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Affine Differential Geometry of Surfaces in $\mathbb{R}^4$

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Abstract. Employing the method of moving frames, i.e. Cartan's algorithm, we find a complete set of invariants for nondegenerate oriented surfaces M^2 in $\mathbb{R}^4$ relative to the action of the general affine group on $\mathbb{R}^4$. The invariants found include a normal bundle, a quadratic form on M^2 with values in the normal bundle, a symmetric connection on M^2 and a connection on the normal bundle. Integrability conditions for these invariants are also determined. Geometric interpretations are given for the successive reductions to the bundle of affine frames over M^2, obtained by using the method of moving frames, that lead to the aforementioned invariants. As applications of these results we study a class of surfaces known as harmonic surfaces, finding for them a complete set of invariants and their integrability conditions. Further applications involve the study of homogeneous surfaces; these are surfaces which are fixed by a group of affine transformations that act transitively on the surface. All homogeneous harmonic surfaces are determined.

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0. Introduction

We use the method of moving frames to study immersions $\mathbf{x}: M^2 \to \mathbb{R}^4$. We wish to find invariants induced on M^2 by $\mathbf{x}$ such that if two immersions of M^2 into $\mathbb{R}^4$ are given whose corresponding induced invariants are the same, then the immersions differ by an affine transformation of $\mathbb{R}^4$. That is, we seek a complete set of invariants for surfaces under the action of the affine group on $\mathbb{R}^4$. Moreover, given such a complete set of invariants on M^2, we wish to find integrability conditions for these invariants that insure the existence of an immersion $\mathbf{x}: M^2 \to \mathbb{R}^4$ whose induced invariants are the given ones. Some work has already been done on this problem, most notably by Klingenberg [1], [2] and Wilkinson [7]. Recently, Nomizu and Vrancken [4] developed a new theory for the corresponding equiaffine problem; of particular interest is that one will find in this paper a comparison of previous approaches. Wilkinson used the method of moving frames but was motivated by Klingenberg and earlier work of Weise [5], [6] in making his reductions of the bundle of all affine frames induced over the surface M^2. We have tried to apply Cartan's 'algorithm' as faithfully as possible in making reductions. In so doing, we succeeded in finding the desired invariants for suitable nondegeneracy conditions. Invariants were also found under geometrically interesting cases where the nondegeneracy conditions fail.

The first section of this paper is primarily concerned with constructing a semi-conformal structure on the oriented surface M^2 induced by the immersion $\mathbf{x}$. What is meant by a semiconformal structure is defined in this section. This structure is not new; in fact, every author dealing with this problem has observed it. Our first nondegeneracy condition – as well as everyone else's – is that the induced semiconformal structure is nondegenerate. This allows for the completion of a second-order reduction of the bundle of affine frames induced over M^2 by $\mathbf{x}$. Depending on whether the nondegenerate semiconformal structure is definite – in which case we refer to it as a conformal structure – or not, we have what is called an elliptic or hyperbolic immersion, respectively.

In Section 2 we proceed with the reduction of the frame bundle assuming that the immersion is hyperbolic. In course of doing so, a number of new invariants are discovered, amongst them some quadratic forms. Other invariants are found as well, including a normal bundle, a normal bundle-valued quadratic form on M^2, a symmetric connection on M^2, and a connection on the normal bundle. To complete the reduction, we must impose an additional nondegeneracy condition. A couple of different such nondegeneracy conditions are considered. In any case, the ultimate third-order reductions produce framings which are specializations of earlier framings found by Klingenberg.

In Section 3 we are primarily concerned with understanding the geometry of the reductions made in the earlier sections. We find structures which are introduced in making the second-order reductions and the conditions imposed on these structures in making third-order reductions. We also prove existence and uniqueness theorems related to the invariants we have introduced. These invariants are a normal bundle-valued quadratic form and connections on M^2 and the normal bundle. The integrability conditions for these invariants consist of four equations, two of which involve just the curvatures of the connections on M^2 and the normal bundle and two of which involve up to the second derivatives of these curvatures.

In Section 4 we look at an interesting collection of immersed elliptic surfaces which we call harmonic immersions. These surfaces are called harmonic because the immersions defining these surfaces are harmonic with respect to natural induced complex structure that exists on every elliptic surface. It turns out that a complete set of invariants for these surfaces is the induced normal bundle-valued quadratic form together with the induced connection on the normal bundle.

Finally in Section 5 we study homogeneous nondegenerate immersions. We classify large numbers of such surfaces and give many examples of such surfaces.

There are occasions where expressions appear which contain terms whose factors have repeated indices, one an upper index and the other a lower index, for which no sum is intended. Because of this, sums over repeated indices are to be reserved to expressions in which the symbol Σ explicitly appears.

1. The Affine Semiconformal Structure

The space $\mathcal{F}$ of affine frames is defined by

$$\mathcal{F} = \{(\mathbf{p}, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4) \in (\mathbb{R}^4)^5 : \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3 \wedge \mathbf{e}_4 \neq 0\}.$$

We denote a frame by $(\mathbf{p}, \mathbf{e})$ so that $\mathbf{e}$ stands for $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4)$. It is clear that $\mathcal{F}$ is an open submanifold of $\mathbb{R}^{20}$ with two connected components. We will use the index range $1 \le r,\ s,\ t \le 4$. If we regard the components of $(\mathbf{p}, \mathbf{e}) \in \mathcal{F}$ as determining $\mathbb{R}^4$-valued functions on $\mathcal{F}$, $\mathbf{p}\colon \mathcal{F} \to \mathbb{R}^4$ and $\mathbf{e}_r : \mathcal{F} \to \mathbb{R}^4$, then we may define 1-forms σ^r and ω^s_r on $\mathcal{F}$ by

$$\begin{aligned} d\mathbf{p} &= \sum \mathbf{e}_r \sigma^r, \\ d\mathbf{e}_r &= \sum \mathbf{e}_s \omega^s_r. \end{aligned} \tag{1.1}$$

The structure equations are

$$\begin{aligned} d\sigma^r &= -\sum \omega^r_t \wedge \sigma^t, \\ d\omega^r_s &= -\sum \omega^r_t \wedge \omega^t_s. \end{aligned} \tag{1.2}$$

If we identify $\mathcal{F}$ with $\mathbf{A}(4,\ \mathbb{R})$, the affine group of $\mathbb{R}^4$, in the usual way, then the twenty 1-forms σ^r, ω^s_r form a basis for the left-invariant 1-forms on $\mathbf{A}(4,\ \mathbb{R})$.

The map $\mathbf{p}\colon \mathcal{F} \to \mathbb{R}^4$ makes $\mathcal{F}$ into a principal fiber bundle over $\mathbb{R}^4$ with fiber $G_{\mathbf{p}} = \mathrm{GL}(4,\ \mathbb{R})$. Clearly the set of forms $\{\sigma^r\}$ span the semibasic forms of the projection $\mathbf{p}\colon\ \mathcal{F} \to \mathbb{R}^4$.

Let M be an oriented surface and $\mathbf{x}\colon\ M \to \mathbb{R}^4$ be a smooth immersion. We define the zeroth-order frame bundle of $\mathbf{x}$, $\mathcal{F}^{(0)}_{\mathbf{x}}$, to be the pullback under $\mathbf{x}$ of the bundle $\mathbf{p}\colon\ \mathcal{F} \to \mathbb{R}^4$, i.e.

$$\mathcal{F}^{(0)}_{\mathbf{x}} = \{(m,\ \mathbf{p},\ \mathbf{e}) \in M \times \mathcal{F} :\ \mathbf{x}(m) = \mathbf{p}\}.$$

We will identify m with $\mathbf{x}(m)$. Thus

$$\mathcal{F}^{(0)}_{\mathbf{x}} = \{(m,\ \mathbf{e}) \in \mathcal{F} :\ m \in M\}.$$

From now on $\mathbf{x}$ will denote, in addition to itself, its pullback to $\mathcal{F}^{(0)}_{\mathbf{x}}$. Thus on $\mathcal{F}^{(0)}_{\mathbf{x}}$, $d\mathbf{x} = \Sigma \mathbf{e}_r \sigma^r$; hence the 1-forms σ^r (restricted to $\mathcal{F}^{(0)}_{\mathbf{x}}$) are semibasic.

We define the first-order frame bundle, $\mathcal{F}^{(1)}_{\mathbf{x}}$, by

$$\mathcal{F}^{(1)}_{\mathbf{x}} = \{(m,\ \mathbf{e}) \in \mathcal{F}^{(0)}_{\mathbf{x}} :\ \sigma^3 = \sigma^4 = 0 \quad \text{and} \quad \sigma^1 \wedge \sigma^2 > 0 \text{ at } (m,\ \mathbf{e})\}.$$

The group of the bundle $\mathbf{p}\colon \mathcal{F}^{(1)}_{\mathbf{x}} \to M,\ G_1$, is defined by

$$G_1 = \left\{ \begin{pmatrix} \mathbf{a} & \mathbf{c} \\ \mathbf{0} & \mathbf{b} \end{pmatrix} :\ \mathbf{c} \in \mathbb{R}^{2\times 2},\ \mathbf{a},\ \mathbf{b} \in \mathrm{GL}(2,\ \mathbb{R}) \text{ and } \det \mathbf{a} > 0 \right\},$$

where $\mathbb{R}^{2\times 2}$ denotes the space of 2×2 matrices.

We will use the index ranges $1 \le i,\ j,\ k \le 2$, and $3 \le \alpha,\ \beta,\ \gamma \le 4$. The structure equations (1.2) along with Cartan's lemma imply that there exist smooth real valued-functions A^α_{ij} on $\mathcal{F}^{(1)}_{\mathbf{x}}$ such that $A^\alpha_{ij} = A^\alpha_{ji}$ and $\omega^\alpha_i = \Sigma A^\alpha_{ij}\sigma^j$. Define a semibasic quadratic form ϕ on $\mathcal{F}^{(1)}_{\mathbf{x}}$ by

$$\phi = \omega^3_1\omega^4_2 - \omega^4_1\omega^3_2.$$

The determinant as a function on $\mathbb{R}^{2\times 2}$ is a quadratic form. We denote the associated symmetric bilinear form by $|,|$ so that $|\mathbf{a},\ \mathbf{a}| = \det \mathbf{a}$. If we let $\boldsymbol{A}_i = (A^\alpha_{ij})$, then

$$\phi(\mathbf{e}_i,\ \mathbf{e}_j) = |\boldsymbol{A}_i,\ \boldsymbol{A}_j|. \tag{1.3}$$

We wish to study how ϕ varies along the fibers of $\mathbf{p}$: $\mathcal{F}^{(1)}_{\mathbf{x}} \to M$. Let $(m,\ \mathbf{e})$ and $(m,\ \tilde{\mathbf{e}})$ denote two frames in $\mathcal{F}^{(1)}_{\mathbf{x}}$. Let ω^α_i and ϕ (respectively, $\tilde{\omega}^\alpha_i$ and $\tilde{\phi}$) denote the values of these forms at $(m,\ \mathbf{e})$ (respectively, $(m,\ \tilde{\mathbf{e}})$).

LEMMA 1.1. *If* $\tilde{\mathbf{e}} = \mathbf{e}\left(\begin{smallmatrix}\mathbf{a} & \mathbf{c}\\ \mathbf{0} & \mathbf{b}\end{smallmatrix}\right)$, *then* $\tilde{\phi} = (\det \mathbf{a}/\det \mathbf{b})\phi$.

Proof. Let

$$\boldsymbol{\mu} = \begin{pmatrix}\omega^3_1 & \omega^3_2\\ \omega^4_1 & \omega^4_2\end{pmatrix} \quad \text{and} \quad \tilde{\boldsymbol{\mu}} = \begin{pmatrix}\tilde{\omega}^3_1 & \tilde{\omega}^3_2\\ \tilde{\omega}^4_1 & \tilde{\omega}^4_2\end{pmatrix}.$$

It is straightforward to show that $\tilde{\boldsymbol{\mu}} = \mathbf{b}^{-1}\boldsymbol{\mu}\mathbf{a}$. Now just take the determinant. □

By a *semiconformal structure compatible with a quadratic form* Q we mean

$$\{rQ : r \text{ is real and } \neq 0\}.$$

Lemma 1.1 says that the quadratic form ϕ on $\mathcal{F}^{(1)}_{\mathbf{x}}$ induces a semiconformal structure on the tangent space at each point of M, or, for short, induces a semiconformal structure on M. We denote this semiconformal structure on M by ϕ and call it the *affine semiconformal structure* induced by $\mathbf{x}$.

We make the first nondegeneracy assumption:

(I) ϕ is everywhere nondegenerate on M

If ϕ is definite, we say that $\mathbf{x}$ is *elliptic.* If ϕ is indefinite, we say that $\mathbf{x}$ is *hyperbolic.*

If $\mathbf{x}$ is elliptic, then the orientation of M and ϕ determine a complex analytic structure on M. For hyperbolic $\mathbf{x}$, the structure ϕ is completely determined by a pair of rank 1 distributions on M which are everywhere transversal; the union of these distributions is, of course, the set of null vectors of ϕ.

For hyperbolic $\mathbf{x}$ we can make the following partial second-order reduction:

$$\mathcal{F}_{\mathbf{x}}^{(\phi)} = \{(m,\ \mathbf{e}) \in \mathcal{F}_{\mathbf{x}}^{(1)} : \phi(\mathbf{e}_i,\ \mathbf{e}_i) = 0\}.$$

If $\mathbf{x}$ is elliptic, we complexify $\mathcal{F}_{\mathbf{x}}^{(0)}$ and replace $\mathcal{F}_{\mathbf{x}}^{(1)}$ as defined above by

$$\mathcal{F}_{\mathbf{x}}^{(1)} = \{(m,\ \mathbf{e}_1,\ \mathbf{e}_{\bar{1}},\ \mathbf{e}_2,\ \mathbf{e}_{\bar{2}}) \in \mathcal{F}_{\mathbf{x}}^{(0)} \otimes \mathbb{C} : \mathbf{e}_{\bar{1}} = \overline{\mathbf{e}_1},\ \mathbf{e}_{\bar{2}} = \overline{\mathbf{e}_2},$$
$$\Re\mathbf{e}_1 \text{ and } \Im\mathbf{e}_1 \text{ are tangent to } M \text{ at } m, \text{ and } -i\mathbf{e}_1 \wedge \mathbf{e}_{\bar{1}} > 0\},$$

where $\Re\mathbf{e}_1$ and $\Im\mathbf{e}_1$ denote the real and imaginary parts of $\mathbf{e}_1$, respectively. Then $\mathcal{F}_{\mathbf{x}}^{(\phi)}$ is defined just as in the hyperbolic case but in this setting $\mathbf{e} = (\mathbf{e}_1,\ \mathbf{e}_{\bar{1}},\ \mathbf{e}_2,\ \mathbf{e}_{\bar{2}})$. Thus, $(m,\ \mathbf{e}) \in \mathcal{F}_{\mathbf{x}}^{(\phi)}$ if and only if $\mathbf{e}_1$ is a vector of type (1, 0).

We will continue to use the same index ranges when $\mathbf{x}$ is hyperbolic. However, for elliptic $\mathbf{x}$ we will use $1,\ \bar{1}$ in place of $1,\ 2$ and $2,\ \bar{2}$ in place of $3,\ 4$. We also introduce the following convention. For a given index n which may have one of two values, $\bar{n}$ will denote the other possible value. For elliptic $\mathbf{x}$, since $\overline{\mathbf{e}_r} = \mathbf{e}_{\bar{r}}$ it follows that $\overline{\sigma^i} = \sigma^{\bar{i}}$, $\overline{\omega_s^r} = \omega_{\bar{s}}^{\bar{r}}$ and $\overline{A_{ij}^{\alpha}} = A_{\bar{i}\bar{j}}^{\bar{\alpha}}$.

2. Completing the Reduction for Hyperbolic Immersions

In this section we assume that $\mathbf{x}$ is hyperbolic. From (1.3) it follows that on $\mathcal{F}_{\mathbf{x}}^{(\phi)}$, $\det \boldsymbol{A}_i = 0$ and $|\boldsymbol{A}_i,\ \boldsymbol{A}_{\bar{i}}|$ is nowhere zero. This implies that rank $\boldsymbol{A}_i = 1$ everywhere on $\mathcal{F}_{\mathbf{x}}^{(\phi)}$. If ${}^t(A_{12}^3\ \ A_{12}^4) \neq 0$, then one can show that ${}^t(A_{11}^3\ \ A_{11}^4)$ and ${}^t(A_{22}^3\ \ A_{22}^4)$ are linearly dependent. But this contradicts $|\boldsymbol{A}_i,\ \boldsymbol{A}_{\bar{i}}| \neq 0$. Hence $A_{i\bar{i}}^{\alpha} = 0$.

LEMMA 2.1. *For all $m \in M$, there exist $(m,\ \mathbf{e}) \in \mathcal{F}_{\mathbf{x}}^{(\phi)}$ at which $\omega_1^3 = \sigma^1$, $\omega_2^3 = 0$, $\omega_1^4 = 0$, $\omega_2^4 = \sigma^2$, i.e. $A_{11}^3 = A_{22}^4 = 1$ and all other $A_{ij}^{\alpha} = 0$.*

Proof. Let $(m,\ \mathbf{e}) \in \mathcal{F}_{\mathbf{x}}^{(\phi)}$. Set $\tilde{\mathbf{e}}_i = \mathbf{e}_i$, $\tilde{\mathbf{e}}_3 = \Sigma\mathbf{e}_{\alpha}A_{11}^{\alpha}$, and $\tilde{\mathbf{e}}_4 = \Sigma\mathbf{e}_{\alpha}A_{22}^{\alpha}$. This gives the required result since $(m,\ \tilde{\mathbf{e}}_1,\ \tilde{\mathbf{e}}_2,\ \tilde{\mathbf{e}}_3,\ \tilde{\mathbf{e}}_4) \in \mathcal{F}_{\mathbf{x}}^{(\phi)}$. □

Now define the second-order frame bundle, $\mathcal{F}_{\mathbf{x}}^{(2)}$, by

$$\mathcal{F}_{\mathbf{x}}^{(2)} = \{(m,\ \mathbf{e}) \in \mathcal{F}_{\mathbf{x}}^{(\phi)} : \text{all the conditions in Lemma 2.1 hold}\}.$$

This is a G_2-bundle over M. If we set $\mathrm{s} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, then

$$G_2 = \left\{ \begin{pmatrix} \mathbf{a} & \mathbf{c} \\ \mathbf{0} & \mathbf{b} \end{pmatrix} \in G_1 : \right.$$
$$\left. \mathbf{a} \text{ or } \mathbf{sa} \text{ is diagonal and } \mathbf{b} = \mathbf{a}^2 \text{ or } \mathbf{sb} = (\mathbf{sa})^2, \text{ resp.} \right\}.$$

LEMMA 2.2. *The forms* $\omega_i^{\bar{\imath}}$, $\omega_\alpha^{\bar{\alpha}}$, $2\omega_1^1 - \omega_3^3$ *and* $2\omega_2^2 - \omega_4^4$ *are semibasic on* $\mathcal{F}_{\mathbf{x}}^{(2)}$.

Proof. In general $\omega_i^\alpha = \Sigma A_{ij}^\alpha \sigma^j$. Take the exterior derivative of this equation to get

$$\sum \left(\mathrm{d}A_{ij}^\alpha - \sum A_{il}^\alpha \omega_j^l - \sum A_{lj}^\alpha \omega_i^l + \sum A_{ij}^\beta \omega_\beta^\alpha\right) \wedge \sigma^j = 0.$$

Thus, by Cartan's lemma and the fact that A_{ij}^α is symmetric in i, j, there exists scalars C_{ijk}^α, symmetric in the three lower indices, such that

$$\mathrm{d}A_{ij}^\alpha - \sum A_{il}^\alpha \omega_j^l - \sum A_{lj}^\alpha \omega_i^l + \sum A_{ij}^\beta \omega_\beta^\alpha = \sum C_{ijk}^\alpha \sigma^k.$$

For the A_{ij}^α on $\mathcal{F}_{\mathbf{x}}^{(2)}$ this becomes

$$\omega_3^3 - 2\omega_1^1 = \sum C_{11i}^3 \sigma^i; \qquad \omega_4^4 - 2\omega_2^2 = \sum C_{22i}^4 \sigma^i; \tag{2.1}$$

$$-\omega_2^1 = \sum C_{12i}^3 \sigma^i; \qquad -\omega_1^2 = \sum C_{21i}^4 \sigma^i; \tag{2.2}$$

$$\omega_4^3 = \sum C_{22i}^3 \sigma^i; \qquad \omega_3^4 = \sum C_{11i}^4 \sigma^i. \quad \square \tag{2.3}$$

LEMMA 2.3. *For all* $m \in M$, *there exist* $(m, \mathbf{e}) \in \mathcal{F}_{\mathbf{x}}^{(2)}$ *at which* $C_{122}^3 = C_{211}^4 = 0$, *i.e.*, $\omega_2^1 \wedge \sigma^1 = \omega_1^2 \wedge \sigma^2 = 0$.

Proof. Let $(m, \mathbf{e})$ be given. Set $\tilde{\mathbf{e}}_i = \mathbf{e}_i$, $\tilde{\mathbf{e}}_3 = -\mathbf{e}_2 C_{211}^4 + \mathbf{e}_3$ and $\tilde{\mathbf{e}}_4 = -\mathbf{e}_1 C_{122}^3 + \mathbf{e}_4$. Then the new frame $(m, \tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2, \tilde{\mathbf{e}}_3, \tilde{\mathbf{e}}_4)$ is of the sort we claimed existed. $\square$

We make a partial third-order reduction:

$$\mathcal{F}_{\mathbf{x}}^{(g)} = \{(m, \mathbf{e}) \in \mathcal{F}_{\mathbf{x}}^{(2)} : C_{122}^3 = C_{211}^4 = 0 \text{ at } (m, \mathbf{e})\}.$$

LEMMA 2.4. *Each of the* 1*-forms* ω_4^1, ω_3^2, $\mathrm{d}C_{121}^3 - C_{121}^3\omega_2^2$, $\mathrm{d}C_{212}^4 - C_{212}^4\omega_1^1$, $\mathrm{d}C_{222}^3 - C_{222}^3(3\omega_2^2 - 2\omega_1^1)$ *and* $\mathrm{d}C_{111}^4 - C_{111}^4(3\omega_1^1 - 2\omega_2^2)$ *is semibasic on* $\mathcal{F}_{\mathbf{x}}^{(g)}$.

Proof. Compute the exterior derivatives of $\omega_2^1 = -C_{121}^3\sigma^1$, $\omega_1^2 = -C_{212}^4\sigma^2$, $\omega_3^4 = C_{111}^4\sigma^1$, and $\omega_4^3 = C_{222}^3\sigma^2$ and apply Cartan's lemma. $\square$

With the help of Lemma 2.4, one sees that, at this point of the reduction process, a number of invariants appear. For example, span$\{\mathbf{e}_1, \mathbf{e}_3\}$ and span$\{\mathbf{e}_2, \mathbf{e}_4\}$ are constant on each connected component of a fiber of $\mathcal{F}_{\mathbf{x}}^{(g)}$ but are interchanged on the two components. Moreover one sees that the quadratic form $\omega_2^1\omega_1^2 = C_{112}^3 C_{221}^4 \sigma^1\sigma^2$ is invariant and thus is the pullback of a quadratic form on M^2. We denote this quadratic form by g and call it the *affine* (*Lorentzian*) *metric.* One may also show that $\omega_3^4\omega_4^3 = C_{222}^3 C_{111}^4 \sigma^1\sigma^2$ is the pullback of a quadratic form which we denote by h and call the *secondary affine* (*Lorentzian*) *metric.* The difference $k = h - g = (C_{222}^3 C_{111}^4 - C_{112}^3 C_{221}^4)\sigma^1\sigma^2$ will also turn out to be important. We call it simply the *difference metric*. We say an immersion is *regular* if k is nowhere zero on M^2. Note that all these metrics are compatible with ϕ wherever they differ from zero.

LEMMA 2.5. *For all $m \in M^2$ there exist $(m, \mathbf{e}) \in \mathcal{F}_{\mathbf{x}}^{(g)}$ at which $C^3_{111} = C^4_{222} = 0$.*

Proof. Compute the exterior derivatives of the equations (2.1) and use Cartan's lemma to obtain

$$\mathrm{d}C^3_{111} - C^3_{111}\omega^1_1 - 3\omega^1_3 \equiv 0 \quad \text{and} \quad \mathrm{d}C^4_{222} - C^4_{222}\omega^2_2 - 3\omega^2_4 \equiv 0 \bmod (\sigma^i). \quad (2.4)$$

The result follows by a standard moving frame argument. □

We make yet another partial third-order reduction:

$$\mathcal{F}_{\mathbf{x}}^{(n)} = \{(m, \mathbf{e}) \in \mathcal{F}_{\mathbf{x}}^{(g)} : C^3_{111} = C^4_{222} = 0 \text{ at } (m, \mathbf{e})\}.$$

LEMMA 2.6. *The 1-forms ω^1_3 and ω^2_4 are semibasic on $\mathcal{F}_{\mathbf{x}}^{(n)}$.*

Proof. This is an immediate consequence of (2.4). □

One immediately sees that $\operatorname{span}\{\mathbf{e}_3\}$ and $\operatorname{span}\{\mathbf{e}_4\}$ are invariant on connected components of any fiber of $\mathcal{F}_{\mathbf{x}}^{(n)}$ and are interchanged on the two components of the fiber. In particular, $\operatorname{span}\{\mathbf{e}_3, \mathbf{e}_4\}$ is constant on each fiber and we may define a bundle over M^2 denoted by NM whose fiber over $m \in M$, N_mM, is the $\operatorname{span}\{\mathbf{e}_3, \mathbf{e}_4\}$ where $(m, \mathbf{e}) \in \mathcal{F}_{\mathbf{x}}^{(n)}$. We call NM the *affine normal bundle* of the immersion $\mathbf{x}$.

On $\mathcal{F}_{\mathbf{x}}^{(n)}$ the quadratic form

$$A = \sum \mathbf{e}_\alpha A^\alpha_{ij}\sigma^j = \mathbf{e}_3(\sigma^1)^2 + \mathbf{e}_4(\sigma^2)^2$$

is invariant; we call A the *affine quadratic form*. Also the cubic form C defined by

$$\begin{aligned} C &= \sum \mathbf{e}_\alpha C^\alpha_{ijk}\sigma^i\sigma^j\sigma^k \\ &= \mathbf{e}_3\sigma^2(C^3_{211}(\sigma^1)^2 + C^3_{222}(\sigma^2)^2) + \mathbf{e}_4\sigma^1(C^4_{111}(\sigma^1)^2 + C^4_{122}(\sigma^2)^2) \end{aligned}$$

is invariant. We refer to it simply as the *cubic form*. The invariance of C will be verified from another point of view quite easily in the next section. Also to be verified there is that the symmetric connection ∇ on M^2 defined by ω^i_j and the connection D on NM defined by ω^α_β are invariant. We call ∇ and D the *affine connection* and the *normal affine connection*, respectively. The curvature forms Ω^i_j of ∇ are defined by

$$\Omega^i_j = \mathrm{d}\omega^i_j + \sum \omega^i_k \wedge \omega^k_j = \frac{1}{2}\sum R^i_{jkl}\sigma^k \wedge \sigma^l,$$

where $R^i_{jkl} = -R^i_{jlk}$ are the components of the curvature operator R of ∇ at $(m,\ \mathbf{e}) \in \mathcal{F}^{(n)}_{\mathbf{x}}$. Also the curvature forms Ω^α_β of D are given by

$$\Omega^\alpha_\beta = \mathrm{d}\omega^\alpha_\beta + \sum \omega^\alpha_\gamma \wedge \omega^\gamma_\beta = \frac{1}{2}\sum N^\alpha_{\beta ij}\sigma^i \wedge \sigma^j,$$

where $N^\alpha_{\beta ij} = -N^\alpha_{\beta ji}$ are the components of the curvature operator $R^\perp$ of D at $(m,\ \mathbf{e}) \in \mathcal{F}^{(n)}_{\mathbf{x}}$. One may obtain still other invariants using the components of these curvature operators; e.g., $R^2_{112}/(C^4_{221})^2$ and $R^1_{221}/(C^3_{112})^2$ are also invariants where $C^3_{221}C^4_{112} \neq 0$.

To complete the reduction we must impose another nondegeneracy condition. Perhaps the most natural such assumption for hyperbolic $\mathbf{x}$ is the following:

(II) g never vanishes on M.

This is equivalent to asserting that $\omega^1_2 \wedge \omega^2_1$ never vanishes on M^2. When (II) holds we may refer to the immersion as being *nonsingular.*

LEMMA 2.7. *If* $\mathbf{x}$ *satisfies* (II), *then for each* $m \in M$ *there exists a frame* $(m,\ \mathbf{e}) \in \mathcal{F}^{(n)}_{\mathbf{x}}$ *at which* $\omega^1_2 = \sigma^1$ *and* $\omega^2_1 = \sigma^2$, *i.e.* $C^3_{112} = C^4_{221} = -1$.

Proof. We leave it to the reader to see how C^3_{112} and C^4_{211} transform under the action of G_n, the group of the fiber of $\mathcal{F}^{(n)}_{\mathbf{x}}$, in order to believe this is so. It is important to realize that G_n has two components. □

Finally, under the assumption (II), we define the complete third-order reduction for hyperbolic $\mathbf{x}$:

$$\mathcal{F}^{(3)}_{\mathbf{x}} = \{(m,\ \mathbf{e}) \in \mathcal{F}^{(n)}_{\mathbf{x}} : \omega^1_2 = \sigma^1 \text{ and } \omega^2_1 = \sigma^2 \text{ at } (m,\ \mathbf{e})\}.$$

One easily shows that G_3, the group of the bundle $\mathcal{F}^{(3)}_{\mathbf{x}}$, is trivial. Hence M^2 is diffeomorphic to $\mathcal{F}^{(3)}_{\mathbf{x}}$ and the unique section of $\mathcal{F}^{(3)}_{\mathbf{x}}$ defines what we call the *canonical framing* $\mathbf{e}$ induced on M^2 by $\mathbf{x}$. When $\mathbf{e}$ is the canonical framing, we will denote the basis dual to $\mathbf{e}_1, \mathbf{e}_2$ by $\theta^1,\ \theta^2$.

On $\mathcal{F}^{(3)}_{\mathbf{x}}$ we may introduce a connection 1-form ψ by

$$\mathrm{d}\theta^1 = -\psi \wedge \theta^1 \quad \text{and} \quad \mathrm{d}\theta^2 = \psi \wedge \theta^2. \tag{2.5}$$

Of course, ψ is a connection form of the Levi-Cevita connection of g relative to the canonical framing. Of importance is the fact that ψ is an invariant. This is also true for the functions a_i defined by

$$\psi = a_1\theta^1 - a_2\theta^2. \tag{2.6}$$

Assuming (II), ψ and a_i are defined everywhere on M^2. The components of R and $R^\perp$ with respect to the canonical framing are additional invariants. One may show the following respect to this framing:

$$\begin{aligned} \omega_1^1 &= (1 - R_{112}^2)\theta^1 + (1 - a_2)\theta^2 \\ \omega_2^2 &= (1 - a_1)\theta^1 + (1 - R_{221}^1)\theta^2 \end{aligned} \tag{2.7}$$

$$\begin{aligned} \omega_3^3 &= 2(1 - R_{112}^2)\theta^1 + (1 - a_2)\theta^2 \\ \omega_4^4 &= (1 - 2a_1)\theta^1 + 2(1 - R_{221}^1)\theta^2 \end{aligned} \tag{2.8}$$

Another nondegeneracy condition which, as the reader will see, leads to particularly nice existence and uniqueness theorems is the assumption that $\mathbf{x}$ is regular. We will not consider any reductions of $\mathcal{F}_{\mathbf{x}}^{(n)}$ under the assumption of regularity.

3. Existence and Uniqueness for Hyperbolic Immersions

One of the main tasks of this section is to interpret geometrically the reductions made in the previous sections.

The only differential geometric structure on $\mathbb{R}^4$ is $\bar{\nabla}$, the standard symmetric connection. It is natural to ask whether $\mathbf{x}$ can induce from $\bar{\nabla}$ a symmetric connection ∇ on M^2 and a connection D on some rank 2 bundle NM such that $TM \oplus NM = M \times \mathbb{R}^4$. In fact, once NM is chosen it is easy to induce such connections. For once we choose a bundle NM such that $TM \oplus NM = M \times \mathbb{R}^4$, we can define a number of connections and tensors in the following fashion. Define a symmetric connection ∇ on M and a symmetric NM-valued covariant 2-tensor A on M by

$$\bar{\nabla}_X Y = \nabla_X Y + A(X, Y), \tag{3.1}$$

where X and Y are vector fields on M. Also, for a vector field X on M and a section y of NM set

$$\bar{\nabla}_X Y = -S_y(X) + D_X y, \tag{3.2}$$

where $S_y(X)$ is a vector field tangent to M and $D_X y$ is a section of NM. Equation (3.2) uniquely defines S and the connection D on NM. We wish to choose NM so that ∇ and D have some relation to other structures induced by $\mathbf{x}$. These structures are determined by $\mathcal{F}_{\mathbf{x}}^{(2)}$ and we shall turn our attention to them.

Let $\mathbb{R}_4$ denote the dual space of $\mathbb{R}^4$ and $\langle\,,\rangle : \mathbb{R}_4 \times \mathbb{R}^4 \to \mathbb{R}$ denote the canonical pairing. Given an immersion $\mathbf{x}$: $M \to \mathbb{R}^4$ we define the vector bundle CM by asserting that for each $m \in M$ its fiber $C_m M$ is the subspace of $\mathbb{R}_4$ that annihilates $T_m M$ (viewed as a subspace of $\mathbb{R}^4$). By means of the natural duality between CM and any choice of bundle NM transverse to TM, we can induce a connection on CM also denoted by D, which depends on the choice of NM.

We do this as follows. For any section ξ of CM and vector field X tangent to M, define $D_X\xi$ by requiring that $\langle D_X\xi,\, y\rangle = X\langle \xi,\, y\rangle - \langle \xi,\, D_X y\rangle$ for any section y of NM.

Assuming $\mathbf{x}$ is hyperbolic, we may define

$$K = \{X \in TM : \phi(X,\, Y) = 0 \text{ for some } Y \neq 0\}.$$

Then K is the union of two transversal rank 1 subbundles which we denote by K_1 and K_2; we denote $K \cap T_m M$ by K_m. Also note that the set of subspaces $\{\text{span}\{\mathbf{e}_1,\, \mathbf{e}_2,\, \mathbf{e}_3\},\, \text{span}\{\mathbf{e}_1,\, \mathbf{e}_2,\, \mathbf{e}_4\}\}$ is invariant on fibers of $\mathcal{F}^{(2)}_{\mathbf{x}}$. That each subspace of this set is invariant on a connected component of a fiber follows from the fact that ω_i^α and $\omega_{\bar\alpha}^\alpha$ are semibasic on $\mathcal{F}^{(2)}_{\mathbf{x}}$; the two subspaces of the set are interchanged on the two components of any fiber. Now define $L = \bigcup_{m\in M} L_m$ where

$$L_m = \{\xi \in C_m M : \xi \text{ annihilates span}\{\mathbf{e}_1,\, \mathbf{e}_2,\, \mathbf{e}_\alpha\}$$
$$\text{where } \alpha = 3 \text{ or } 4 \text{ and } (m,\, \mathbf{e}) \in \mathcal{F}^{(2)}_{\mathbf{x}}\}.$$

Again L is the union of two transversal rank 1 subbundles of CM; we denote these subbundles by L_3 and L_4. Let K_i' and L_α' denote K_i and L_α with their 0-sections removed, respectively; also let $K' = K_1' \cup K_2'$ and $L' = L_3' \cup L_4'$.

Using A (for any NM) we can define a bundle map $A' : K' \to L'$ as follows: For $X \in (K')_m$, the fiber of K' over m, define $A'(X) \in (L')_m$, the fiber of L' over m, by

$$\langle A'(X),\, A(X,\, X)\rangle = 1.$$

This equation uniquely defines $A'(X)$ for the following reasons: (i) Even though $A(X,\, X)$ depends on the choice of $N_m M$, its value modulo $T_m M$ is independent of this choice. (ii) $\langle \xi,\, A(X,\, X)\rangle = 0$ either for all $\xi \in (L_3)_m$ or for all $\xi \in (L_4)_m$. From now on we label L_3 and L_4 so that $A'(K_i') \subset L_{2+i}'$.

Before going on, we mention that if $(m,\, \mathbf{e}) \in \mathcal{F}^{(2)}_{\mathbf{x}}$ and $\boldsymbol{\epsilon} = (\epsilon^1,\, \epsilon^2,\, \epsilon^3,\, \epsilon^4)$ is dual to $\mathbf{e}$, then $A'(\mathbf{e}_i) = \epsilon^{2+i}$. Moreover, if we let $N_m M = \text{span}\{\mathbf{e}_3,\, \mathbf{e}_4\}$, then at m it is the case that $A = \mathbf{e}_3(\sigma^1)^2 + \mathbf{e}_4(\sigma^2)^2$ but A varies with the choice of $(m,\, \mathbf{e}) \in \mathcal{F}^{(2)}_{\mathbf{x}}$. However,

$$A' = \epsilon^{2+i}/(\sigma^i)^2 : (K_i')_m \to (L_{2+i}')_m$$

is invariant of this choice.

One would like to require that the induced connection ∇ has some relation to the subbundles K_i. For example, one might want to require that each K_i is parallel, i.e. for any section X of K_i, $(\nabla X) \wedge X = 0$. The results of Section 2 show that, in general, this is impossible since $\omega_i^{\bar{i}}$ need not be identically zero. The reduction to $\mathcal{F}_{\mathbf{x}}^{(g)}$ is obtained by requiring that any integral curve of K_i, denoted by γ_i, is geodesic with respect to the connection ∇. This is so since the reduction is defined by requiring that $\omega_{\bar{i}}^{i}(\mathbf{e}_{\bar{i}}) = 0$, i.e., $C_{\bar{i}\bar{i}\bar{i}}^{2+i} = 0$; c.f. equation (2.2). We get for free that the subbundle L_{2+i} is parallel along γ_i; this follows since $C_{\bar{i}\bar{i}\bar{i}}^{2+i} = 0$ implies $\omega_{2+\bar{i}}^{2+i}(\mathbf{e}_i) = 0$. Since A' gives an inverse quadratic relation between K_i' and L_{2+i}', it is natural to want to impose the condition that

$$\langle D_X A'(X),\, X\rangle = -2\langle A'(X),\, \nabla_X X\rangle$$

for any section X of K_i. This is precisely the condition that defines the reduction to $\mathcal{F}_{\mathbf{x}}^{(n)}$ since that condition is equivalent to asserting that $\omega_{2+i}^{2+i}(\mathbf{e}_i) = 2\omega_i^i(\mathbf{e}_i)$ on $\mathcal{F}_{\mathbf{x}}^{(n)}$, i.e. $C_{iii}^{2+i} = 0$; cf. equation (2.1).

The conditions $C_{\bar{i}\bar{i}\bar{i}}^{2+i} = C_{iii}^{2+i} = 0$ are known as Weise's apolarity conditions; cf. [7] where the notation is somewhat different. Since these conditions reduce $\mathcal{F}_{\mathbf{x}}^{(2)}$ to $\mathcal{F}_{\mathbf{x}}^{(n)}$ they determine the affine normal bundle NM and thus $A,\ \nabla,\ D$ and S.

Let $\tilde{\nabla}$ be the connection induced on $TM \otimes TM \otimes NM$ by ∇ and D. It turns out that $\tilde{\nabla}A$ is the cubic form C. Thus the Weise's apolarity conditions are conditions on $\tilde{\nabla}A$.

We suppose from here on that NM is chosen so that Weise's apolarity conditions hold, that is, by means of $\mathcal{F}_{\mathbf{x}}^{(n)}$, and turn to uniqueness questions. It is clear from (3.1) and (3.2) that if S is determined by $A,\ \nabla$, and D, then $\mathbf{x}$ is uniquely determined up to an affine transformation of $\mathbb{R}^4$ by $A,\ \nabla$ and D. By taking covariant derivatives of (3.1) and (3.2) we find the following.

LEMMA 3.1. *Let X, Y and Z be vector fields tangent to M and let y be a section of M. Then*

(1) $R(X,\, Y)Z = S_{A(Y,Z)}(X) - S_{A(X,Z)}(Y)$,

(2) $\tilde{\nabla}A$ *is a cubic form,*

(3) $R^{\perp}(X,\, Y)y = A(X,\, S_y(Y)) - A(Y,\, S_y(X))$,

(4) $(\nabla_X S_y)(Y) - S_{D_X y}(Y) = (\nabla_Y S_y)(X) - S_{D_Y y}(X)$.

Let $\mathbf{e}$ be a section of $\mathcal{F}_{\mathbf{x}}^{(n)}$ which we will call a *standard section* and define scalar functions $S_{\alpha j}^{i}$ by setting $S_{\mathbf{e}_\alpha} = S_\alpha = \Sigma \mathbf{e}_i S_{\alpha j}^{i} \sigma^j$, i.e. $\omega_\alpha^i = -\Sigma S_{\alpha j}^{i} \sigma^j$. Also let R_{jkl}^{i} and $N_{\beta ij}^{\alpha}$ be the components of R and $R^{\perp}$ with respect to this section standard section.

LEMMA 3.2. *The following relations hold for any section $\mathbf{e}$ of $\mathcal{F}_{\mathbf{x}}^{(n)}$.*

$$S_{32}^{1} = R_{121}^{1} = N_{312}^{3}, \qquad S_{41}^{2} = R_{212}^{2} = N_{412}^{4}, \tag{3.3}$$

$$S_{32}^{2} = R_{121}^{2}, \qquad S_{41}^{1} = R_{212}^{1}, \tag{3.4}$$

$$S_{31}^{2} = N_{321}^{4}, \qquad S_{42}^{1} = N_{412}^{3}. \tag{3.5}$$

Proof. These equations follow by setting $X = \mathbf{e}_1$, $Y = \mathbf{e}_2$ and $Z = \mathbf{e}_1$ or $\mathbf{e}_2$ in (1) of Lemma 3.1 or $X = \mathbf{e}_1$, $Y = \mathbf{e}_2$ and $y = \mathbf{e}_3$ or $\mathbf{e}_4$ in (3) of Lemma 3.1. □

Clearly equations (3.3)–(3.5) are equivalent to (1) and (3) of Lemma 3.1. It remains to be seen whether or not S^1_{31} and S^2_{42} can be determined.

For any standard section $\mathbf{e}$, we define scalar functions Γ^i_{jk} by $\omega^i_j = \Sigma\Gamma^i_{jk}\sigma^k$. Also, for any function $f : M \to \mathbb{R}$, define $f_{,i}$ by

$$\mathrm{d}f = \sum f_{,i}\sigma^i.$$

Finally, a semicolon, used in a similar fashion, will denote covariant differentiation.

The next lemma is a restatement of (2) of Lemma 3.1 in terms of a standard section $\mathbf{e}$.

LEMMA 3.3. *Let* $\mathbf{e}$ *be any section of* $\mathcal{F}^{(n)}_{\mathbf{x}}$. *Then*

$$\left.\begin{aligned} C^3_{222}S^1_{31} - C^3_{112}S^2_{42} &= R^1_{212;2} - C^3_{211}R^1_{212} - N^3_{412;1} - C^4_{221}N^3_{412} \\ C^4_{221}S^1_{31} - C^4_{111}S^2_{42} &= R^2_{112;1} - C^4_{122}R^2_{112} - N^4_{312;2} - C^3_{112}N^4_{312} \end{aligned}\right\} \tag{3.6}$$

$$\left.\begin{aligned} &S^1_{31,2} - 2(\Gamma^1_{12} + C^3_{112})S^1_{31} \\ &\quad = R^1_{121,1} - (\Gamma^1_{11} + \Gamma^2_{21} + C^4_{221})R^1_{121} - C^3_{112}R^2_{121} - C^4_{111}N^3_{412} \\ &S^2_{42,1} - 2(\Gamma^2_{21} + C^4_{221})S^2_{42} \\ &\quad = R^2_{212,2} - (\Gamma^1_{12} + \Gamma^2_{22} + C^3_{112})R^2_{212} - C^4_{221}R^1_{212} - C^3_{222}N^4_{321} \end{aligned}\right\} \tag{3.7}$$

Proof. We get these equations by taking the exterior derivatives of ω^i_α and applying the structure equations. Lemma 3.2 is used to replace some $S^i_{\alpha j}$ by curvatures. □

Notice that equations (3.6) may be used to determine S^1_{31} and S^2_{42} on M if $C^3_{222}C^4_{111} - C^3_{112}C^4_{211}$ is never zero on M. But this means that $\mathbf{x}$ is a regular hyperbolic immersion. This gives our first uniqueness result.

THEOREM 3.4. *If* $\mathbf{x}$: $M^2 \to \mathbb{R}^4$ *is a regular hyperbolic immersion of an oriented surface, then* $\mathbf{x}$ *is determined up to an affine transformation of* $\mathbb{R}^4$ *by* A, ∇ *and* D.

We now wish to consider a uniqueness theorem of nonsingular hyperbolic immersions. For these considerations all of our calculations are in terms of the components of the canonical framing. In particular, $C^3_{112} = C^4_{221} = -1$ on M. We say a nonsingular hyperbolic immersion $\mathbf{x}$ is *exceptional* if everywhere on M

(1) $k = 0$ (i.e. $g = h$ or equivalently $C^3_{111}C^4_{222} = 1$),
(2) $\square \log C^4_{111} = 4 * d * \psi$,

where $*$ denotes the Hodge star operator and $\square = * d * d$ is the d'Alembertian, or wave, operator associated with the affine Lorentzian metric g.

THEOREM 3.5. *Let* $\mathbf{x}$: $M^2 \to \mathbb{R}^4$ *be a nonsingular hyperbolic immersion. If* $\mathbf{x}$ *is not exceptional, the* $\mathbf{x}$ *is determined up to an affine transformation of* $\mathbb{R}^4$ *by* A, ∇ *and* D. *If* $\mathbf{x}$ *is exceptional, then* $\mathbf{x}$ *is determined up to an affine transformation of* $\mathbb{R}^4$ *by* A, ∇, D *and either* $S^1_{31}(m_0)$ *or* $S^2_{42}(m_0)$, *for some* $m_0 \in M$.

Proof. Again we show that under the hypotheses of this theorem S^1_{31} and S^2_{42} can be determined on M by the other data.

Let $M_1 = \{m \in M\colon C^3_{112}C^4_{221} = 1 \text{ at } m\}$. On $M \backslash M_1$ we may use (3.6) to determine S^1_{31} and S^2_{42}. If M_1 is nowhere dense, then S^1_{31} and S^2_{42} are determined on M. Thus we suppose that M_1 has interior points and restrict our attention to $\overset{\circ}{M}_1$, the interior of M_1. Rewrite (3.6) and (3.7) in the following form:

$$S_1 - C^3_{112}S_2 = -P_1, \qquad C^4_{221}S_1 - S_2 = P_2, \tag{3.6$'$}$$

$$S_{1,2} + 2a_2S_1 = Q_1, \qquad S_{2,1} + 2a_1S_2 = Q_2, \tag{3.7$'$}$$

where S_1 and S_2 stand for S^1_{31} and S^2_{42}, respectively, and $-P_1$, P_2 and Q_i are the right-hand sides of (3.6) and (3.7), respectively. We have used (2.7) to evaluate Γ^1_{12} and Γ^2_{21}. From equations (3.6′) and (3.7′) we obtain

$$S_{1,1} + [2a_1 - (\log C^4_{111})_{,1}]S_1 = (P_{2,1} + 2a_1P_2 - 2 - Q_2)C^4_{111}. \tag{3.8}$$

A similar equation, arrived at by interchanging the indices 1 and 2, holds for S_2 on $\overset{\circ}{M}_1$. Equation (3.8) and the first equation of (3.7′) imply that on $\overset{\circ}{M}_1$

$$\mathrm{d}S_i + \mu_iS_i = \rho_i, \tag{3.9}$$

taking into account the symmetry of the indices, where

$$\mu_i = [2a_i - (\log C^{2+i}_{iii})_{,i}]\sigma^i + 2a_{\bar{\imath}}\sigma^{\bar{\imath}} \tag{3.10}$$

and

$$\rho_i = [P_{\bar{\imath},i} + 2a_iP_{\bar{\imath}} - Q_{\bar{\imath}}]C^{2+i}_{iii}\sigma^i + Q_i\sigma^{\bar{\imath}}. \tag{3.11}$$

Compute the exterior derivative of (3.9) and substitute $\rho_i - \mu_iS_i$ for $\mathrm{d}S_i$ to obtain

$$S_i\,\mathrm{d}\mu_i = \mathrm{d}\rho_i + \mu_i \wedge \rho_i. \tag{3.12}$$

Let $U = \{m \in \overset{\circ}{M}_1 : \mathrm{d}\mu_{i|m} = 0\}$. One may show that for any $m \in \overset{\circ}{M}_1$, either both or neither of $\mathrm{d}\mu_{i|m}$ are zero. On $\overset{\circ}{M}_1 \backslash U$, we may solve (3.12) for S_1 and S_2. Hence if U is nowhere dense in $\overset{\circ}{M}_1$, S_1 and S_2 are determined on M. Thus we may suppose that m_0 is an interior point of U. If there exists m_1 not in the interior of U, let c: $[0, 1] \to M$ be a curve with $c(0) = m_0$ and $c(1) = m_1$. Since the S_i are known at m_1, we may use (3.9) along c to determine S_i at m_0. Hence if m_1 exists, S_1 and S_2 are determined on M. If no such m_1 exists then $U = M$; thus on M, $C^4_{111}C^3_{222} = 1$ and $\mathrm{d}\mu_i = 0$. The last condition is precisely (2) of the definition of an exceptional hyperbolic immersion. Hence if no such m_1 exists then **x** is exceptional. It should now be clear as well that if **x** is exceptional then the only additional information we need to determine the S_i is a value of one of them at some point of M. □

Remark. It is worth noting that when **x** is a nonsingular hyperbolic immersion and not exceptional then it is determined up to an affine transformation of $\mathbb{R}^4$ by the 1-forms θ^i and the real-valued functions C^{2+i}_{iii} and $R^{\bar{i}}_{i i \bar{i}}$; cf. equations (2.7) and (2.8). From this one sees, under the same conditions on **x**, that A, C and R determine **x** up to an affine transformation of $\mathbb{R}^4$ (cf. Wilkinson [7]). This follows since C and A determine the canonical framing when the hyperbolic immersion **x** is nonsingular.

We now turn to existence theorems. Let M be an oriented surface and BM be a rank 2 vector bundle over M. Assume ∇ is a symmetric connection on M and D is a connection on BM. Let A be a section of $T^*M \otimes T^*M \otimes BM$ which is symmetric in its arguments when viewed as a bundle-valued bilinear form. For short, we will simply say that the triple (A, ∇, D) is *associated* with M and BM. We say A is *hyperbolic* on M if, for all $m \in M$, the image of $A_{|m}$ spans B_mM and there exist linearly independent $X, Y \in T_mM$ such that $A(X, Y) = 0$. When A is hyperbolic, a frame $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4$ of $T_mM \oplus B_mM$ is said to be *standard* if $\mathbf{e}_i \in T_mM$, $\mathbf{e}_\alpha \in B_mM$, $\mathbf{e}_1 \wedge \mathbf{e}_2 > 0$, $A(\mathbf{e}_1, \mathbf{e}_2) = 0$ and $\mathbf{e}_{2+i} = A(\mathbf{e}_i, \mathbf{e}_i)$. Let $C = \tilde{\nabla}A$, where $\tilde{\nabla}$ is induced from ∇ and D, and suppose C is a cubic form. Then C is said to satisfy Weise's apolarity condition if the components of C with respect to a standard frame satisfy

$$C^{i+2}_{iii} = C^{i+2}_{i\bar{i}\bar{i}} = 0.$$

The triple (A, ∇, D) is said to be *regular* on M if with respect to any standard frame $C^3_{222}C^4_{111} - C^3_{112}C^4_{221}$ is never zero. Also, the triple (A, ∇, D) is said to be *nonsingular* on M if there exists a standard framing of M with respect to which $C^3_{112} = C^4_{221} = -1$ everywhere on M; such a standard framing is unique if it exists and is called the *canonical* framing. Let θ^1, θ^2 be a framing dual to the canonical framing $\mathbf{e}_1$, $\mathbf{e}_2$; then $\theta^1\theta^2$ is a Lorentzian metric on M. Because of this metric, it is possible to define, in a fashion similar to the way we define an immersion **x** to be exceptional, the idea that a triple (A, ∇, D) is exceptional.

By a *realization* y: $BM \to \mathbb{R}^4$ of BM we mean a smooth map such that, for all $m \in M$, y: $B_mM \to \mathbb{R}^4$ is a nonsingular linear map. A bundle NM over M of 2-dimensional subspaces of $\mathbb{R}^4$ is said to be the *image* of BM under the realization y if $y(B_mM) = N_mM$, for all $m \in M$.

THEOREM 3.6. *Let* (A, ∇, D) *be a triple associated with the open simply connected oriented surface* M *and the rank 2 bundle* BM. *Assume that* A *is hyperbolic and* $C = \tilde{\nabla} A$ *is a cubic form that satisfies Weise's apolarity condition. If* (A, ∇, D) *is regular and, with respect to a standard framing,*

(1) $R^i_{i\bar{i}\bar{i}} = N^{2+i}_{(2+i)\bar{i}\bar{i}}$, *and*
(2) S^1_{31} *and* S^2_{41}, *which are defined by* (3.6), *satisfy* (3.7),

(*where* $\nabla_{\mathbf{e}_j}\mathbf{e}_i = \Sigma \mathbf{e}_k \Gamma^k_{ij}$), *then there exists a regular hyperbolic immersion* $\mathbf{x}\colon M \to \mathbb{R}^4$ *and a realization* $y\colon BM \to \mathbb{R}^4$ *whose image is* NM, *the normal bundle of* $\mathbf{x}$, *such that the induced affine connection is* ∇. *Moreover, identifying* NM *with* BM *by means of* y, *the induced affine normal connection is* D *and the induced affine quadratic form is* A.

The proof is straightforward so the details are left to the reader.

In general, the existence theory for triples which are not regular but are nonsingular is messy. However, the assumption that the triple is nonsingular and exceptional can be dealt with quite reasonably. We define 1-forms μ_i as in (3.10), where a_i are defined by (2.5) and (2.6) using the canonical framing. Because the triple is exceptional, $d\mu_i = 0$ on M. Assuming M is simply connected, there exist smooth functions $f_i\colon M \to \mathbb{R}$ such that $df_i = \mu_i$. Define P_i and Q_i by comparing (3.6) and (3.7) with (3.6′) and (3.7′); also define ρ_i using (3.11). In order for there to exist S^1_{31} and S^2_{42} satisfying (3.6′) it is necessary that $C^3_{112}P_2 + P_1 = 0$. In order for any solution $S_1 = S^1_{31}$, $S_2 = S^2_{42}$ of (3.6′) to satisfy (3.7′) it is also necessary that (3.9) holds. But (3.9) may be written

$$d(e^{f_i} S_i) = e^{f_i}\rho_i. \tag{3.13}$$

For (3.13) to have a solution S_i it is necessary that $d\rho_i + \mu_i \wedge \rho_i = 0$. Because of these observations the next theorem should seem quite reasonable. The details of the proof are left to the reader.

THEOREM 3.7. *Let* (A, ∇, D) *be a triple associated to the open simply connected surface* M *and the rank 2 vector bundle* BM. *Assume that* A *is hyperbolic and* C *is a cubic form satisfying Weise's apolarity conditions. Assume also that the triple* (A, ∇, D) *is nonsingular and exceptional. Suppose (with respect to the canonical framing)* $R^i_{i\bar{i}\bar{i}} = N^{2+i}_{(2+i)\bar{i}\bar{i}}$, $C^3_{112}P_2 + P_1 = 0$, *and* $d\rho_1 + \mu_1 \wedge \rho_1 = 0$ *on* M. *Let* $m_0 \in M$; *then for each* $S \in \mathbb{R}$ *there exists an exceptional nonsingular hyperbolic immersion* $\mathbf{x}\colon M \to \mathbb{R}^4$ *and a realization* $y\colon BM \to \mathbb{R}^4$ *whose image is* NM *which induces the affine connections* ∇ *and* D, *the affine quadratic form* A, *and for which* $S^2_{42}(m_0) = S$.

4. Elliptic Immersions (Including some Nongeneric Cases)

Everything we did in Sections 2 and 3 for hyperbolic immersions can be done for elliptic immersions. The results in Section 2 can be transformed into results for elliptic immersions by replacing the indices 2, 3, 4 by $\bar{1}$, 2, $\bar{2}$, respectively. Also indices of the form $2+i$ are to be replaced by $1+i$ in Sections 2 and 3, with the understanding that $1+\bar{1}=\bar{2}$. The only differences to note in the elliptic setting are that the fibers and groups are connected and the metrics g and h are Riemannian where they are not zero. The interpretations of the conditions that lead to the reduction of $\mathcal{F}_{\mathbf{x}}^{(2)}$ to $\mathcal{F}_{\mathbf{x}}^{(n)}$ are similar when one works with the complexifications of TM, NM, etc. More will be said about this later in this section. The existence and uniqueness theorems of Section 3 have the same form in the elliptic setting with the exception that one deals with just S^1_{21}, whose real and imaginary parts correspond to S^1_{31} and S^2_{42}.

We wish to examine some nongeneric immersions in the elliptic case. The most interesting nongeneric situation is the one in which the subbundles K_1 and $K_{\bar{1}}$ (the elliptic analogues of K_1 and K_2) are parallel on M^2. As stated in Section 3, this is a natural condition to want to impose on ∇ but this condition does not hold in general. Certainly, elliptic or hyperbolic immersions for which this condition holds should be of interest. Before we begin the study of such surfaces, we mention that in the hyperbolic case, if K_1 and K_2 are parallel then $\mathbf{x}$ (at least locally) is a translation surface, i.e., it is the sum of the immersions of two curves in $\mathbb{R}^4$. Also, while it is possible for just one of K_1 or K_2 to be parallel on M in the hyperbolic case, the corresponding situation is not possible in the elliptic case. This is so because K_1 and $K_{\bar{1}}$ are conjugate to one another and ∇ commutes with conjugation.

Let $\mathbf{x}$ be an elliptic immersion for which K_1 is parallel on M. Note that K_1 is parallel if and only if $\omega^{\bar{1}}_i=0$ on M relative to a standard (elliptic) framing and this condition is equivalent to the affine (Riemannian) metric $g=0$ on M. Recall that when $\mathbf{x}$ is elliptic it induces a complex analytic structure on M with σ^1 of type (1, 0) (meaning that for any standard framing $\mathbf{e}$: $M\to\mathcal{F}_{\mathbf{x}}^{(n)}$, $\mathbf{e}^*(\sigma^1)$ is of type $(1,0)$).

PROPOSITION 4.1. *Suppose $g=0$ on M; then $\mathbf{x}$: $M\to\mathbb{R}^4$ is harmonic, i.e. $\bar{\partial}\partial\mathbf{x}=0$.*

Proof. Just note that $\partial\mathbf{x}=\mathbf{e}_1\sigma^1$ so that $\bar{\partial}\partial\mathbf{x}=d\partial\mathbf{x}=(\mathbf{e}_1\omega^1_1+\mathbf{e}_3\sigma^1)\wedge\sigma^1-\mathbf{e}_1\omega^1_{\bar{1}}\wedge\sigma^1=0$ since $\omega^{\bar{1}}_1=0$. □

Consequently, we will call such an elliptic immersion a *harmonic* immersion.

Since $\omega^{\bar{1}}_1=0$ for harmonic immersions, $C^{\bar{2}}_{\bar{1}1}=0$ on $\mathcal{F}_{\mathbf{x}}^{(n)}$. This implies

LEMMA 4.2. *If $\mathbf{x}$ is harmonic, then, on $\mathcal{F}_{\mathbf{x}}^{(n)}$, $\omega^2_2=2\omega^1_1$.*

This last lemma shows that A and D determine ∇ since $\omega^1_1=\frac{1}{2}\omega^2_2$ and $\omega^{\bar{1}}_1=0$ on $\mathcal{F}_{\mathbf{x}}^{(n)}$. Of course, if $\mathbf{x}$ is regular then A, ∇ and D determine $\mathbf{x}$ up to an affine transformation of $\mathbb{R}^4$. Thus, we obtain

THEOREM 4.3. *Let $\mathbf{x}$ be a regular harmonic immersion. Then $\mathbf{x}$ is determined by A and D up to an affine transformation of $\mathbb{R}^4$.*

If $\mathbf{x}$ is harmonic and regular then $C^{\bar{2}}_{\bar{1}11}$ is never zero on M. Because of this, for each $m \in M$, we can find a frame $(m, \mathrm{e}) \in \mathcal{F}^{(n)}_{\mathbf{x}}$ at which $C^{\bar{2}}_{\bar{1}11} = 1$. This leads to a new third-order reduction $\hat{\mathcal{F}}^{(3)}_{\mathbf{x}}$ defined by

$$\hat{\mathcal{F}}^{(3)}_{\mathbf{x}} = \{(m, \mathrm{e}) \in \mathcal{F}^{(n)}_{\mathbf{x}} : \omega^{\bar{2}}_{2} = \sigma^1\}.$$

One may show that $\hat{G}_3$, the group of the bundle, is isomorphic to $\mathbb{Z}_5$. We call any section of this bundle a canonical harmonic framing. With respect to such a framing $h = \sigma^1\sigma^{\bar{1}}$.

Let us now turn to existence results. Suppose M is an oriented surface and BM is a rank 2 vector bundle over M. Let A be a section of $T^*M \otimes T^*M \otimes BM$ which is symmetric when viewed as a BM-valued bilinear form and let D be a connection on BM. For short, we will say that the pair (A, D) is *associated* to M and BM. Let B^*M be the bundle which is dual to BM and let D also denote the connection induced on B^*M by means of this duality. We say that A is *elliptic* on M if, for all $m \in M$, $A(X, Y) = 0$ for some $X, Y \in T_mM$ implies that X or Y is zero. We complexify TM, BM, and B^*M and denote any complexification by the same symbol. We define $(K)_m$ by

$$(K)_m = \{X \in T_mM : A(X, Y) = 0 \text{ for some nonzero } Y \in T_mM\}$$

and let $K = \bigcup_{m\in M}(K)_m$. Define $K_1 = \{X \in K : -iX \wedge \bar{X} > 0\}$; this is a rank 1 subbundle of TM. If we set $K_{\bar{1}} = \overline{K_1}$, then K_1 and $K_{\bar{1}}$ are transversal and $K = K_1 \bigcup K_{\bar{1}}$. We define a complex analytic structure on TM by asserting that K_1 is the bundle of all vectors of type $(1, 0)$. Let $(K_1)_m$ and $(K_{\bar{1}})_m$ denote the fiber of K_1 and $K_{\bar{1}}$ over m, respectively. Define L_2 to be the rank 1 subbundle of B^*M whose fiber over m, $(L_2)_m$, is the annihilator of $A((K_{\bar{1}})_m, (K_{\bar{1}})_m)$. Likewise, $L_{\bar{2}} = \overline{L_2}$ is the bundle of annihilators of $A(K_1, K_1)$. We say the pair (A, D) satisfies the *harmonic apolarity condition* if L_2 is $(1, 0)$ parallel, i.e. $D_{\partial/\partial z}\xi \wedge \xi = 0$ for every section ξ of L_2 and any local complex coordinate z. We say the pair (A, D) is *regular harmonic* if $D_{\partial/\partial \bar{z}}\xi \wedge \xi$ is not zero anywhere the section ξ of L_2 is not zero.

A frame field $\mathbf{e} = (\mathrm{e}_1, \mathrm{e}_{\bar{1}}, \mathrm{e}_2, \mathrm{e}_{\bar{2}})$ of $TM \oplus BM$ is said to be *standard* if $\mathrm{e}_{\bar{r}} = \overline{\mathrm{e}_r}$, e_1 is a vector field of type $(1, 0)$, and $\mathrm{e}_2 = A(\mathrm{e}_1, \mathrm{e}_1)$. Given a pair (A, D) associated to M and BM with A elliptic, we can define a connection ∇ on M as follows: Let $\mathbf{e}$ be a standard framing of $TM \oplus BM$; then ∇ is the connection whose connection forms ω^i_j with respect to e_1, $\mathrm{e}_{\bar{1}}$ are given by

$$\omega^1_1 = \tfrac{1}{2}\omega^2_2 \quad\text{and}\quad \omega^{\bar{1}}_1 = 0 \tag{4.1}$$

on M. Finally, note that if

$$\mathrm{e}_1 \cdot \omega^2_2(\mathrm{e}_{\bar{1}}) - \mathrm{e}_{\bar{1}} \cdot \omega^{\bar{2}}_{\bar{2}}(\mathrm{e}_1) = 2[\mathrm{e}_1, \mathrm{e}_{\bar{1}}] \tag{4.2}$$

holds for any standard framing then it holds for all standard framings.

LEMMA 4.4. *The connection* ∇ *is well-defined. It is symmetric if* (4.2) *holds for any standard framing. Moreover, if* $(A,\ D)$ *satisfies the harmonic apolarity condition, then* $C = \tilde{\nabla} A$ *is a cubic form and the triple* $(A,\ \nabla,\ D)$ *which is associated to* M *and* BM *satisfies Weise's apolarity condition.*

Proof. One must check that the connection forms ω^i_j transform in the appropriate fashion under a change of standard framing; this is easy to do. Thus the connection is well-defined. It is also easy to show that if (4.2) holds for any standard framing, then ∇ is symmetric. Note that K_1 is parallel with respect to ∇ and, by definition, L_2 is parallel in the $(1,\ 0)$ direction. Since $\omega^2_2 = 2\omega^1_1$ as well, Weise's apolarity condition holds. One may show that

$$C = \mathbf{e}_2 C^2_{\bar{1}\bar{1}\bar{1}}(\sigma^{\bar{1}})^3 + \mathbf{e}_{\bar{2}} C^{\bar{2}}_{111}(\sigma^1)^3$$

with respect to a standard framing so it is a cubic form. □

When $(A,\ D)$ satisfies the harmonic apolarity condition and is regular on M, one can find standard framings of $TM \oplus BM$ for which $\omega^{\bar{2}}_2 = \sigma^1$. Such framings are called *canonical harmonic* framings.

THEOREM 4.5. *Let* $(A,\ D)$ *denote a pair associated to the open simply connected oriented surface* M *and the rank 2 bundle* BM *with* A *elliptic,* $(A,\ D)$ *satisfying the harmonic polarity condition and* (4.2) *holding for any standard framing. Suppose* $(A,\ D)$ *is regular and let* $\mathbf{e}$ *be a canonical harmonic framing. If, with respect to that framing,*

$$N^2_{21\bar{1}} = -\tfrac{1}{3}, \tag{4.3}$$

$$\begin{aligned} N^2_{\bar{2}\bar{1}1,1,\bar{1}} &- \tfrac{1}{2}(3\Gamma^{\bar{2}}_{\bar{2}1} - \Gamma^2_{21})N^2_{\bar{2}\bar{1}1,\bar{1}} - \Gamma^2_{2\bar{1}}N^2_{\bar{2}\bar{1}1,1} \\ &- \tfrac{1}{2}[3\Gamma^{\bar{2}}_{\bar{2}1,\bar{1}} - \Gamma^2_{21,\bar{1}} - \Gamma^2_{21}(3\Gamma^{\bar{2}}_{\bar{2}1} - \Gamma^2_{21}) - 1]N^2_{\bar{2}\bar{1}1} \\ &= \tfrac{1}{6}(\Gamma^2_{21} + \Gamma^{\bar{2}}_{\bar{2}1}), \end{aligned} \tag{4.4}$$

where $D_{\mathbf{e}_i}\mathbf{e}_\beta = \Sigma \mathbf{e}_\alpha \Gamma^\alpha_{\beta i}$, *then there exists a regular harmonic immersion* $\mathbf{x} :\ M \to \mathbb{R}^4$ *and a realization* $y :\ BM \to \mathbb{R}^4$ *with image* NM *such that* A *and* D *are the affine quadratic form and the affine normal connection, respectively.*

Proof. We define the symmetric connection ∇ using (4.1) for a canonical harmonic framing. By Lemma 4.4, $C = \tilde{\nabla} A$ is a cubic form and Weise's apolarity condition holds for the triple $(A,\ \nabla,\ D)$. Since $(A,\ D)$ is regular, it follows that $(A,\ \nabla,\ D)$ is regular. One shows that $R^1_{1\bar{1}1} = -\frac{1}{3}$, using $\omega^2_2 = 2\omega^1_1$, for a canonical harmonic framing, so that $N^1_{11\bar{1}} = R^1_{1\bar{1}1}$, which is the elliptic analogue of condition (1) in Theorem 3.6. One also shows that the elliptic analogue of (3.6), for a canonical harmonic framing, implies

$$S^1_{21} = N^2_{\bar{2}\bar{1}1,1} - \tfrac{1}{2}(3\Gamma^{\bar{2}}_{\bar{2}1} - \Gamma^2_{21})N^2_{\bar{2}\bar{1}1}. \tag{4.5}$$

Then the elliptic analogue of (3.7) becomes (4.4). Hence condition (2) of the elliptic analogue of Theorem 3.6 holds. □

We could consider elliptic immersions for which $h = 0$ on M and $g \neq 0$ anywhere on M. A complete set of invariants in this case consists of A and ∇, since, in this case, A and ∇ determine D and the immersion is regular. Results similar to Theorems 4.3 and 4.5 can be obtained.

Before we consider elliptic immersions for which $h = g = 0$ on M we need some preliminary results which are of general interest.

LEMMA 4.6. *In terms of a section of* $\mathcal{F}_{\mathbf{x}}^{(n)}$, *i.e. a standard framing, for an elliptic immersion* $\mathbf{x}$ *the following hold:*

$$\begin{aligned} 3R^1_{1\bar{1}1} &= C^2_{11\bar{1};1} - 3|C^2_{11\bar{1}}|^2 - |C^{\bar{2}}_{111}|^2 \\ R^1_{\bar{1}1\bar{1}} &= C^2_{1\bar{1}\bar{1};\bar{1}} - 2(C^2_{11\bar{1}})^2 \\ N^{\bar{2}}_{2\bar{1}1} &= C^{\bar{2}}_{111;\bar{1}} - 2C^2_{11\bar{1}}C^{\bar{2}}_{111} \end{aligned}$$

Proof. We take the exterior derivatives of $\omega^2_2 = \omega^1_1 + C^2_{11\bar{1}}\sigma^{\bar{1}}$, $\omega^1_{\bar{1}} = -C^2_{\bar{1}11}\sigma^1$, and $\omega^{\bar{2}}_2 = C^{\bar{2}}_{111}\sigma^1$ and use the elliptic analogues of (3.3)–(3.5). □

LEMMA 4.7. *If* $C = 0$ *on* M, *then* $R = R^\perp = 0$ *and, for any standard framing,* $S_2 = \mathbf{e}_1 \cdot S^1_{21}\sigma^1$.

Proof. Lemma 4.6 implies $R = R^\perp = 0$ on M. The elliptic analogue of Lemma 3.2 implies $S_2 = \mathbf{e}_1 \cdot S^1_{21}\sigma^1$. □

It is obvious that $g = h = 0$ on M if and only if $C = 0$ on M, when $\mathbf{x}$ is an elliptic immersion. Under the assumption that $C = 0$, it is the case that A and ∇ determine D but S is not determined by them since the elliptic analogue of (3.6) degenerates to the equation $0 = 0$, by Lemma 4.7. However, S has a very simple form as can be seen from Lemma 4.7. One may show that $-\omega^1_2\sigma^1 = S^1_{21}(\sigma^1)^2$ is a holomorphic quadratic differential; let us denote it by $\hat{S}$. Hence, we obtain

THEOREM 4.8. *Any elliptic immersion* $\mathbf{x}$ *for which* $C = 0$ *is determined up to an affine transformation of* $\mathbb{R}^4$ *by* A, ∇, $\hat{S}$.

If we suppose that $\hat{S}$ is never zero on M we can introduce a third-order reduction $\hat{\mathcal{F}}^{(3)}_{\mathbf{x}}$ defined by

$$\hat{\mathcal{F}}^{(3)}_{\mathbf{x}} = \{(m, \mathbf{e}) \in \mathcal{F}^{(n)}_{\mathbf{x}} : \omega^1_2 = \sigma^1 \text{ at } (m, \mathbf{e})\}.$$

The group of $\hat{\mathcal{F}}^{(3)}_{\mathbf{x}}$ is $\mathbb{Z}_2$. For any section of $\hat{\mathcal{F}}^{(3)}_{\mathbf{x}}$ one may show that $\omega^1_2 = \mathrm{d}z$ and $\omega^1_1 = k\,\mathrm{d}z$ where z is a local complex coordinate and k is holomorphic. With respect to the coordinate z, $\mathbf{x}$ satisfies the differential equation

$$\frac{\mathrm{d}^3\mathbf{x}}{\mathrm{d}z^3} - 3k\frac{\mathrm{d}^2\mathbf{x}}{\mathrm{d}z^2} + \left(2k^2 - \frac{\mathrm{d}k}{\mathrm{d}z} - 1\right)\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}z} = 0. \tag{4.6}$$

If $C = S = 0$ on M, then $\mathbf{x}$ is affine equivalent to the immersion $\mathbf{q} : U \to \mathbb{R}^4$ given by $\mathbf{q}(u, v) = (u, v, uv, u^2 - v^2)$ and U is a connected open subset of $\mathbb{R}^2$. Viewing $\mathbb{R}^4$ as $\mathbb{C}^2$ with coordinates z^i, the image of $\mathbf{q}$ is part of the graph of $z^2 = (z^1)^2$.

One may show there are no integrability conditions. Hence, given any open simply connected oriented surface M, a rank 2 vector bundle BM, an elliptic symmetric section A of $T^*M \otimes T^*M \otimes BM$, a flat symmetric connection ∇ on M, and $\hat{S}$, a holomorphic quadratic differential with respect to the complex structure induced on M by A, there exists an elliptic immersion $\mathbf{x} : M \to \mathbb{R}^4$ with A the induced affine quadratic form, ∇ the induced affine connection on M, $C = 0$ and S determined by $\hat{S}$ in the obvious manner.

Remark. Similar results hold for hyperbolic immersions. There are more possibilities to consider since all the members of any subset of the set $\{C^3_{112}, C^3_{222}, C^4_{221}, C^4_{111}\}$ of components of C with respect to a standard framing may be set equal to zero.

5. Homogeneous Surfaces

We look for immersions $\mathbf{x} : M \to \mathbb{R}^4$ with a large group of affine symmetries. Let M be oriented and simply connected and suppose $\mathbf{x} : M \to \mathbb{R}^4$ is a nondegenerate immersion. Let $A_0(4, \mathbb{R})$ denote the group of orientation preserving affine transformations of $\mathbb{R}^4$. Suppose G is a Lie group that acts effectively on M and for each $\tau \in G$ (viewing as a diffeomorphism of M) there exists $T \in A(4, \mathbb{R})$ (with necessarily $T \in A_0(4, \mathbb{R})$) such that

$$T \circ \mathbf{x} = \mathbf{x} \circ \tau. \tag{5.1}$$

When a diffeomorphism τ of M satisfies the preceding condition we say that it is *compatible with an affine transformation.* Any compatible τ must preserve invariant 1-forms or interchange them. Such forms exist if C or S is not zero on M; thus, when that is so, the dimension of G is at most 2. If $C = S = 0$ on M, then G is 3-dimensional; for that case, the immersions have simple explicit parametrizations and there is not much to say after that. Thus we will focus on the case where G is 2-dimensional. We say that $\mathbf{x}$ is *homogeneous* if there exists a Lie group G of dimension at least 2 acting effectively on M by compatible diffeomorphisms. If G contains an orientation-reversing diffeomorphism then we say that $\mathbf{x}$ is *symmetric.*

We begin by considering homogeneous elliptic and hyperbolic immersions which are nonsingular and thus possess invariant 1-forms θ^i. Let us suppose that G is a 2-dimensional Lie group and denote the connected component of the identity of G by G_0. Since each $\tau \in G_0$ must preserve θ^i, G_0 acts simply and transitively on M. Any $\tau \in G$ is an isometry of the induced affine metric g. The curvature K of the induced affine metric g must be constant and in the elliptic case $K \leq 0$ since each nontrivial isometry of $\mathbb{S}^2$ has a fixed point. If $K = 0$, then G_0 is the group of all translations of either the Euclidean or Minkowski plane. If $K \neq 0$, then G_0 is the group of all orientation-preserving isometries that preserve a field of geodesics (that foliate M).

Any $\tau \in G\backslash G_0$ must interchange θ^i and $\theta^{\bar{\imath}}$. In the elliptic case this means that $\Re e_1$ must be preserved by such τ. For the hyperbolic case, this means that $e_1 + e_2$ is preserved by such τ. Since every orientation-reversing isometry of a semi-Riemannian surface of constant curvature is a reflection or a glide reflection in a geodesic, the vector field $\Re e_1$ in the elliptic case or $e_1 + e_2$ in the hyperbolic case is foliated by geodesics if $G\backslash G_0 \neq \emptyset$. In fact, if $G\backslash G_0 \neq \emptyset$, then G consists of all the isometries of M that preserve those geodesics.

Consider a homogeneous (nonsingular) elliptic immersion $\mathbf{x}$. All the invariant functions induced on M by $\mathbf{x}$ are constant. In particular, with respect to the canonical framing, the connection form ψ is given by $\psi = a_1\theta^1 - a_{\bar{1}}\theta^{\bar{1}}$, where $a_1 =$ constant. Also $K = -4|a_1|^2$, $R^{\bar{1}}_{11\bar{1}}$, $C^{\bar{2}}_{\bar{1}11}$, and S^1_{21} are constant. We introduce a change of notation and write a, r, c, and s for a_1, $R^{\bar{1}}_{11\bar{1}}$, $C^{\bar{2}}_{\bar{1}11}$, and S^1_{21}, respectively. By the elliptic analogue of (2.7), $\Gamma^1_{11} = 1 - r$ and $\Gamma^{\bar{1}}_{\bar{1}1} = 1 - a$, where $\omega^i_j = \Sigma\Gamma^i_{jk}\theta^k$. Then

$$R^1_{11\bar{1}} = 1 + a - \bar{a} + \bar{a}r - |a|^2 \quad \text{and} \quad N^2_{21\bar{1}} = a - 2\bar{a} + 2\bar{a}r - 2|a|^2 - |c|^2.$$

The integrability condition $R^1_{11\bar{1}} = -N^2_{21\bar{1}}$ implies

$$|c|^2 = 1 + 2\bar{a} - 3a - 3|a|^2 + 3a\bar{r}. \tag{5.2}$$

One may show that $N^{\bar{2}}_{21\bar{1}} = (2\bar{r} - 3\bar{a} - 1)c$. Then the elliptic analogues of (3.6) and (3.7) become

$$s + c\bar{s} = (1 - 2r)r - (3\bar{a} - \bar{r})(3\bar{a} - 2\bar{r} + 1)c, \tag{5.3}$$

$$2\bar{a}s = (1 - r - a)(1 + a - \bar{a} + ar - |a|^2) - r + (2r - 3a - 1)|c|^2. \tag{5.4}$$

If the affine Riemannian metric has curvature $K = 0$, then $a = 0$ and the system of equations (5.2)–(5.4) reduce to

$$\left.\begin{aligned} c &= -\frac{s + r(2r-1)}{\overline{s + r(2r-1)}} && \text{if } s \neq r(1-2r) \\ |c| &= 1 && \text{if } s = r(1-2r). \end{aligned}\right\} \tag{5.5}$$

Also, if $K = 0$, M must be $\mathbb{C}$ and we may suppose that $\theta^1 = dz$, where z is the standard coordinate on $\mathbb{C}$. Conversely, if $\mathbb{C}$ is given along with complex constants c, r, and s satisfying (5.5), then there exists a nondegenerate elliptic immersion $\mathbf{x} : M \to \mathbb{R}^4$ with $\theta^1 = dz$ and $C^{\bar{2}}_{\bar{1}11} = c$, $R^{\bar{1}}_{11\bar{1}} = r$ and $S^1_{21} = s$, with respect to the canonical framing for which $e_1 = \partial/\partial z$. Moreover, $\mathbf{x}$ is homogeneous and the induced affine metric on $\mathbb{C}$ is flat. The images of the immersions that were just discussed will be called *flat homogeneous elliptic surfaces.*

PROPOSITION 5.1. *The classifying space for flat homogeneous elliptic surfaces (mod affine transformations of $\mathbb{R}^4$) is $\mathbb{C}^2$ with coordinates r, s projectively blown-up along the curve $s = r - 2r^2$.*

Flat symmetric (nonsingular) elliptic immersions are characterized among the homogeneous ones by the condition that c, r and s are real. For such immersions

$$c = \pm 1 \quad \text{and} \quad c = 1 \Rightarrow s = r - 2r^2.$$

We call the images of such immersions *flat symmetric elliptic surfaces.*

PROPOSITION 5.2. *The classifying space for flat symmetric elliptic surfaces (mod affine transformations of $\mathbb{R}^4$) is $\mathbb{R}^2 \cup \mathbb{R}$ with $\mathbb{R}^2$ (respectively, $\mathbb{R}$) corresponding to the case $c = 1$ (respectively, $c = -1$).*

Among the flat homogeneous elliptic immersions there is precisely one (modulo affine transformations) which is algebraic with respect to a Cartesian coordinate system on $M = \mathbb{C}$ and, in fact, it is a symmetric immersion. This immersion corresponds to $r = \frac{5}{3}$, $s = \frac{19}{9}$, and $c = -1$. Modulo an affine transformation of $\mathbb{R}^4$ and an isometry of $\mathbb{C}$, this immersion $\mathbf{x} : \mathbb{C} \to \mathbb{R}^4$ is given by

$$\mathbf{x}(u + iv) = {}^t(v,\ u + 2v^2,\ uv + \tfrac{2}{3}v^3,\ u^2 + (4u - 1)v^2 + \tfrac{4}{3}v^4), \tag{5.6}$$

where $\mathrm{d}u + i\ \mathrm{d}v = \theta^1$. Because $r = \frac{5}{3}$ (and $a = 0$) one may show that this immersion is 'homogeneous with respect to the action of the equiaffine group on $\mathbb{R}^4$'. This immersion was also found by Jiangfan Li [3]; there it is pointed out that this immersion is a critical point of an equiaffine area functional but no mention is made of the fact that it is equiaffine homogeneous.

Let us turn our attention to homogeneous immersions that are not flat. If $a \neq 0$, then $K < 0$; a natural choice for the domain of such a homogeneous immersion $\mathbf{x}$ is $\mathbb{H} = \{z \in \mathbb{C} : \Im z > 0\}$. Necessarily the domain of such an immersion is conformally equivalent to $\mathbb{H}$. When the domain is $\mathbb{H}$, one may show that

$$\theta^1 = -\frac{\mathrm{d}z}{a(z - \bar{z})}. \tag{5.7}$$

Moreover, c, r, and s are solutions of (5.2)–(5.4). Conversely, if a 1-form is given by the right side of (5.7) on $\mathbb{H}$ and a, c, r, and s satisfy (5.2)–(5.4), then there exists a nondegenerate elliptic immersion $\mathbf{x} : \mathbb{H} \to \mathbb{R}^4$ with θ^1 satisfying (5.7), $C^{\bar{2}}_{111} = c$, $R^{\bar{1}}_{11\bar{1}} = r$, and $S^1_{\bar{2}1} = s$ with respect to the canonical framing. Also the induced connection form $\psi = a\theta^1 - \bar{a}\theta^{\bar{1}}$. The images of such immersions will be called *negatively curved homogeneous elliptic surfaces.*

An explicit computation (with the help of a computer) involving (5.2)–(5.4) shows that the following hold:

PROPOSITION 5.3. *There exists at least one negatively curved homogeneous elliptic surface with any value for* $a \neq 0$. *There are at most 5 such surfaces, which are affinely inequivalent, with any particular value for* $a \neq \pm 1$. *There exists one such surface, up to affine equivalence, with* $a = \pm 1$ *and* $c = 0, 2, -3$ *or* $e^{i\theta}$, $\theta \in \mathbb{R}$, *and no such surfaces with* $a = \pm 1$ *and other values for c.*

PROPOSITION 5.4. *A negatively curved homogeneous elliptic surface with a real and not equal to* ± 1 *or* $a = \pm 1$ *and c real is symmetric. For a not equal to* $\pm \frac{2}{3}$ *or* ± 2, *there are 5 such affinely inequivalent surfaces. For* $a = \pm \frac{2}{3}$ *(respectively,* ± 2*), there exists 4 (respectively, 3) such affinely inequivalent surfaces.*

We will now consider a homogeneous regular harmonic immersion $\mathbf{x} : M^2 \to \mathbb{R}^4$. Let $\mathbf{e}$ denote a canonical framing associated with the immersion. Denote the $(1,0)$-form dual to $\mathbf{e}_1$ by θ^1; thus $h = \theta^1\theta^{\bar{1}}$. Define ψ and a in the same manner as we did earlier for this θ^1. The only other 'primary' invariant is $N^{\bar{2}}_{21\bar{1}}$, which will be denoted by n. Of course, a and n are constant and will change by a factor which is a fifth root of unity if the canonical framing is changed. Define Γ^r_{sk} by $\omega^r_s = \Sigma\Gamma^r_{sk}\sigma^k$. From the definition of ψ, it follows that $\Gamma^1_{1\bar{1}} = -\bar{a}$. Using the facts that $C^2_{11\bar{1}} = 0$ for a harmonic immersion and $C^2_{\bar{1}1\bar{1}} = 1$ for a canonical framing, the third equation of Lemma 4.6 implies that $\Gamma^1_{11} = -\frac{1}{2}(\bar{n} + 3a)$. Thus

$$\omega^1_1 = -\tfrac{1}{2}(\bar{n} + 3a)\theta^1 - \bar{a}\theta^{\bar{1}}.$$

We use this to compute $R^1_{11\bar{1}}$. The integrability condition (4.3) becomes

$$an + |a|^2 = \tfrac{2}{3} \tag{5.8}$$

By (4.5), $S^1_{21} = \frac{1}{2}(3a - \bar{n})\bar{n}$. Recall that $\Gamma^2_{2i} = 2\Gamma^1_{1i}$. The integrability condition (4.4) becomes

$$3|a|^2 n - an^2 - \tfrac{5}{6}(n - \bar{a}) = 0. \tag{5.9}$$

From (5.8) observe that $a \neq 0$. Hence, for all homogeneous regular harmonic immersions the metric h has constant negative curvature. Of course, $\mathbb{H}$ is the natural domain for such immersions and θ^1 may again be given by (5.7). Simultaneously solving (5.8) and (5.9) we find the following, where a *homogeneous regular harmonic surface* is the image of the same kind of immersion.

PROPOSITION 5.5. *For every homogeneous regular harmonic surface,* $|a| = 1$ *or* $\frac{1}{2}$. *Conversely, if* $|a| = 1$ *or* $\frac{1}{2}$ *then there exists a homogeneous regular harmonic surface with that value for a which is unique up to an affine transformation of* $\mathbb{R}^4$.

We have found examples of parametrizations for all the homogeneous regular harmonic surfaces. An immersion X for which $a = e^{i\varphi}$ is given by

$$X(z) = \Re(z,\ iz,\ e^{-5/2i\varphi}z^2,\ e^{-5/2i\varphi}z^3) \quad \text{for } z \in \mathbb{H}.$$

An immersion X for which $a = \frac{1}{2}e^{i\varphi}$ is given by

$$X(z) = \Re[i\, e^{-5/2i\varphi}(z,\ z^2,\ z^3,\ z^4)] \quad \text{for } z \in \mathbb{H}$$

Due to the fact that the group of the fiber of $\hat{\mathcal{F}}^{(3)}_{\mathbf{x}}$ is $\mathbb{Z}_5$, the isotropy subgroup of the stabilizer group of a homogeneous regular harmonic surface contains a subgroup isomorphic to $\mathbb{Z}_5$. The isotropy subgroup is isomorphic to $\mathbb{Z}_5$ precisely when a is not a fifth root of ± 1 or $\pm\frac{1}{2}$, otherwise the isotropy subgroup is isomorphic to the dihedral group D_5.

Finally, consider homogeneous harmonic immersions $\mathbf{x}$ for which $C = 0$ but $S \neq 0$. Modulo reparametrizations, they all have domain $\mathbb{C}$ and satisfy (4.6) with k a complex constant and z the standard coordinate on $\mathbb{C}$. If we consider $\mathbb{R}^4 = \mathbb{C}^2$ with coordinates z^i, then the images of these immersions, modulo affine transformations of $\mathbb{R}^4$, consist of the graphs of $z^2 = \log z^1$, $z^2 = z^1 \log z^1$ and $z^2(z^1)^{-R} = 1$, where $R \in \mathbb{C}\backslash\{\frac{1}{2}, 1, 2\}$. (Here we view $\log z^1$ and $(z^1)^{-R}$ as multiple-valued functions where appropriate.) For comparison it is worth remarking that when $C = S = 0$ the corresponding surface is the graph of the complex parabola $z^2 = (z^1)^2$.

Very similar results hold for homogeneous hyperbolic immersions in the non-singular case and in comparable singular cases where the immersion is a translation surface.

References

1. Klingenberg, W.: Zur affinen Differentialgeometrie. Teil I: Über p-dimensionale Minimalflächen und Sphären im n-dimensional Raum, *Math. Z.* **54** (1951), 65–80.
2. Klingenberg, W.: Zur affinen Differentialgeometrie. Teil II: Über zweidimensionale Flächen im vierdimensionalen Raum, *Math. Z.* **54** (1951), 184–216.
3. Li, J.: Harmonic surfaces in Affine 4-space (preprint).
4. Nomizu, K. and Vrancken, L.: A new equiaffine theory for surfaces in $\mathbb{R}^4$, *Int. J. Math.* **4** (1993), 127–165.
5. Weise, K.: Der Berührungstensor zweier Flächen und die Affingeometrie der F_p im A_m (Teil I), *Math. Z.* **43** (1938), 469–480.
6. Weise, K.: Der Berührungstensor zweier Flächen und die Affingeometrie der F_p im A_m (Teil II), *Math. Z.* **44** (1939), 161–184.
7. Wilkinson, S.: General affine differential geometry for low codimension immersions, *Math. Z.* **197** (1988), 583–594.