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A Minkowski-Type Theorem for Covering Minima in the Plane

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Abstract. For a convex body $K \subset E^2$ and a lattice $L \subset E^2$ let $\mu_i(K, L)$, $i = 1, 2$, denote its covering minima introduced by Kannan and Lovasz. We show $\mu_1(K, L)\mu_2(K, L)V(K) \geq \frac{3}{4}\det(L)$, where V denotes the area. This inequality is tight and there are five different cases of equality.

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1. Introduction and Results

In the following let E^d, $d \geq 2$, denote the Euclidean d-space, and the set of lattices $L \subset E^d$ with lattice determinant $\det(L) \neq 0$ is denoted by $\mathcal{L}^d$. For $L \in \mathcal{L}^d$ let $L^* = \{y \in E^d: x \cdot y \in \mathbb{Z}, \text{for all } x \in L\}$ denote the dual lattice. An unimodular transformation is defined by an integral matrix with determinant ± 1. An unimodular transformation maps a lattice onto itself. Let $\mathcal{K}^d$ denote the set of convex bodies – compact convex sets – in E^d and let $\mathcal{K}_0^d$ denote the subset of the 0-symmetric convex bodies, i.e. $K \in \mathcal{K}^d$ with $K = -K$. For $K \in \mathcal{K}^d$ let $V(K)$ denote its volume (for simplicity we use V also for $d = 2$, where it denotes the area).

For $K \in \mathcal{K}^d$ and $L \in \mathcal{L}^d$ Kannan and Lovasz (cf. [8]) introduced the covering minima

$$\begin{aligned}\mu_i(K, L) = \ &\min\{t > 0: tk + L \text{ meets every} \\ &(d-i)\text{-dim. affine subspace of } E^d\}, \quad i = 1, \dots, d.\end{aligned}$$

Clearly $0 = \mu_0 \leq \mu_1 \leq \cdots \leq \mu_d$ and $\mu_d(K, L)$ is the usual covering radius [3].

These functionals have been (implicitly) mentioned first by G. Fejes Tóth [4], who posed the following problem: Find the thinnest d-dimensional lattice of spheres such that every k-dimensional subspace $(0 \leq k \leq d-1)$ intersects some closed sphere of the lattice. For $k = d - 1$ this problem corresponds to the notion of a non-separable lattice of convex bodies (cf. [11], [5]).

The covering minima are a counterpart to the successive minima $\lambda_i(K, L)$, $1 \leq i \leq d$, for $K \in \mathcal{K}_0^d$, with respect to a lattice $L \in \mathcal{L}^d$, defined by

$$\lambda_i(K, L) = \min\{\lambda > 0: \dim(\lambda K \cap L) \geq i\}.$$

For the volume V and the successive minima hold Minkowski's fundamental theorems in *Geometry of Numbers* (cf. [6, p. 123]). For $K \in \mathcal{K}_0^d$ and $L \in \mathcal{L}^d$ holds

$$\lambda_1(K, L)^d V(K) \leq 2^d \det(L), \tag{1}$$

$$\lambda_1(K, L) \cdot \dots \cdot \lambda_d(K, L) V(K) \leq 2^d \det(L). \tag{2}$$

Both inequalities are tight and (2) is an improvement of (1) since $\lambda_1(K, L) \leq \dots \leq \lambda_d(K, L)$.

For $K \in \mathcal{K}^d$, $L \in \mathcal{L}^d$ and $u \in L$ we call $\max_{x,y \in K} u(x-y)$ the *lattice breadth* of K in the direction u. If we minimize over all $u \in L^*$ we obtain the *lattice width* λ_1^* and it holds (cf. [8], [11])

$$\frac{1}{\mu_1(K, L)} = \lambda_1^* = \lambda_1((K-K)^*, L^*) = \min_{u \in L^*} \max_{x,y \in K} u(x-y), \tag{3}$$

where $K^* = \{y : x \cdot y \leq 1, \text{ for all } x \in K\}$ denotes the polar reciprocal body of $K \in \mathcal{K}_0^d$. Equation (3) was first proved by Makai Jr [11] in terms of non-separable lattices and lattice packings.

G. Fejes Tóth [4], Mahler [9] and Makai Jr [11] raised the problem to find an inequality for the covering minima and the volume analogous to (1), i.e. they gave inequalities of the type

$$\mu_1(K, L)^d V(K) \geq e_d \cdot \det(L). \tag{4}$$

The best known results quoted by Betke, Henk and Wills [1] are immediate consequences of the theorems by Rogers and Shepard [12], Bourgain and Milman [2] and (1):

$$\mu_1(K, L)^d V(K) \geq \binom{2d}{d}^{-1} \left(\frac{c_1}{2d}\right)^d, \text{ for } K \in \mathcal{K}^d,$$

$$\mu_1(K, L)^d V(K) \geq \left(\frac{c_1}{4d}\right)^d, \text{ for } K \in \mathcal{K}_0^d.$$

Makai Jr conjectured that $e_d = 1/d!$ and $e_d = (d+1)/(2^d d!)$ are valid in (4) for $K \in \mathcal{K}_0^d$ and $K \in \mathcal{K}^d$, respectively. For the crosspolytope and the simplex respectively and suitable lattices he showed equality and so these constants could not be improved. The conjecture for the centrosymmetric case would follow, including the case of equality, if Mahler's conjecture [10], $V(K)V(K^*) \geq 4^d/d!$, would hold.

Both conjectures hold true in the Euclidean plane. The centrosymmetric case follows from the fact that Mahler's conjecture is true for $d = 2$ (cf. [6, p. 113]) and the general case,

$$\mu_1(K, L)^2 V(K) \geq \tfrac{3}{8} \det(L), \quad K \in \mathcal{K}^2, L \in \mathcal{L}^2, \tag{5}$$

has been proved by L. Fejes Toth and Makai Jr [5] with equality for the triangle in Figure 5 (and $L = \mathbb{Z}^2$).

The question to find an inequality for the covering minima and the volume analogous to (2),

$$\mu_1(K, L) \cdot \cdots \cdot \mu_d(K, L)V(K) \geq f_d \cdot \det(L), \tag{6}$$

has been raised by Betke, Henk and Wills [1]. Of course, every e_d in (4) is a valid f_d in (6).

This paper deals with the planar case. A result by Hurkens [7] states that $\mu_2(K, L) \leq (1 + \frac{2}{3}\sqrt{3})\mu_1(K, L)$ for $K \in \mathcal{K}^d$. If $K = -K$ then it follows from a result by Kannan and Lovasz (cf. [8]) that $\mu_2(K, L) \leq 2\mu_1(K, L)$. Here we give the following tight inequality of the type (6).

THEOREM. (a) *For $K \in \mathcal{K}^2$ and $L \in \mathcal{L}^2$ holds*

$$\mu_1(K, L)\mu_2(K, L)V(K) \geq \tfrac{3}{4} \det(L).$$

(b) *For each $L \in \mathcal{L}^2$ there is up to translations, dilatations and unimodular transformations exactly one triangle, one parallelogram, one trapezoid, one pentagon and one hexagon such that equality holds.*

For $L = \mathbb{Z}^2$ the polygons in (b) are

(A) the 0-symmetric hexagon

$$K = \operatorname{conv}\{\pm(\tfrac{1}{3}, \tfrac{1}{3}); \pm(-\tfrac{2}{3}, \tfrac{1}{3}); \pm(\tfrac{1}{3}, -\tfrac{2}{3})\},$$

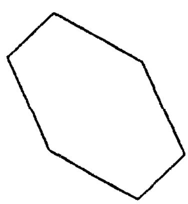

Fig. 1.

(B) the 0-symmetric parallelogram

$$K = \operatorname{conv}\{\pm(\tfrac{3}{8}, \tfrac{3}{8}); \pm(-\tfrac{3}{4}, \tfrac{3}{4})\},$$

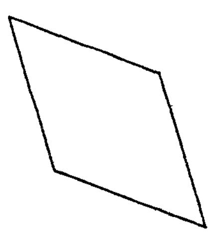

Fig. 2.

(C) the asymmetric pentagon

$$K = \operatorname{conv}\{\pm(b_0/4,\ b_0/4);\ (1 - b_0/4,\ -b_0/4);\ (b_0/4,\ b_0/4 - 1);\\ (1 - 5/4b_0,\ 5/4b_0 - 1)\},$$

where $b_0 = 1 + \frac{1}{5}\sqrt{5}$

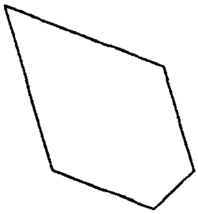

Fig. 3.

(D) the asymmetric trapezoid

$$K = \operatorname{conv}\{(\tfrac{9}{10}, \tfrac{-1}{10});\ (\tfrac{1}{10}, \tfrac{-9}{10});\ (\tfrac{-7}{10}, \tfrac{3}{10});\ (\tfrac{-3}{10}, \tfrac{7}{10})\},$$

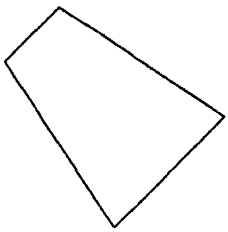

Fig. 4.

(E) the asymmetric triangle

$$K = \operatorname{conv}\{(1, -1);\ (-1, 0);\ (0, 1)\}.$$

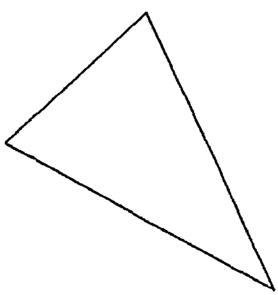

Fig. 5.

Since the hexagon and the parallelogram are 0-symmetric, the theorem cannot be improved for $K \in \mathcal{K}_0^2$.

2. Proof of the Theorem

Since all occurring sizes are equi-affine invariant it suffices to consider the lattice $L = \mathbb{Z}^2$. So in the following we will write $\mu_i(K)$ or just μ_i instead of $\mu_i(K, L)$. Further we can assume that $\mu_2 = 1$.

If $\mu_1 \geq \frac{3}{4}$ then $\mu_1\mu_2 V(K) = (\mu_1/\mu_2)V(\mu_2 K) \geq \frac{3}{4}$ since $K + \mathbb{Z}^2 = E^2$.

If $\mu_1 \leq \frac{1}{2}$ then $\mu_1\mu_2 V(K) = (\mu_2/\mu_1)V(\mu_1 K) \geq \frac{3}{4}$ because of (5).

Thus we set $b := \lambda_1^* = 1/\mu_1$ and in the following we consider the case

$$\tfrac{4}{3} < b < 2.$$

It is sufficient to prove that

$$V(K) \geq \frac{3b}{4}.$$

Since $\mathbb{Z}^2$ is a covering lattice for K, there is a centrally symmetric hexagon (possibly a quadrangle) H contained in K, such that $\mathbb{Z}^2 + H$ is a lattice tiling (cf. [6, Lemma 22.3]). We can assume that H has the center 0. Obviously $\mathbb{Z}^2$ is a critical lattice for $2H$ and hence the boundary of H contains three points $u,\ v,\ u-v \in \frac{1}{2}\mathbb{Z}^2$, where $\{2u,\ 2v\}$ is a basis of $\mathbb{Z}^2$ (cf. [6, Lemma 22.2]). We can assume that $u = (\frac{1}{2},\ 0)$ and $v = (0,\ \frac{1}{2})$ (else we apply a suitable unimodular transformation). Therefore H contains the hexagon $H' = \mathrm{conv}\{\pm(\frac{1}{2},\ 0);\ \pm(0,\ \frac{1}{2});\ \pm(\frac{1}{2},\ -\frac{1}{2})\}$. There is a point $p_0 = (x_0,\ y_0)$ in $W := \{(x,\ y)\colon |x|,\ |y| \leq \frac{1}{2}\}$ which is contained in the boundary of H and of two further translates. p_0 is contained either in the boundary of H' or in the interior of $W\backslash H'$ (the case that p_0 is contained in the boundary of $W\backslash H'$ can be reduced to the first one).

In the first case we can apply an unimodular transformation so that $y_0 = \frac{1}{2}$. The corresponding two translates are $e_2 + H$ and $(e_2 - e_1) + H$ and H is a parallelogram with two horizontal sides with length 1 and height 1. Since the breadth of K in the direction e_2 is $\geq b$ it follows that $K\backslash H$ contains two triangles with basis length 1 and total height of at least $(b-1)$ and hence $V(K) \geq 1 + (b-1)/2 > 3b/4$ since $b < 2$.

In the following we consider the case

$$0 < x_0,\ y_0 < \tfrac{1}{2};\ x_0 + y_0 > \tfrac{1}{2}.$$

The two translates which contain p_0 are $e_1 + H$ and $e_2 + H$ (else there are two overlapping translates). Hence the points $p_1 = p_0 - e_1$ and $p_2 = p_0 - e_2$ belong to H and we obtain the 0-symmetric convex hexagon $H = \mathrm{conv}\{\pm p_0,\ \pm p_1,\ \pm p_2\} \subset K$. Let g_1 be the line containing $-p_2$ and p_0, g_2 the line containing $-p_1$ and p_2 and g_3 the line containing $-p_0$ and p_1. Further, let $s_1,\ s_2,\ s_3$ be the points of intersection of the lines g_1 and g_2, g_2 and g_3 and g_3 and g_1, respectively (Figure 6).

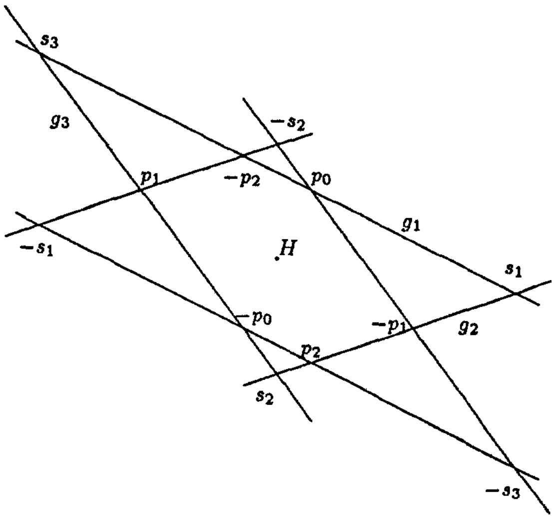

Fig. 6.

We can calculate the coordinates x_i, y_i of s_i elementary and we obtain

$$x_1 = x_0 + \frac{2x_0(1-2x_0)}{1-2y_0}, \qquad y_1 = 2x_0 + y_0 - 1,$$

$$x_2 = 1 - x_0 - 2y_0, \qquad y_2 = -y_0 - \frac{2y_0(1-2y_0)}{1-2x_0},$$

$$x_3 = x_0\frac{2x_0-2y_0-1}{2x_0+2y_0-1}, \qquad y_3 = y_0\frac{2x_0-2y_0+1}{2x_0+2y_0-1}.$$

Obviously the lattice breadths of H in the directions e_1, e_2 and e_1+e_2 equal $2(1-x_0)$, $2(1-y_0)$ and $2(x_0+y_0)$, respectively.

(a) If $1 - x_0 < b/2$ then $K\backslash H$ contains two triangles with basis $\overline{p_0p_2}$ and $-(\overline{p_0p_2})$, respectively, with total height at least $b - 2x_0$ less the triangles $\pm\text{conv}\{p_0, p_2, -p_1\}$. Hence

$$V(K\backslash H) \geq \frac{b}{2} - 1 + x_0.$$

(b) If $1 - y_0 < b/2$ then we have analogously

$$V(K\backslash H) \geq \frac{b}{2} - 1 + y_0.$$

(c) If $x_0 + y_0 < b/2$ then we can conclude like in (a). Here we use the triangles with basis $\overline{p_1p_2}$ and $-(\overline{p_1p_2})$, respectively, with length $\sqrt{2}$ and we obtain

$$V(K\backslash H) \geq \frac{b}{2} - (x_0 + y_0).$$

At least one of the cases (a), (b) or (c) occurs since from $(1 - x_0) \geq b/2$ and $(1 - y_0) \geq b/2$ follows $x_0 + y_0 \leq 2 - b < b/2$. We have to consider the following 7 cases

(1) $1 - x_0 \geq b/2$ and $1 - y_0 \geq b/2$,

(2) $1 - x_0 \geq b/2$ and $x_0 + y_0 \geq b/2$,

(3) $1 - x_0 \geq b/2$ and $x_0 + y_0 \geq b/2$,

(4) $x_0 + y_0 \geq b/2$ and $1 - x_0,\ 1 - y_0 < b/2$,

(5) $1 - y_0 \geq b/2$ and $x_0 + y_0,\ 1 - x_0 < b/2$,

(6) $1 - x_0 \geq b/2$ and $x_0 + y_0,\ 1 - y_0 < b/2$,

(7) $x_0 + y_0,\ 1 - x_0,\ 1 - y_0 < b/2$.

The substitution $v_0 = 1 - x_0 - y_0$, $w_0 = x_0$ is corresponding to an unimodular transformation and reduces the case (2) to (1) and (5) to (4). Analogously $v_0 = 1 - x_0 - y_0$, $w_0 = y_0$ reduces (3) to (1) and (6) to (4) and so it suffices to consider the cases (1), (4) and (7).

In case (1) we have $x_0 + y_0 \leq 2 - b < b/2$ and in (c) we obtain an additional area of at least $b/2 - (x_0 + y_0) \geq 3b/2 - 2$. Since $b > \frac{4}{3}$ it follows $V(K) \geq 3b/2 - 1 > 3b/4$.

In case (4) one of the additional areas in (a) and (b) is as least as large as their mean value $b/2 - 1 + \frac{1}{2}(x_0 + y_0)$. It follows $V(K) \geq b/2 + \frac{1}{2}(x_0 + y_0) \geq 3b/4$.

For case (7) we need some extra investigations. If p_0 and $-p_0$ are contained in the interior of K then $\mu_2(K) < 1$. Hence let $-p_0$ be contained in the boundary of K and by the same reason $-p_1$ and $-p_2$ are contained in the boundary of K, too. It follows that $K\backslash H$ can be decomposed into three parts that are contained in the semi-stripes

the p_1-side of the stripe between g_1 and $-g_1$,
the p_0-side of the stripe between g_2 and $-g_2$,
the p_2-side of the stripe between g_3 and $-g_3$.

In each of these semi-stripes can be contained parts of the areas from (a), (b) and (c), which possibly may overlap. We can restrict ourselves to the case that K contains in each semi-stripe at most three further vertices which guarantee that the lattice breadths in the directions e_1, e_2 and $e_1 + e_2$ are at least b. Now it happens that the parts that participated

in (b) and (c) in the first semi-stripe,
in (a) and (b) in the second semi-stripe,
in (a) and (c) in the third semi-stripe,

do not overlap (they possibly have one side in common). Thus the areas in (a), (b) and (c) are counted at most twice and it follows

$$V(K\backslash H) \geq \frac{1}{2}\left(\frac{b}{2} - 1 + x_0 + \frac{b}{2} - 1 + y_0 + \frac{b}{2} - x_0 - y_0\right) = \frac{3b}{4} - 1,$$

and hence $V(K) \geq 3b/4$.

We still have to investigate the case of equality.

If $b = \frac{4}{3}$, equality only holds for $V(\mu_2 K) = 1$, i.e. $K = H$. Since $1 - x_0$, $1 - y_0$, $x_0 + y_0 \geq b/2 = \frac{2}{3}$ it follows that $x_0 = y_0 = \frac{1}{3}$ and we obtain the hexagon described in (A).

If $b = 2$, equality only holds for $V(\mu_1 K) = \frac{3}{8}$. From the equality case in (5) it follows that K is equivalent to the triangle described in (E).

For $\frac{4}{3} < b < 2$ equality cannot hold if H is a parallelogram and in cases (1)–(3).

For equality in (4) we must have $x_0 + y_0 = b/2$ and the areas in (a) and (b) must be identical, in particular $x_0 = y_0 = b/4$. Further, in each of the estimations equality holds. This is only possible by the addition of the vertices $\pm s_3$ or one of it to H and if the lattice breadths in the directions e_1 and e_2 are exactly b.

(α) In the first case (addition of s_3 and $-s_3$) we have $-x_3 = y_3 = b/2$ and hence $b/2 = x_0/(4x_0 - 1) = b/(4(b-1))$, i.e. $b = \frac{3}{2}$. We obtain the symmetric parallelogram $\text{conv}\{\pm p_0, \pm s_3\}$ described in (B).

(β) In the second case (addition of s_3) we have $1 - x_0 - x_3 = b = y_3 - y_0 + 1$ and it follows that $1 - x_0 + x_0/(4x_0 - 1) = b$ or $1 + b/(4(b-1)) = 5b/4$. The only possible solution is $b = 1 + \frac{1}{5}\sqrt{5} = b_0$ and we obtain the pentagon $\text{conv}\{\pm p_0, -p_1, p_2, s_3\}$ described in (C).

For equality in (7) all parts of $K\backslash H$ in (a), (b) and (c) must be counted twice and in the corresponding estimations equality must hold. This is only possible by the addition of some points of the set $\{\pm s_1, \pm s_2, \pm s_3\}$, more exactly either two or three of these points. Further, the lattice breadths in the directions e_1, e_2 and $e_1 + e_2$ are exactly b.

(α) If we add three points we obtain the triangle $K = \text{conv}\{s_1, s_2, s_3\}$ (or $K = \text{conv}\{-s_1, -s_2, -s_3\}$). Since $K + \mathbb{Z}^2$ is a lattice covering and the thinnest lattice covering with triangles has density $\frac{3}{2}$ (cf. [6, Th. 22.7]), it follows that $V(K) = V(K)/\det(\mathbb{Z}^2) \geq \frac{3}{2} > 3b/4$.

(β) If we add two points then one of the areas in (a), (b) and (c) can be decomposed in two parts which are identical with the areas added in the other cases. We can assume (by an unimodular transformation) that the area corresponding to (c) can be decomposed and we obtain the trapezoid $K = \text{conv}\{s_1, s_2, p_1, -p_2\}$. Further, we have $b/2 - x_0 - y_0 = 3b/4 - 1$, i.e. $x_0 + y_0 = 1 - b/4$. From the condition $1 - y_0 - y_2 = b = x_1 - x_0 + 1$ it follows that $x_0(1 - 2x_0)^2 = y_0(1 - 2y_0)^2$. An elementary discussion of the function $f(t) = t(1-2t)^2$ together with $x_0 + y_0 > \frac{1}{2}$ gives that $x_0 = y_0 = \frac{1}{2} - b/8$.

On the other hand, from the condition $x_1 + y_1 - x_2 - y_2 = b$ it follows with $x_0 = y_0$ that $x_0 = (b+2)/12$. From the two representations of x_0 it follows that $b = \frac{8}{5}$, $x_0 = y_0 = \frac{3}{10}$ and we obtain the trapezoid described in (D).

At the end one easily verifies that for each body given in (A)–(E) it holds that $\mu_2 = 1$ and $V = 3b/4$.

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