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The Surveillance-Evasion Game of Degree

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Abstract. The game of degree is analyzed in a surveillance-evasion problem: the evader strives to escape as soon as possible from the pursuer's detection circle, while the pursuer's desire is the opposite. The evader moves with constant speed and is capable of instantaneous direction changes. The pursuer has a minimum turn-radius, independent of speed, and can move forward with any speed not exceeding a maximum greater than the evader's speed.

The solution is surprisingly complex, including regions where the pursuer's speed is optional, *switch envelopes*, *focal lines*, as well as *chattering* by the pursuer to prevent the crossing of certain *barriers*.

Key Words. Pursuit-evasion problems, differential games, optimal strategies, bounded-state problems, barriers.

1. Introduction

The first description in the open literature of the surveillance-evasion problem was by Dobbie (Ref. 1). A pursuer wishes to keep an evader within a specified detection-radius. The evader is assumed to move with constant speed but to be capable of instantaneous direction changes. The pursuer has a minimum turn-radius, independent of speed, and can move forward at any speed not exceeding a given maximum speed greater than the evader's speed. The pursuer can stop but cannot reverse.

Dobbie gave the conditions (with certain minor typographical errors) on the speed-ratio and on the ratio of detection-radius to pursuer's turn-radius, in order that there exist a *surveillance region* from which

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the evader cannot escape. He thus solved the *game of kind* for the surveillance–evasion problem.

This paper considers the *game of degree*, in which the evader, if he can escape, strives to do so in minimum time, while the pursuer strives to maximize this time.

2. Basic Equations for the Surveillance–Evasion Game of Degree

A pursuer P wishes to keep an evader E within a specified detection radius r_D. E is assumed to move with constant speed but to be capable of instantaneous direction changes. P has a minimum turn-radius R, independent of speed, and can move at any forward speed not exceeding a given maximum speed. He can stop but not reverse.

If γ, $\gamma < 1$, is the ratio of evader's speed to pursuer's maximum speed, and if β is the ratio r_D/R, the equations of motion of E relative to P, in suitable units in which $R = 1$, are

$$\dot{x} = -\varphi_1\varphi_2 y + \gamma \sin \psi,$$

$$\dot{y} = -\varphi_1 + \varphi_1\varphi_2 x + \gamma \cos \psi.$$

Here, the y-axis is oriented along the instantaneous direction of P's motion (or of his possible motion in case he is standing still); ψ is E's direction measured clockwise from the positive y-axis; φ_1, $0 \leqslant \varphi_1 \leqslant 1$, is P's normalized speed; and φ_2, $-1 \leqslant \varphi_2 \leqslant 1$, is the normalized curvature of P's path, $+1$ indicating a minimum-radius right turn. We assume that E wishes to minimize the time τ to reach

$$\sqrt{(x^2 + y^2)} = r > \beta$$

and that P wishes to maximize this time and, if possible, to postpone it indefinitely.

E's optimal direction ψ^* is given in terms of the gradient of the optimal $\tau(x, y)$ by

$$\sin \psi^*/\tau_x = \cos \psi^*/\tau_y = -1/\sqrt{(\tau_x^2 + \tau_y^2)},$$

so that E runs perpendicular to the isochrones, and P's optimal controls are:

$$\varphi_1^* = \begin{cases} 1 & \text{if } \; |x\tau_y - y\tau_x| - \tau_y > 0, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\varphi_2{}^* = \operatorname{sign}(x\tau_y - y\tau_x) \qquad \text{when} \quad \varphi_1{}^* > 0.$$

As in the homicidal chauffeur game (Ref. 2), in the interior of any region in which τ has a continuous gradient,

$$\dot{\psi}^* = -\varphi_2 ,$$

so that E is travelling along a straight line. Also, as in the homicidal chauffeur game, E is running radially outward at game termination, the gradient (t_x , τ_y) being in this case radially inward. However, this game only terminates on that part of the circle $r = \beta$ on which

$$\theta = \tan^{-1}(x/y)$$

lies *outside*, rather than inside, of the interval

$$-\cos^{-1} \gamma \leqslant \theta \leqslant \cos^{-1} \gamma.$$

In this game, moreover, $\varphi_1{}^* = 0$ at termination if $y < 0$; whereas, if $y > 0$, $\varphi_1{}^* = 1$ and $\varphi_2{}^* = \operatorname{sign} x$ immediately prior to termination, implying that P is turning toward E.

3. Solution in Some Simple Cases

The paths and the isochrones for a game for which $\beta^2 + \gamma^2 < 1$ are shown in Fig. 1. The unshaded regions are obtained by simple backward construction of the paths which terminate. The situation in the shaded region is more interesting. It should be noted first that the upper right-hand path, for example, of the lower unshaded region and the lowest path of the upper right-hand unshaded region are both described with E running directly toward Q, the center of curvature of P's minimum-radius right turn. It should then be observed that, if E runs directly toward Q from any point in the right half of the shaded region, his rate of approach towards Q is independent of P's speed φ_1 if P uses $\varphi_2 = +1$ and is increased by any other choice of φ_2. This rate is reduced, of course, if E does not run directly toward Q. P can moreover choose a time-history for his speed φ_1 so as to force the relative position of E through the point A, φ_1 being of course nonunique, and this clearly yields the maximum duration τ that P can obtain when E runs toward Q. Should E reach the lower boundary of the upper right unshaded region at a point above A, this boundary constitutes a compulsory switch-line for P who must

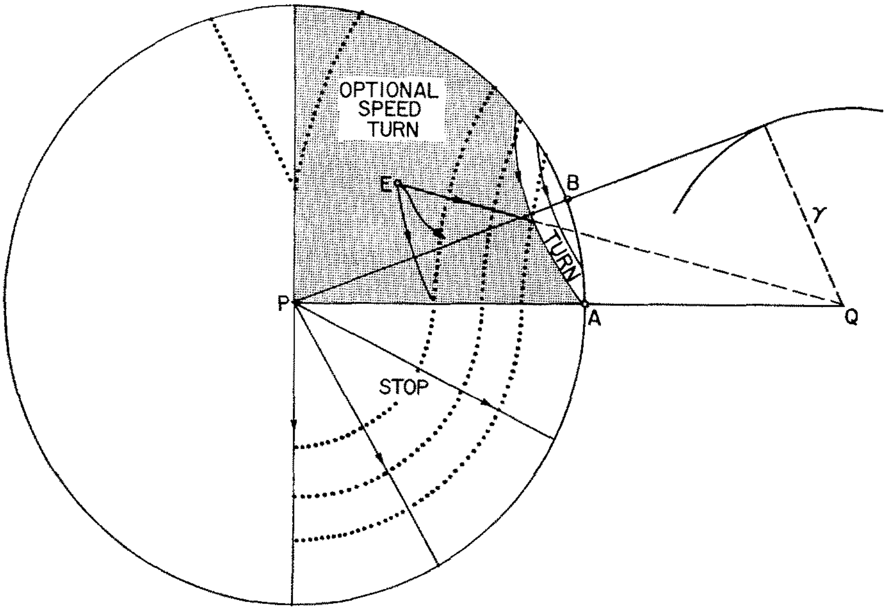

Fig. 1. Solution for $\beta < \sqrt{(1-\gamma^2)}$; →— optimal paths; ...... isochrones.

now turn at maximum speed. This switch-line is of the type called *transition surface* by Isaacs, across which the isochrones have continuous direction. The isochrones in Fig. 1 have a discontinuous direction only on the positive y-axis, which is a *dispersal line* (DL) from which the direction of relative motion is determined by E's choice of Q or its left image as a goal. This DL we may, then, further designate as EDL.

A quite different case arises for large β (see Fig. 2), more specifically for

$$\beta > 1 + \sqrt{(1-\gamma^2)} + \gamma(\pi + \sin^{-1}\gamma).$$

In this case, the path terminating at the upper end B ($\theta = \cos^{-1}\gamma$) of the usable terminal arc has infinitely large $\sqrt{(\tau_x^2 + \tau_y^2)}$, and thus constitutes a *barrier*, which may be obtained, as shown[3] by Isaacs in the homicidal chauffeur game, by winding a string around a circle $\mathscr{C}$ with center Q and radius γ, the string starting along a radius from P. The reason for this can be seen in Fig. 3, in which the paths of P and E are shown in a fixed reference frame, together with the path of a fictitious point P' moving on $\mathscr{C}$ directly between P and Q. Since the arc P_1P_f is $(1/\gamma)$(arc $P_1'P_f'$), we see that the arc $P_1'P_f'$ is equal in length to the line segment E_1E_f, which implies the winding string property.

[3] But, in that game, the string unwinds.

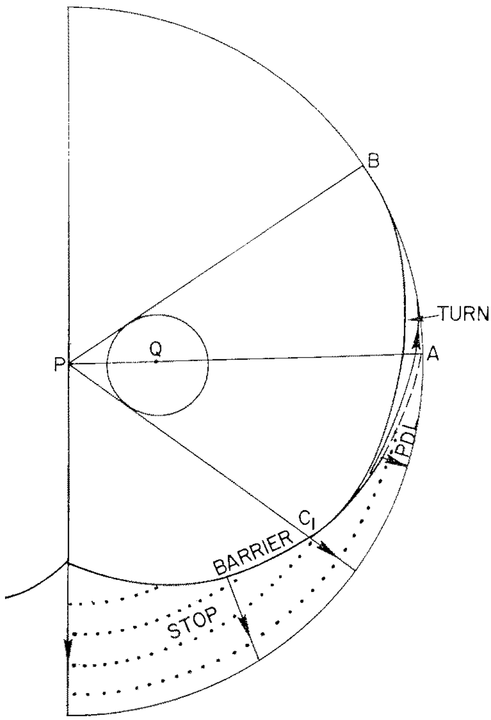

Fig. 2. Solution for $\beta > f_3(\gamma)$; →— optimal paths; ...... isochrones.

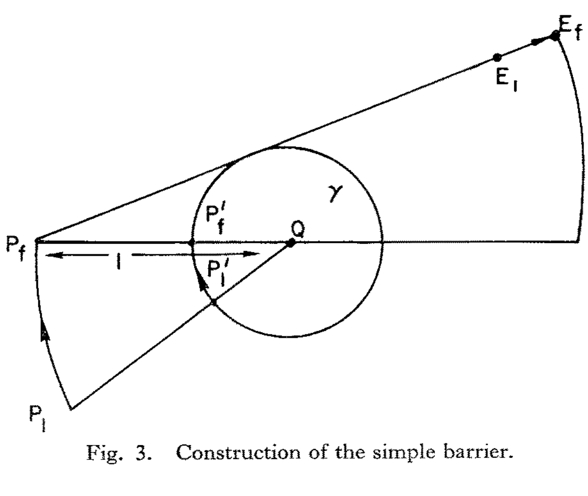

Fig. 3. Construction of the simple barrier.

This barrier reaches the negative y-axis, and hence, together with its left-hand image, forms a closed barrier if

$$\beta \geqslant 1 + \sqrt{(1-\gamma^2)} + \gamma(\pi + \sin^{-1}\gamma) \triangleq f_3(\gamma).$$

Optimal paths are defined only below this closed barrier, since E cannot escape from the upper enclosed region provided that P makes the appropriate full speed turn when E approaches the barrier from above. In this case, since Q is inside the detection circle, the paths which terminate above the x-axis with a turn arrive from below the x-axis. A dispersal line (PDL), shown by a dotted line, separates P's STOP region from his TURN region, the choice being P's. Note that this PDL reaches the barrier at a point C_1 on the extended lower tangent from P to $\mathscr{C}$. This follows from the observation that the angle wound by the string from B to C_1 is equal to $(1/\gamma)(PB - PC_1)$, which is equal in turn to the time of a radial path from C_1 to the detection circle.

A more interesting case arises if

$$f_2(\gamma) < \beta < f_3(\gamma),$$

where

$$f_2(\gamma) = \sqrt{(1-\gamma^2)} + \gamma(\pi + 2\sin^{-1}\gamma).$$

In this case (see Fig. 4), the barrier does not reach the negative y-axis, but it recrosses the lower tangent from P to C at a point C_2 between P and

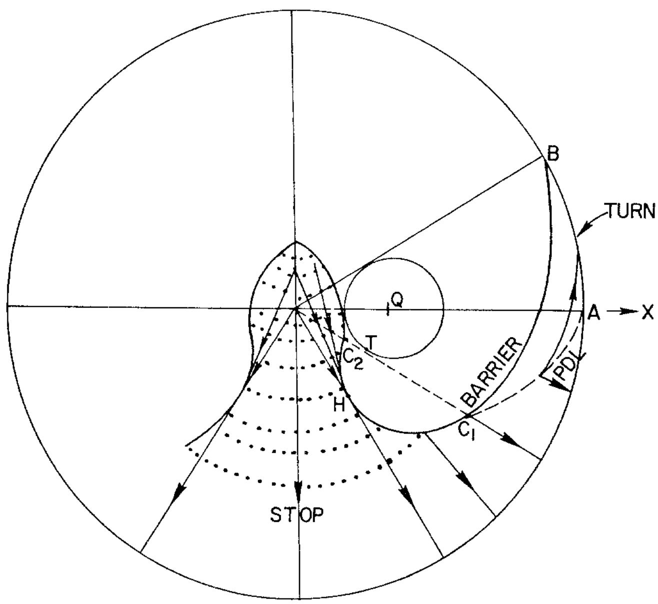

Fig. 4. Solution for $f_2(\gamma) < \beta < f_3(\gamma)$; →— optimal paths; ...... isochrones.

the point T of contact with $\mathscr{C}$. At C_2, the barrier is perpendicular to the radius from P, and so φ_2 changes sign, the barrier being here continued by unwinding a string around the left-hand image $\mathscr{C}'$ of $\mathscr{C}$. This continuation meets its reflection on the positive y-axis, thus again completing a closed barrier. The optimal paths below the barrier in the STOP region on either side of P are straight lines bent around the barrier as far as its tangent from P, the barrier here imposing a *state constraint* on E who must avoid, at all costs, entering the region above the barrier. Note that, although the barrier above C_2 is a left-turn path, implying a left turn by P for points immediately *above* the barrier, this left turn is of no advantage below the barrier. The part of the positive y-axis below the barrier constitutes again an EDL from which E can choose a left or a right path tangent to the barrier. The isochrones have continuous direction, except on the EDL and the left and right PDL's.

4. Solution in More Difficult Cases

We come now to those cases in which there is a simple barrier but it is open. We take up first the relatively simple case in which

$$\beta < 1, \qquad \text{but} \quad \beta^2 + \gamma^2 > 1.$$

Here, Q is just outside the detection circle and there is a very short barrier (see Fig. 5). Any further analytic continuation of this barrier would reveal a cusp at the circle $\mathscr{C}$, and hence and end to the semipermeable character required. The situation is very similar to that in Fig. 1 where there is no barrier, the short barrier being circumvented by adjacent turn paths.

We take up next the cases in which

$$1 < \beta < f_1(\gamma),$$

with

$$f_1(\gamma) = \sqrt{(1-\gamma^2)} + \gamma\left(1 + \frac{\pi}{2} - 2\sin^{-1}\gamma\right).$$

If β is not too much greater than 1, the situation is as illustrated in Fig. 6. Here, the PDL which ends at A is constructible backwards as far as a point D at which its tangent passes through P. At D, it is continued backward by a rather short *switch envelope* (SE), with straight lines arriving tangent on the lower side, and turn paths leaving the upper side towards termination on the detection circle. The switch envelope phenomenon seems to have appeared first in Ref. 3. The SE is here

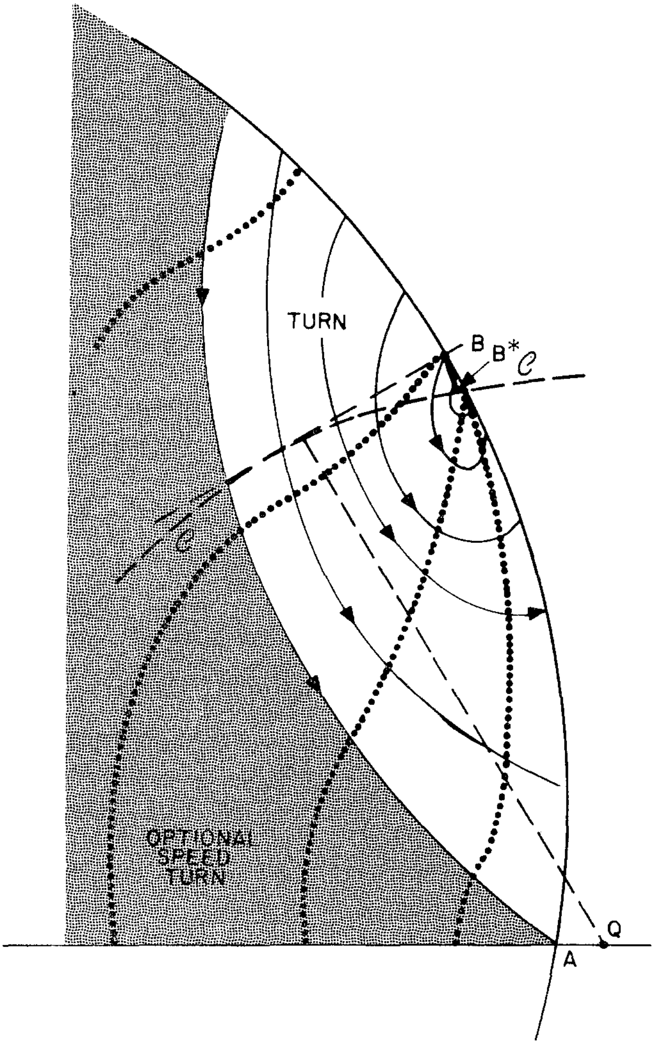

Fig. 5. Solution for $\sqrt{(1-\gamma^2)} < \beta < f_3(\gamma)$; →— optimal paths; ...... isochrones.

continued backward to a point G where paths are tangent both above and below. G, in turn, is the end of a *focal line* (FL), which starts at Q.

The isochrone directions are discontinuous across the FL, forming in this case a *ravine* which P, the maximizer, endeavors to leave to one side or the other, but is frustrated by E, who counteracts P's control in order to remain on the FL. For the duration of this motion along the FL toward G, E faces a *perpetuated dilemma*, in that he must react immediately to any changes of strategy by P.

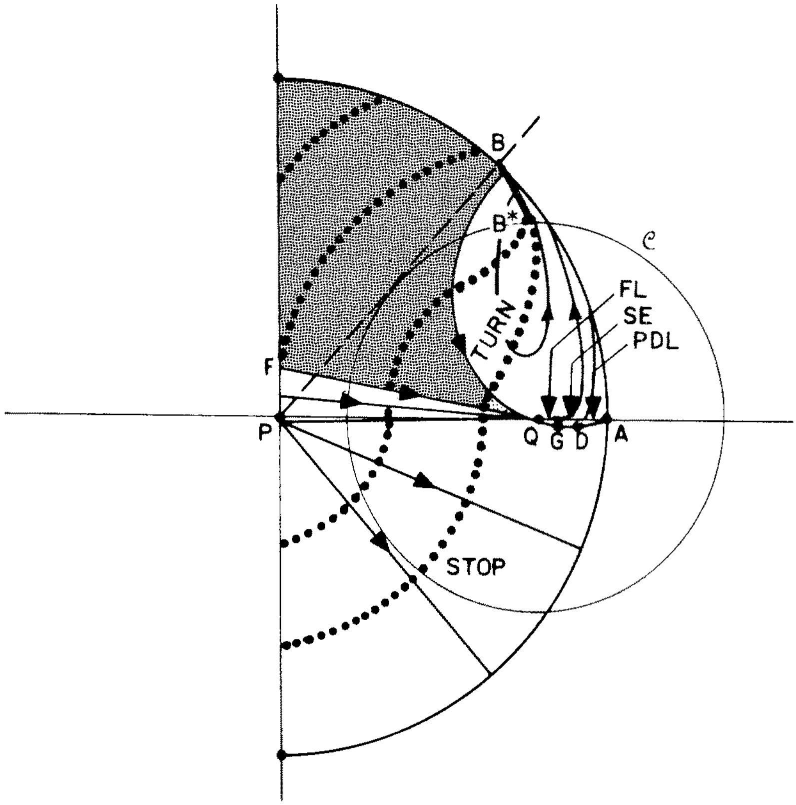

Fig. 6. Solution for $1 < \beta < f_1(\gamma)$, β not close to $f_1(\gamma)$; →— optimal paths; ...... isochrones.

The FL between Q and G is, moreover, a circular arc of radius γ. To see this, consider a position E, distant r' from Q, and construct O at distance γ from both Q and E (see Fig. 7). Clearly,

$$\overrightarrow{EO} = \overrightarrow{EQ} + \overrightarrow{QO};$$

but clockwise rotation of $\overrightarrow{EQ}$ through 90° gives the velocity $\vec{v}_3$ of E, due solely to P's right turn. Clockwise rotation of $\overrightarrow{EO}$ and $\overrightarrow{QO}$ through 90° thus yields two velocities $\vec{v}_1$ and $\vec{v}_2$, of magnitude γ, such that $\vec{v}_1 = \vec{v}_2 + \vec{v}_3$. Therefore, E can guarantee motion perpendicular to $\overrightarrow{EO}$, i.e., along a circle with center 0 and radius γ, by running perpendicular to $\overrightarrow{QO}$ if P turns. This proves that the FL is an arc of radius γ and center O, and E's direction, in case P turns, is perpendicular to $\overrightarrow{OQ}$ for all points on this FL.

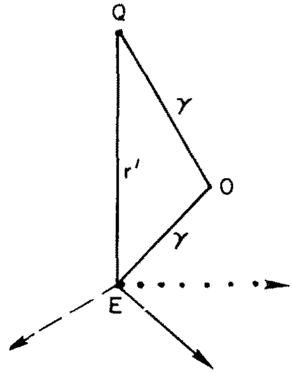

Fig. 7. Construction of the focal line.

Returning to Fig. 6, the straight and curved relative paths arriving tangent to the FL at Q form, as in Figs. 1 and 5, the boundary of a region (shaded) in which E runs toward Q and P turns hard right but in which P's speed is optional. Note that as $\beta \to 1$ from above, Q approaches A and the FL, SE, and P DL all shrink to the point A, with the OPTIONAL SPEED region rotating into its position in Figs. 1 and 5.

A further complication arises (see Fig. 8) if

$$1 < \beta < f_1(\gamma),$$

but β is now not too close to the indicated lower limit. The region above the barrier near B is now a TURN-AWAY region, where a left turn is advantageous for P. The TURN-AWAY region is separated from the TURN region by a P DL which reaches the barrier at the start of that right-turn path whose turn switch-function starts at zero.

The left-turn paths, however, reach the barrier from above. Here, we must assume that P reacts immediately with a right turn to any attempt by E to cross the barrier. Therefore, E finds such an attempt fruitless and instead chooses ψ to more leftward along the barrier as fast as possible, with P using an intermediate

$$\varphi_2 = \tilde{\varphi}(\psi, x, y), \qquad \text{with} \quad -1 < \tilde{\varphi} < 1,$$

so as to keep the path immediately adjacent to the barrier. The use of a control $\tilde{\varphi}(\psi, x, y)$ by P, depending on knowledge of E's control ψ as well as the state (x, y), may be justified by replacing this by a disconitnuous closed-loop strategy

$$\varphi(x, y) = \begin{cases} -1 & \text{above} \\ +1 & \text{below} \end{cases} \text{ a line } \mathscr{L} \text{ parallel to the barrier}$$

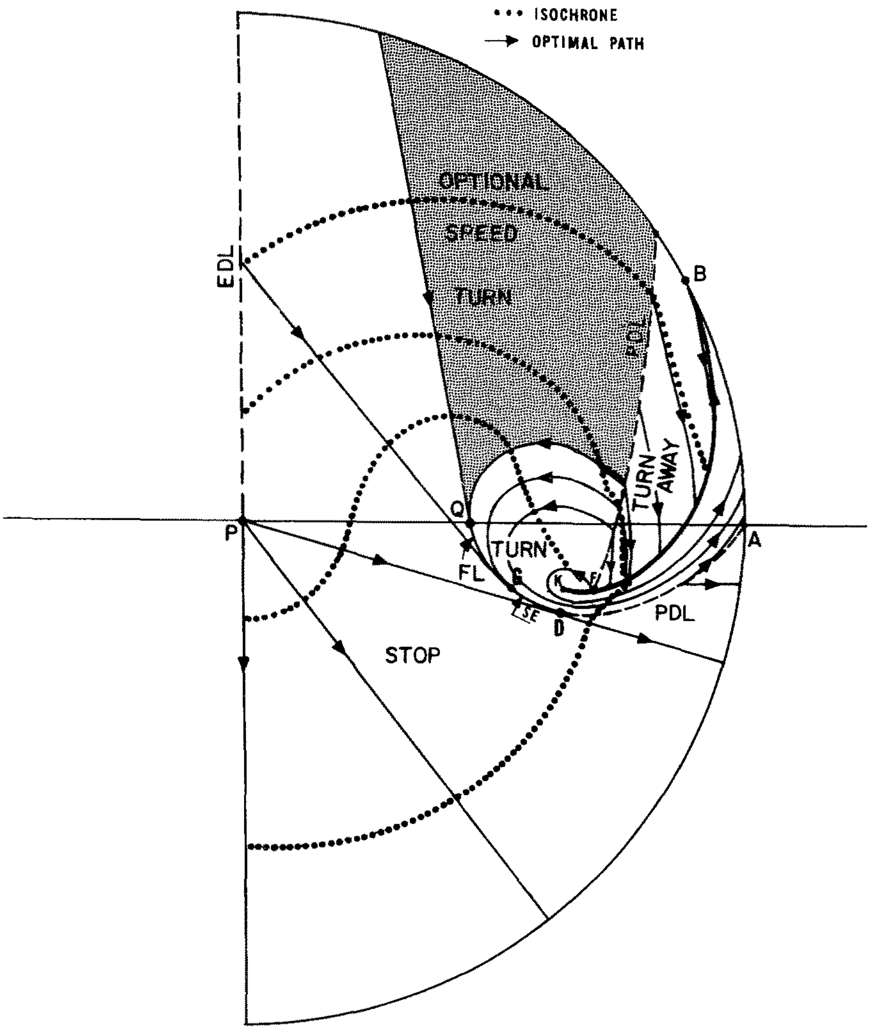

Fig. 8. Solution for $1 < \beta < f_1(\gamma)$, β not close to 1; →— optimal paths; ...... isochrones.

and a short distance ϵ above it. This not only prevents E from crossing the barrier but produces a chatter along $\mathscr{L}$, at a rate equal, in the limit as $\epsilon \to 0$, to the rate along the barrier produced by $\tilde{\varphi}(\psi, x, y)$. The optimal ψ, which maximizes this rate, is then calculable. This ψ is also E's direction on arrival at the barrier from above on paths which, it should be noted, are *not* tangent to the barrier. This is because the barrier here, in contrast to Fig. 4, imposes a *state constraint* on P, whose vectogram is straight. Indeed, if x' denotes distance along the barrier and y' denotes a distance above the barrier,

$$\max_{\varphi} \min_{\psi}(\tau_{x'}\dot{x}' + \tau_{y'}\dot{y}' + 1) = 0 = \min_{\substack{\psi \\ \varphi(\psi)\to\dot{y}'=0}} (\tau_{x'}\dot{x}' + 1) = 0,$$

which implies not only the continuity of ψ, but also the vanishing of P's turn switch-function on arrival at the barrier, so that ψ corresponds to direct pursuit of P by E. The continuity of E's direction, together with the discontinuity in φ_2, provides a discontinuity in the direction of the relative paths. Knowledge of ψ next determines the arriving paths, which are then constructible back to the P DL, together with the isochrones, after making use of the time-history along the barrier.

The TURN-AWAY region extends above the x-axis, and some paths will follow the barrier from points somewhat above the x-axis; and, as long as $y > 0$, P uses $\varphi_2 = -1$ with intermediate speed $\tilde{\varphi}_1(\psi, x, y)$. Now, ψ corresponds to motion by E directly towards Q', the left-hand image of Q, since P's speed switch-function must now vanish on arrival at the barrier from the left. The line $\mathscr{L}$ is now replaced by two lines $\mathscr{L}_1$, $\mathscr{L}_2$ which converge as $y \to 0$. The region between $\mathscr{L}_1$ and $\mathscr{L}_2$ is a STOP region, and chatter occurs on the line $\mathscr{L}_1$ further from the barrier. $\mathscr{L}_2$ merely serves to prevent E from crossing the barrier.

Finally, we take up the case

$$f_1(\gamma) < \beta < f_2(\gamma).$$

Here, the barrier crosses the lower tangent from P to $\mathscr{C}$ at two points C_1, C_2 beyond T (see Fig. 9) before arriving (backwards) at $\mathscr{C}$. The tangent from P to this barrier touches the barrier at H between C_1 and C_2. The part of the barrier between H and C_1 is the source of straight paths which proceed radially to the detection circle. Between H and a nearby point G, the barrier is the envelope of arriving straight paths. G is the termination, as in Figs. 6 and 8, of an FL which is an arc of a circle through Q with radius γ. In this case, moreover, G is the unique point of tangency of such an arc with the barrier itself.

Above and to the right of Q, another mutation takes place. The chattering paths along the barrier cease to progress leftward as soon as the barrier crosses below the lower tangent from P to $\mathscr{C}$. In this case, indeed, as pointed out by Dobbie (Ref. 1), P can prevent escape by E, and the resulting SURVEILLANCE region is bounded on the right by our previous right-turn barrier and on the left by a new left-turn barrier ending at C_2, where the corresponding turn switch-function changes sign. Note that, if this switch-function had not changed sign, E would

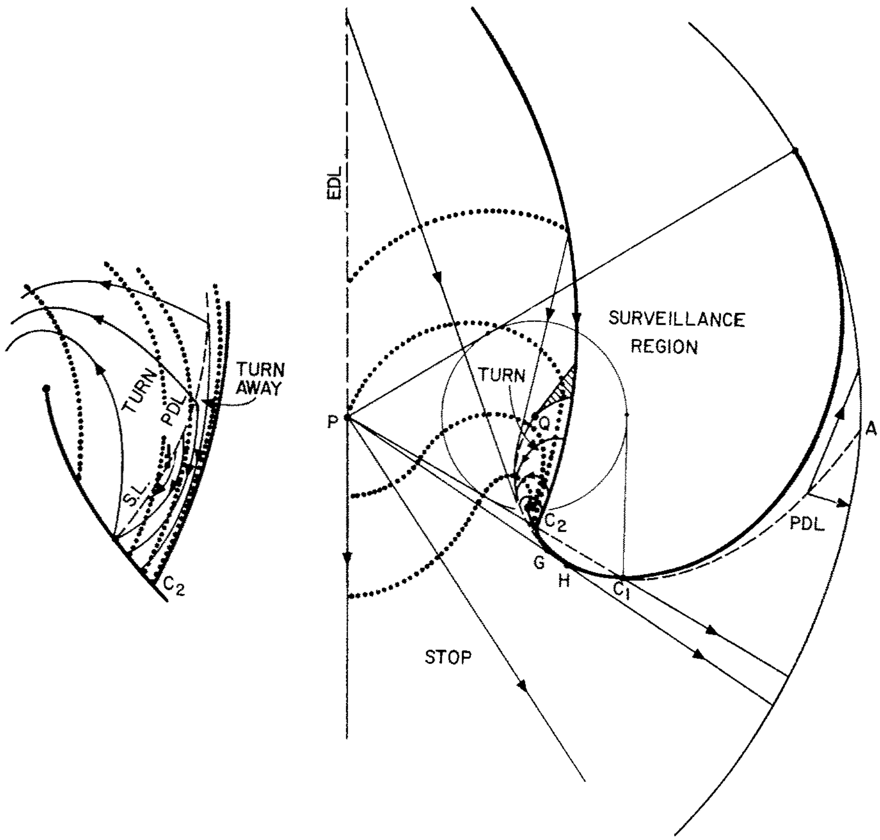

Fig. 9. Solution for $f_1(\gamma) < \beta < f_2(\gamma)$; →— optimal paths; ...... isochrones.

escape by continuing to run leftward and perpendicular to the left-turn barrier after arrival at an intersection with the right-turn barrier.

The new barrier, like the previous barriers, is obtained by winding or unwinding (in this case unwinding) a string around the circle $\mathscr{C}$ or its left-hand image $\mathscr{C}'$ (in this case $\mathscr{C}'$), the winding or unwinding depending on which side of the barrier E is striving to avoid. This is apparent from Fig. 10, where, for semipermeability, E's direction is perpendicular to the resulting relative motion, and the velocity triangle is seen to be congruent to, and rotated through 90° from, the triangle formed by E, Q', and the tangent from E to $\mathscr{C}'$.

To the left of the left-turn barrier, the paths in Fig. 9 are somewhat as in Fig. 8, although the TURN-AWAY region is now only a very narrow strip adjacent to the left-turn barrier along its entire length. This TURN-AWAY region is separated from the TURN region at its lower

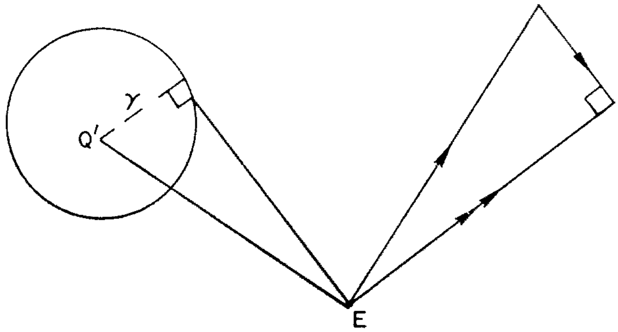

Fig. 10. Construction for all barriers.

end by a short switch line (SL) across which isochrone directions are continuous. The SL is continued upwards by the P DL. Note that τ and $\nabla\tau$ become infinitely large as we approach the left-turn barrier from the left: P finds it advantageous to get as close as possible to C_2, where progress leftward along the barrier is infinitely slow.

The justification for the figures in this section, as in the earlier sections, rests on the satisfaction of Isaacs' main equation at all points where isochrones have been constructed, as well as on the semipermeability of the barriers separating the remaining region, if any. This includes the composite closed barrier of Fig. 9. The indicated strategies are *globally* optimal for the game of degree, because P is crossing the isochrones as slowly as possible, and E as rapidly as possible. Dobbie's composite closed barrier in Fig. 9, as well as Dobbie's closed simple barriers in Figs. 3 and 4 constitute the solution to the *game of kind*. Without, however, some knowledge of the game of degree, we have no assurance that the game of kind has been solved. This knowledge should consist either of the complete solution to the game of degree as in Fig. 9, or else of some upper bound, guaranteeable by E but depending on position, to escape time from any position in the supposed ESCAPE region. Thus, it is possible to construct another composite barrier by means of a left-turn barrier ending at C_1, where the turn switch-function also changes sign. The resulting SURVEILLANCE region is not the largest that P can maintain.

The rather complex figures in this paper have been obtained purely by graphical construction. Except in the case of the SE, whose geometrical construction must rest on a construction for the direction at each point, leading to a geometric *integration* which could introduce substantial error, the isochrones and paths retain reasonable geometric accuracy. The requirement, however, that the SE be in turn tangent to an FL which is a circular arc through Q of known radius, preserves a good accuracy, not only for the SE but also for the direction of the FL at Q, which determines the rest of the figure to the right of Q. These figures were constructed using $\gamma = \frac{1}{2}$ and a few representative values for β, and the reader may trace a certain evolution with increasing β, starting with Fig. 1 and proceeding with Figs. 5, 6, 8, 9, 4, and 2. Nevertheless, computer-aided experience (Refs. 3–4) with the homicidal chauffeur game suggests that it might be rash to assert that no further interesting variations can arise in the present game for some β, γ'.

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