_2, t) \end{Bmatrix} \end{aligned} \qquad (i = 1, 2) \tag{30}$$

$$\Delta_t = \begin{Bmatrix} s_1 s_4 X_7(t) - s_2 s_3 X_8(t) \\ [\delta_1{}^0 \sinh (2\alpha_4 t h) + \delta_2{}^0 \sin (2\overline{\alpha}_5 t h)]/2 \end{Bmatrix}$$

in which

$$X_{i+4}(x_2, t) = \cosh(\varkappa_i t x_2)/\cosh(\varkappa_i t h), \qquad X_{i+6}(t) = \tanh(\varkappa_i t h)$$

$$w_{2i-1}^{(j)}(t) = f_{2i-1,2j-1}X_1(h, t) + f_{2i-1,2j}X_2(h, t)$$

$$w_{2i}^{(j)}(t) = f_{2i,2j-1}X_3(h, t) + f_{2i,2j}X_4(h, t)$$

$$C_{11} = s_4\varkappa_1, \quad C_{12} = s_3\varkappa_2, \quad C_{21} = s_2\varkappa_1, \quad C_{22} = s_1\varkappa_2$$

$$C_{31} = s_4\delta_3, \quad C_{32} = s_3\delta_4, \quad C_{41} = s_2\delta_3, \quad C_{42} = s_1\delta_4$$

$$\delta_{2+i} = \delta_i + \nu_2\varkappa_i \qquad (i, j = 1, 2) \qquad (31)$$

$$f_{11} = \bar{\alpha}_5 r_1 - \alpha_4 r_2, \quad f_{12} = -\alpha_4 r_1 - \bar{\alpha}_5 r_2, \quad f_{13} = -f_{12}, \quad f_{14} = f_{11}$$

$$f_{21} = \alpha_4 r_3 - \bar{\alpha}_5 r_4, \quad f_{22} = -\alpha_4 r_4 - \bar{\alpha}_5 r_3, \quad f_{23} = f_{22}, \quad f_{24} = -f_{21}$$

$$f_{31} = \nu_2 f_{11} - \varepsilon_2 r_2 - \varepsilon_1 r_1, \quad f_{32} = \nu_2 f_{12} + \varepsilon_1 r_2 - \varepsilon_2 r_1, \quad f_{33} = -f_{32}, \quad f_{34} = f_{31}$$

$$f_{41} = \nu_2 f_{21} + \varepsilon_2 r_3 + \varepsilon_1 r_4, \quad f_{42} = \nu_2 f_{22} + \varepsilon_1 r_3 - \varepsilon_2 r_4, \quad f_{43} = f_{42}, \quad f_{44} = -f_{41}$$

$$\delta_1^0 = -r_1 r_4 - r_2 r_3, \quad \delta_2^0 = r_1 r_3 - r_2 r_4, \quad \nu_2 = A_{12}/A_{11}.$$

3. Solution for Fixed-End Condition

We now consider the case in which the end of the strip is fully constrained, as defined by (2). According to (10) and (26)

$$U(x_2) = \bar{U}(p) = I_1^U(t) = I_2^U(t) = 0. \qquad (32)$$

Using the second of boundary conditions (2) and expression (28) it can then be shown that the unknown function $q(y)$ satisfies the integral equation

$$\int_{-h}^{h} \left[\frac{1}{x_2 - y} + M_1(x_2, y)\right] q(y)\, dy = R_1(x_2) \qquad (33)$$

where

$$M_1(x_2, y) = \int_0^\infty M_1^*(x_2, y, t)\, dt$$

$$R_1(x_2) = -\frac{4A_{22}}{\gamma_2}\int_0^\infty \frac{1}{\Delta_t}\,[w_1(x_2, t)\,\bar{P}_1(t) + w_2(x_2, t)\,\bar{P}_2(t)]\, dt \qquad (34)$$

and

$$M_1^*(x_2, y, t) = \begin{Bmatrix} \Omega_I[z_1(x_2, t), z_2(x_2, t)] \\ 2\Omega_{II}[w_1(x_2, t), w_2(x_2, t)]/\gamma_2 \end{Bmatrix} \tag{35}$$

$$\Omega_I[z_n(x_2, t), z_{n+1}(x_2, t)] = z_n(x_2, t)\, e^{-\varkappa_2(h-y)t} + z_{n+1}(x_2, t)\, e^{-\varkappa_1(h-y)t} \tag{36}$$

$$\begin{aligned} &\Omega_{II}[w_n(x_2, t), w_{n+1}(x_2, t)] \\ &= \frac{1}{\Delta_t} e^{-\alpha_4(h-y)t}\{[m_1 w_n(x_2, t) - w_{n+1}(x_2, t)] \cos[\bar{\alpha}_5(h-y)\, t] \\ &\qquad - [m_2 w_n(x_2, t) - m_3 w_{n+1}(x_2, t)] \sin[\bar{\alpha}_5(h-y)\, t]\} \quad (\text{here} \quad n = 1) \end{aligned} \tag{37}$$

and also

$$z_i(x_2, t) = \frac{1}{\Delta_t} \{[l_{i1} - l_{i2} X_8(t)]\, X_5(x_2, t) - [l_{i3} - l_{i4} X_7(t)]\, X_6(x_2, t)\} \qquad (i, j = 1, 2) \tag{38}$$

$$l_{i,2j-1} = [m_1 + (-1)^i\, m_2]\, C_{1j}/\gamma_2, \qquad l_{i,2j} = -[1 + (-1)^i\, m_3]\, C_{2j}/\gamma_2.$$

The kernel $M_1(x_2, y)$ in (33) is not bounded as $y \to h$ and $x_2 \to \pm h$. The singular points can be extracted, following the procedure given in [9], by expressing $M_1(x_2, y)$ in the form

$$M_1(x_2, y) = M_1^\infty(x_2, y) + M_f(x_2, y) \tag{39}$$

where $M_f(x_2, y)$ is bounded in $[-h, h]$, while $M_1^\infty(x_2, y)$ can be obtained by using the asymptotic values of $M_1^*(x_2, y, t)$ as $t \to \infty$ in the form

$$\begin{aligned} M_1^\infty(x_2, y) = -\frac{1}{s_5} \Bigg[&\frac{d_1}{y - h - \lambda(h - x_2)} + \frac{d_1}{y - h - \lambda(h + x_2)} \\ &+ \frac{d_2}{y - (2h - x_2)} + \frac{d_2}{y - (2h + x_2)} \\ &+ \frac{d_3}{y - h - (h - x_2)/\lambda} + \frac{d_3}{y - h - (h + x_2)/\lambda} \Bigg] \end{aligned} \tag{40}$$

where

$$s_5 = s_1 s_4 - s_2 s_3, \qquad \lambda = \varkappa_1/\varkappa_2$$
$$d_1 = \frac{l_{11} - l_{12}}{\varkappa_2}, \quad d_2 = \frac{l_{21} - l_{22}}{\varkappa_1} - \frac{l_{13} - l_{14}}{\varkappa_2}, \quad d_3 = \frac{l_{24} - l_{23}}{\varkappa_1}. \tag{41}$$

Expression (40) is valid in the case of material type I; similar results can be obtained for material type II. The integral Eq. (33) can now be written as

$$\int_{-h}^{h} \left[\frac{1}{x_2 - y} + M_1^\infty(x_2, y)\right] q(y)\, dy = R_1^*(x_2) \tag{42}$$

where $R_1^*(x_2)$ is bounded in $[-h, h]$ (the case of loads $P_1(x_1)$, $P_2(x_1)$ concentrated along $x_1 = 0$ is not considered here).

According to [10], $q(y)$ may be expressed in the form

$$q(y) = \frac{q^*(y)}{(h^2 - y^2)^\alpha} \tag{43}$$

where $0 \leqq \mathrm{Re}\,(\alpha) < 1$, and $q^*(y)$ satisfies the Hölder condition near and at $\pm h$.

Next we consider the sectionally holomorphic function

$$\Phi(z) = \frac{1}{\pi} \int\limits_{-h}^{h} \frac{q(y)}{y - z}\, dy. \tag{44}$$

According to [10] we have

$$\Phi\{h + \lambda[h + (-1)^i\, x_2]\} = -\frac{q^*(h)}{(2h)^\alpha\, \lambda^\alpha [h + (-1)^i\, x_2]^\alpha \sin \pi\alpha} + \Phi_i{}^0(x_2)$$

$$\text{as} \quad x_2 \to (-1)^{i-1}\, h \quad (i = 1, 2) \tag{45}$$

$$\Phi(x_2) = \frac{\cot \pi\alpha}{(2h)^\alpha} \left[\frac{q^*(-h)}{(x_2 + h)^\alpha} - \frac{q^*(h)}{(x_2 - h)^\alpha\, e^{-i\pi\alpha}}\right] + \Phi_3{}^0(x_2), \quad \text{as} \quad x_2 \to \pm h$$

where

$$|\Phi_j{}^0(z)| < \frac{c_j}{(z \pm h)^{\alpha_0}}, \quad \mathrm{Re}\,\alpha_0 < \mathrm{Re}\,\alpha \tag{46}$$

in which c_j $(j = 1, 2, 3)$ are real constants. Substituting expressions (40) and (45) into (42), and taking into account the fact that $q(y)$ is an odd function of $y(q(-y) = -q(y))$ leads to the following characteristic equations for determining the power of the singularity α

$$\left\{\begin{array}{l} s_5 \cos \pi\alpha + d_1\lambda^{-\alpha} + d_2 + d_3\lambda^\alpha = 0 \\ \gamma_2\delta_1{}^0\alpha_2{}^2 \cos \pi\alpha + \phi_1(\alpha) = 0 \end{array}\right\} \tag{47}$$

where

$$\begin{aligned} \phi_i(\alpha) = r_5{}^{(i)}\alpha_4 - r_6{}^{(i)}\bar{\alpha}_5 + (r_8{}^{(i)}\alpha_4 + r_7{}^{(i)}\bar{\alpha}_5) \cos \left[2\alpha \tan^{-1}\left(\frac{\bar{\alpha}_5}{\alpha_4}\right)\right] \\ + (r_8{}^{(i)}\bar{\alpha}_5 - r_7{}^{(i)}\alpha_4) \sin \left[2\alpha \tan^{-1}\left(\frac{\bar{\alpha}_5}{\alpha_4}\right)\right] \quad (\text{here} \quad i = 1) \end{aligned} \tag{48}$$

and

$$\begin{aligned} & r_5{}^{(i)} = r_9{}^{(i)} - r_{12}^{(i)}, \quad r_6{}^{(i)} = r_{10}^{(i)} + r_{11}^{(i)}, \quad r_7{}^{(i)} = r_{10}^{(i)} - r_{11}^{(i)}, \quad r_8{}^{(i)} = r_9{}^{(i)} + r_{12}^{(i)} \\ & r_9{}^{(i)} = m_1 f_{2i-1,1} - f_{2i,2}, \quad r_{10}^{(i)} = -m_1 f_{2i-1,2} + f_{2i,1} \\ & r_{11}^{(i)} = m_2 f_{2i-1,1} - m_3 f_{2i,2}, \quad r_{12}^{(i)} = -m_2 f_{2i-1,2} + m_3 f_{2i,1}. \end{aligned} \tag{49}$$

By applying L'Hôpital's rule it can be shown that in the special case of material isotropy both of Eqs. (47) reduce to

$$(3 - 4\nu) \cos \pi\alpha + 2(\alpha - 1)^2 - 8\nu^2 + 12\nu - 5 = 0 \tag{50}$$

where ν denotes Poisson's ratio. This equation is identical to that derived by Gupta [3] for an isotropic semi-infinite strip.

Of particular interest is the stress component $\sigma_{11}(0, x_2)$. Using (29) and changing the order of integration gives

$$\sigma_{11}(0, x_2) = \frac{1}{\pi} \int_{-h}^{h} \left[\frac{\gamma_4}{y - x_2} + M_2(x_2, y) \right] q(y)\, dy + R_2(x_2) \tag{51}$$

where

$$M_2(x_2, y) = \int_0^\infty M_2^*(x_2, y, t)\, dt$$
$$R_2(x_2) = \frac{2A_{11}}{\pi} \int_0^\infty \frac{1}{\Delta_t} [w_3(x_2, t)\, \overline{P}_1(t) + w_4(x_2, t)\, \overline{P}_2(t)]\, dt \tag{52}$$

and

$$M_2^*(x_2, y, t) = \begin{Bmatrix} \Omega_I[z_3(x_2, t), z_4(x_2, t)] \\ \alpha_2{}^4 \Omega_{II}[w_3(x_2, t), w_4(x_2, t)] \end{Bmatrix}. \tag{53}$$

Here Ω_I and Ω_{II} are given respectively by Eqs. (36) and (37) with $n = 3$; z_i $(i = 3, 4)$ are given by (38); and furthermore

$$l_{i,2j-1} = [m_1 - (-1)^i\, m_2]\, C_{3j}\alpha_2{}^4/2, \quad l_{i,2j} = -[1 + (-1)^i\, m_3]\, C_{4j}\alpha_2{}^4/2$$
$$(i = 3, 4;\ j = 1, 2). \tag{54}$$

Following a procedure similar to that applied earlier to the integral Eq. (33), it can be shown from (51) that the dominant (unbounded) portion of $\sigma_{11}(0, x_2)$ can be expressed as

$$\sigma_{11}^d(0, x_2) = -\frac{q^*(h)}{(2h)^\alpha \sin \pi\alpha} \left[\frac{1}{(h - x_2)^\alpha} + \frac{1}{(h + x_2)^\alpha} \right] \psi(\alpha) \tag{55}$$

in which

$$\psi(\alpha) = \begin{Bmatrix} \gamma_4 \cos \pi\alpha + d_4\lambda^{-\alpha} + d_5 + d_6\lambda^\alpha \\ \gamma_4 \cos \pi\alpha - \alpha_2{}^2\, \phi_2(\alpha)/(2\delta_1{}^0) \end{Bmatrix} \tag{56}$$

where

$$d_4 = \frac{l_{32} - l_{31}}{s_5\varkappa_2}, \quad d_5 = \frac{l}{s_5} \left[\frac{l_{33} - l_{34}}{\varkappa_2} - \frac{l_{41} - l_{43}}{\varkappa_1} \right], \quad d_6 = \frac{l_{43} - l_{44}}{s_5\varkappa_1} \tag{57}$$

and where $\phi_2(\alpha)$ is given by (48) when $i = 2$.

Finally, we define stress-intensity factors T_1 and T_2 as

$$T_1 = \lim_{x_2 \to h} (2h)^\alpha (h - x_2)^\alpha \sigma_{11}(0, x_2)$$
$$T_2 = \lim_{x_2 \to h} (2h)^\alpha (h - x_2)^\alpha \sigma_{12}(0, x_2) \tag{58}$$

and let T_3 represent the ratio $T_3 = -T_2/T_1$. In this case it is found that

$$T_1 = -\frac{q^*(h)\,\psi(\alpha)}{\sin \pi\alpha}, \quad T_2 = q^*(h), \quad T_3 = \frac{\sin \pi\alpha}{\psi(\alpha)}. \tag{59}$$

4. Solution for Prescribed-Stress End Condition

For the case of boundary conditions (3) in which stresses are prescribed over the end $x_1 = 0$, Eq. (10) gives $q(x_2) = Q(x_2)$. From the first of conditions (3) and expression (29) we obtain, for material type I, the following equation governing $U'(x_2) \equiv dU/dx_2$

$$\int_{-h}^{h} \left[\frac{1}{x_2 - y} + M_3(x_2, y)\right] U'(y)\,dy = R_3(x_2) \tag{60}$$

in which

$$M_3(x_2, y) = \int_0^\infty [z_5(x_2, t)\,e^{-\varkappa_2(h-y)t} + z_6(x_2, t)\,e^{-\varkappa_1(h-y)t}]\,dt$$

$$R_3(x) = \frac{\pi P(x_2)}{A_{11}\gamma_3} - \frac{\pi\gamma_4}{A_{11}\gamma_3}\int_0^\infty \bar{Q}(p)\cos px_2\,dp$$
$$- \frac{2}{\gamma_3}\int_0^\infty \frac{1}{\Delta_t}\{w_3(t, x_2)\,[\bar{P}_1(t) + I_1^Q(t)] + w_4(t, x_2)\,[\bar{P}_2(t) + I_2^Q(t)]\}\,dt. \tag{61}$$

Here z_i $(i = 5, 6)$ are given by Eq. (38), and

$$l_{i,2j-1} = [m_4 + (-1)^i\,m_5]\,C_{3j}/4\gamma_3, \qquad l_{i,2j} = (-1)^i\,m_6 C_{4j}/4\gamma_3$$
$$(i = 5, 6;\ j = 1, 2) \tag{62}$$
$$m_4 = \lambda_1/\alpha_4, \qquad m_5 = \lambda_1/\alpha_5, \qquad m_6 = 2\lambda_2/\alpha_3.$$

The structure of the kernel $M_3(x_2, y)$ is the same as that of $M_1(x_2, y)$. An analysis similar to that presented earlier leads to the characteristic equation

$$s_5 \cos \pi\alpha_1 + \frac{(l_{51} - l_{52})\,\lambda^{-\alpha_1} - l_{53} + l_{54}}{\varkappa_2} + \frac{l_{61} - l_{62} - (l_{63} - l_{64})\,\lambda^{\alpha_1}}{\varkappa_1} = 0 \tag{63}$$

for determination of the singularity power α_1. In arriving at (63) we have taken $U'(y) = U^*(y)/(h^2 - y^2)^{\alpha_1}$ where $U^*(y)$ is bounded in $[-h, h]$. Eq. (63) is identical to the characteristic equation derived in [7] for the case of an edge crack. Based upon numerical calculations covering a wide range of orthotropic elastic constants, it can be concluded that Eq. (63) has no roots in the interval [0, 1]. Hence, in the case of the prescribed-stress boundary conditions (3) stress singularities do not occur at the corners of the strip. We shall not devote further attention to this case, but instead will examine numerical results for the fixed-end case.

5. Numerical Results and Discussion

The functions $M_k^*(x_2, y, t)$ $(k = 1, 2)$ have t^{-1} singularities at $t \to 0$. However since $q(y)$ is an odd function we can remove the singularities if, instead of Eq. (33) and (51), we write the equivalent expressions

$$\int_{-h}^{h} \left[\frac{1}{x_2 - y} + \Psi_1(x_2, y)\right] q(y)\, dy = R_1(x_2) \tag{33'}$$

$$\sigma_{11}(0, x_2) = \frac{1}{\pi} \int_{-h}^{h} \left[\frac{\gamma_4}{y - x_2} + \Psi_2(x_2, y)\right] q(y)\, dy + R_2(x_2) \tag{51'}$$

where

$$\begin{aligned}\Psi_k(x_2, y) = \int_0^1 &\{[z_{2k-1}(x_2, t) - z^*_{2k-1}(x_2, t)]\, e^{-\varkappa_2(h-y)t} \\ &+ [z_{2k}(x_2, t) - z^*_{2k}(x_2, t)]\, e^{-\varkappa_1(h-y)t}\}\, dt + \int_1^\infty M_k^*(x_2, y, t)\, dt \\ &+ \sum_{j=1}^{n} \frac{1}{j!j} \sum_{n=1}^{2} \{l_{2k-1,1}\varkappa_2^j [y - h + (-1)^n \lambda x_2]^j \\ &+ (l_{2k,1}\varkappa_1^j - l_{2k-1,3}\varkappa_2^j)\, [y - h + (-1)^n x_2]^j \\ &- l_{2k,3}[y - h + (-1)^n x_2/\lambda]^j\}/(2s_6) \qquad (k = 1, 2)\end{aligned} \tag{64}$$

in which

$$\begin{aligned} z_i^*(x_2, t) &= [l_{i1} \cosh(\varkappa_1 t x_2) - l_{i3} \cosh(\varkappa_2 t x_2)]/(t s_6) \qquad (i = 1, 2, 3, 4) \\ s_6 &= h(s_1 s_4 \varkappa_1 - s_2 s_3 \varkappa_2). \end{aligned} \tag{65}$$

Eq. (64) applies in the case of material type I; a similar expression can be obtained for material type II. For a solution to the integral Eq. (33′) we use a numerical method based upon the Gauss-Jacobi integration formula [9]. Eq. (33′)

must be solved subject to the following equilibrium condition

$$\int_{-h}^{h} q(y)\, dy = 0. \tag{66}$$

The corresponding system of algebraic equations is

$$\sum_{j=1}^{n} A_j \left[\frac{1}{z_m - \tau_j} + h\Psi_1(hz_m, h\tau_j)\right] G(h\tau_j) = R_1(x_m) \qquad \sum_{j=1}^{n} A_j G(h\tau_j) = 0 \tag{67}$$

where z_m $(m = 1, 2, \ldots, n-1)$ and τ_j $(j = 1, 2, \ldots, n)$ are the zeros of Jacobi polynomials $P_{n-1}^{(1-\alpha,1-\alpha)}(z)$ and $P_n^{(-\alpha,-\alpha)}(\tau)$, respectively. Here

$$G(h\tau_j) = (1 - \tau_j^2)^\alpha\, q(h\tau_j) \tag{68}$$

and the coefficients A_j are found to be

$$A_j = \frac{2^{1-2\alpha}\Gamma^2(n + 1 - \alpha)}{n!\,\Gamma(n + 1 - 2\alpha)\,(1 - \tau_j^2)\,[P_n'^{(-\alpha,-\alpha)}(\tau_j)]^2}. \tag{69}$$

Similarly, Eq. (51′) leads to the following approximate formula for $\sigma_{11}(0, x_2)$

$$\sigma_{11}(0, hz_m) = \frac{1}{\pi} \sum_{j=1}^{n} \left[\frac{\gamma_4}{\tau_j - z_m} + h\Psi_2(hz_m, h\tau_j)\right] G(h\tau_j) + R_2(hz_m). \tag{70}$$

In order to illustrate the effects of the orthotropy upon the behavior of the fixed-end strip, numerical results have been calculated for various combinations of the stiffness ratios $b_1 = A_{11}/A_{66}$, $b_2 = A_{22}/A_{66}$ and $b_3 = A_{12}/A_{66}$. Fig. 2 and 3 show that variations of the singularity power α (solid lines) and the stress-intensity factor ratio T_3 (dashed lines) associated with variations in b_2 and b_3, for the cases of $b_1 = 10$ and $b_1 = 28$, respectively. The circles and squares on these lines indicate, respectively, the limit points defining materials types *I* and *II*. From these figures it is evident that material orthotropy can have a significant influence upon both the power of the stress singularity and the stress-intensity factors. It is seen, for example, that an increase in $b_1 = A_{11}/A_{66}$ for fixed values of b_2 and b_3 results in a substantial decrease in α and T_3. It is noted also that as the material behavior approaches isotropy ($b_1 = b_2 = 2 + b_3$), the present results agree fully with those given in [3].

Figs. 4 and 5 show the variations of the stresses acting on the fixed end of a strip of half-width $h = 1$ caused by a pair of concentrated longitudinal forces

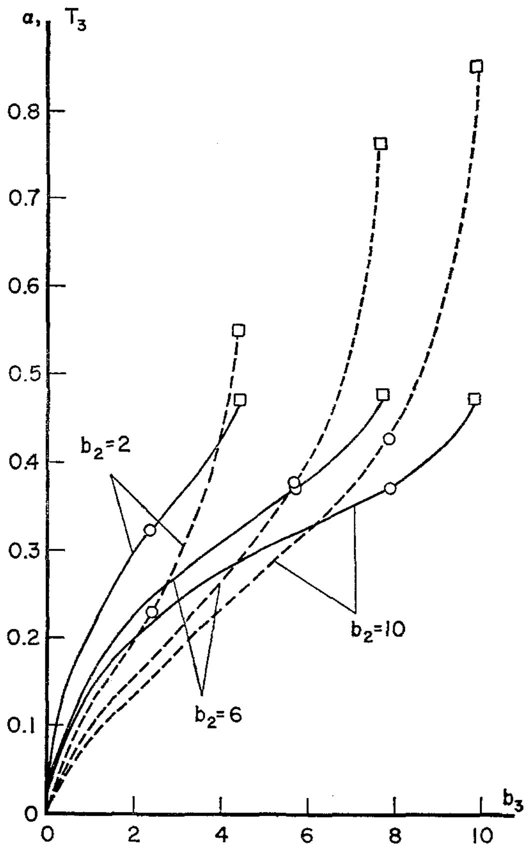


Fig. 2. Singularity power α and stress-intensity factor ratio T_3 for $b_1 = 10$

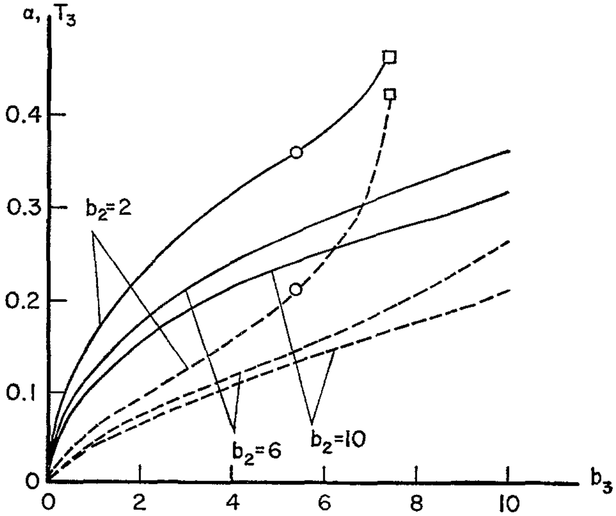
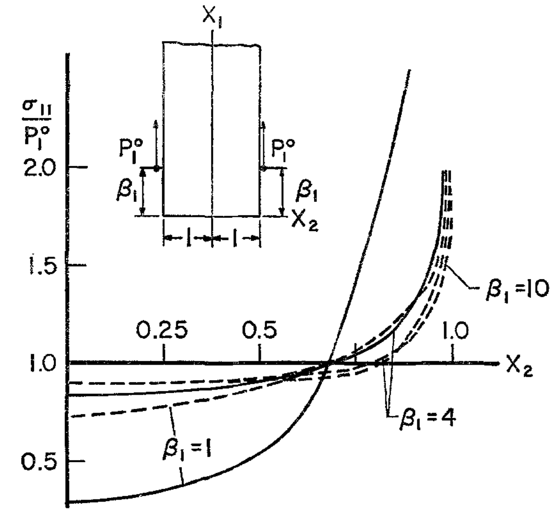


Fig. 3. Singularity power α and stress-intensity factor ratio T_3 for $b_1 = 28$



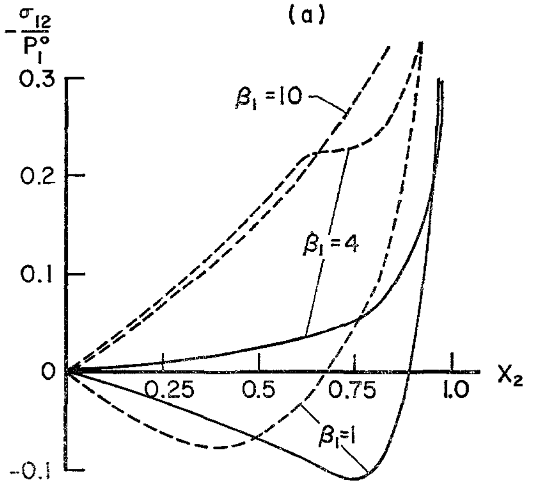
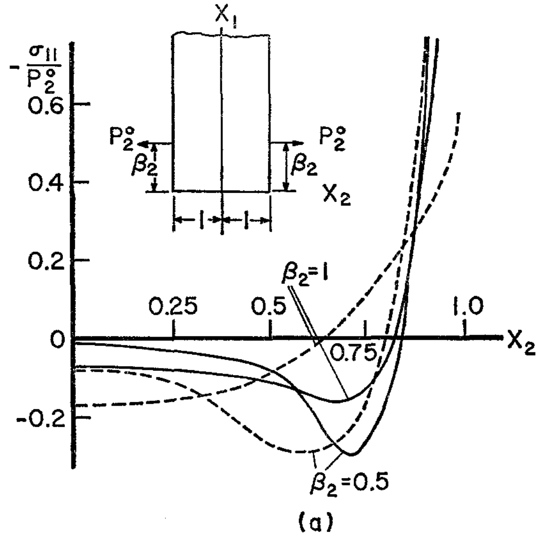


Fig. 4. Distributions of (a) normal stress and (b) shear stress due to longitudinal forces $P_1{}^0$

$(P_1(x_1) = P_1{}^0\delta(x_1 - \beta_1)$ with $P_2(x_1) = 0)$ and transverse forces $(P_1(x_1) = 0$ and $P_2(x_1) = P_2{}^0\delta(x_1 - \beta_2))$, respectively. The solid lines correspond to a typical boron-epoxy composite material ($b_1 = 26.9$, $b_2 = 3.6$, $b_3 = 3.35$, $\alpha = 0.248$), while the dashed lines relate to a nearly isotropic material ($b_1 = 4.33$, $b_2 = 4.35$, $b_3 = 2.33$, $\alpha = 0.315$). In accordance with St. Venant's principle, the distribution of stress in the case of a pair of longitudinal forces (Fig. 4) acting at a large distance ($\beta_1 = 10$) from the fixed end are in reasonably good agreement with the corresponding results for a strip loaded with a statically-equivalent uniformly-distributed force at $x_1 = \infty$ [3].



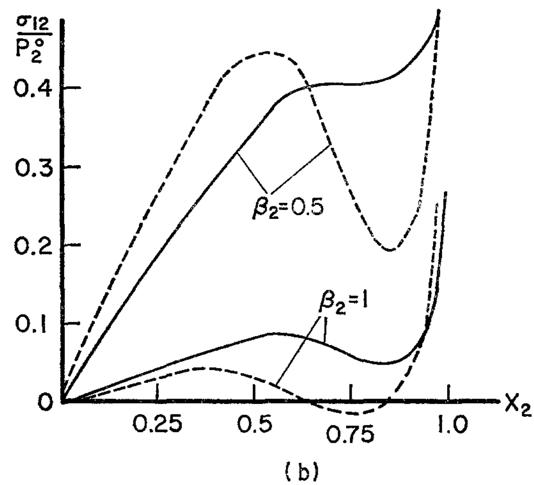


Fig. 5. Distributions of (a) normal stress and (b) shear stress due to transverse forces P_2^0

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