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On Functions Whose Local Minima Are Global

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Abstract. In this paper, necessary and sufficient conditions for a local minimum to be global are derived. The main result is that a real function, defined on a subset of R^n, has the property that every local minimum is global if, and only if, its level sets are lower-semicontinuous point-to-set mappings.

Key Words. Global optimality, nonconvex programming, point-to-set maps, mathematical programming, nonlinear optimization.

1. Introduction

The family of real functions on R^n whose local minima are global is of considerable interest in optimization. The purpose of this work is to present characterizations of such functions by a property, called lower semicontinuity of their level sets. Semicontinuity of point-to-set mappings has been used in optimization theory for various purposes [see, for example, Dantzig, Folkman, and Shapiro (Ref. 1) Meyer (Ref. 2), Hogan (Ref. 3), and Zangwill (Ref. 4)]. Here, we propose still another use of this concept.

2. Definitions

Let f be a real function on a subset C of R^n and let α be a real number. Consider the *level sets* of f

$$L_f(\alpha) = \{x: x \in C, f(x) \leqslant \alpha\} \tag{1}$$

and the set

$$G = \{\alpha: \alpha \in R, L_f(\alpha) \neq \varnothing\}. \tag{2}$$

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It follows that $L_f(\alpha)$ is a point-to-set mapping of points in G into subsets of R^n. We have then (Ref. 2) the following definition.

Definition 2.1. The point-to-set mapping $L_f(\alpha)$ is said to be lower semicontinuous (lsc) at a point $\alpha \in G$ if $x \in L_f(\alpha)$, $\{\alpha^i\} \subset G$, $\{\alpha^i\} \to \alpha$ imply the existence of a natural number K and a sequence $\{x^i\}$ such that

$$x^i \in L_f(\alpha^i), \qquad i = K, K+1,..., \qquad \text{and} \qquad \{x^i\} \to x. \tag{3}$$

If $L_f(\alpha)$ is lsc at every $\alpha \in G$, it is said to be lsc on G.

Let $B_\delta(x) \subset R^n$ denote an open ball with radius δ centered around x.

Definition 2.2. A point $\bar{x} \in C$ is a local minimum of f if there exists a $\delta > 0$ such that

$$f(x) \geqslant f(\bar{x}) \tag{4}$$

for every $x \in C \cap B_\delta(\bar{x})$, and it is a global minimum of f on C if (4) holds for every $x \in C$.

3. Results

Theorem 3.1. Let f be a real function on $C \subset R^n$, and let $\bar{\alpha} = f(\bar{x})$, $\bar{x} \in C$. Suppose that $L_f(\alpha)$ is lsc at $\bar{\alpha}$. If $\bar{x}$ is a local minimum of f, then it is also a global minimum of f on C.

Proof. Suppose that the hypotheses hold and $\bar{x}$ is not a global minimum of f on C. Hence, there exists a point $\tilde{x} \in C$ such that

$$f(\tilde{x}) < f(\bar{x}). \tag{5}$$

Define the sequence $\{\alpha^i\}$ by

$$\alpha^i = [(1/i) f(\tilde{x}) + (1 - 1/i) f(\bar{x})], \qquad i = 1, 2,... . \tag{6}$$

Clearly,

$$\lim_{i\to\infty}\{\alpha^i\} = f(\bar{x}) = \bar{\alpha} \tag{7}$$

and $\bar{x} \in L_f(\bar{\alpha})$. From (5) and (6), it follows that

$$f(\tilde{x}) \leqslant \alpha^i < f(\bar{x}), \qquad i = 1, 2,..., \tag{8}$$

and $\tilde{x} \in L_f(\alpha^i)$, $i = 1, 2,...,$ hence $\{\alpha^i\} \subset G$.

Since $L_f(\alpha)$ is assumed to be lsc at $\bar{\alpha}$, there exists a natural number K and a sequence $\{x^i\}$ converging to $\bar{x}$ such that $x^i \in L_f(\alpha^i)$ for $i = K, K+1,\ldots$. Hence,

$$f(x^i) \leqslant \alpha^i, \qquad i = K, K+1,\ldots; \tag{9}$$

and, by (8),

$$f(x^i) < f(\bar{x}), \qquad i = K, K+1,\ldots. \tag{10}$$

Since $\{x^i\} \to \bar{x}$, for a sufficiently small $\delta > 0$ there exists a natural number K_δ such that $x^i \in C \cap B_\delta(\bar{x})$, $i = K_\delta, K_\delta + 1,\ldots$; and, by the hypotheses,

$$f(x^i) \geqslant f(\bar{x}), \qquad i = K_\delta, K_\delta + 1,\ldots, \tag{11}$$

contradicting (10). □

Lemma 3.1. Let f be a real function on $C \subset R^n$, and let $\bar{\alpha} \in G$, $\{\alpha^i\} \subset G$, $\{\alpha^i\} \to \bar{\alpha}$. If $f(x) < \bar{\alpha}$, then there exists a natural number K and a sequence $\{x^i\}$ such that (3) holds.

Proof. Since $f(x) < \bar{\alpha}$ and $\{\alpha^i\} \to \bar{\alpha}$, there is a natural number K such that

$$f(x) \leqslant \alpha^i, \qquad i = K, K+1,\ldots. \tag{12}$$

Hence, the sequence $\{x^i\}$, constructed by letting $x^i = x$ for all i, satisfies (3). ◁

Corollary 3.1. Let f and $\bar{\alpha}$ be defined as in Lemma 3.1. If $f(x) < \bar{\alpha}$ for every $x \in L_f(\bar{\alpha})$, then $L_f(\alpha)$ is lsc at $\bar{\alpha}$.

Proof. Follows directly from Definition 2.1 and Lemma 3.1. □

Theorem 3.2. Let f be a real function on $C \subset R^n$. If every $x \in C$ satisfying $f(x) = \bar{\alpha}$ is either a global minimum of f on C or it is not a local minimum of f, then $L_f(\alpha)$ is lsc at $\bar{\alpha}$.

Proof. We have to show that, for every $x \in L_f(\bar{\alpha})$ and $\{\alpha^i\} \to \bar{\alpha}$, $\{\alpha^i\} \subset G$, there exists a sequence $\{x^i\}$ with the required properties for lower semicontinuity of $L_f(\alpha)$ at $\bar{\alpha}$. The existence of such a sequence for $x \in L_f(\bar{\alpha})$ such that $f(x) < \bar{\alpha}$ is assured by Lemma 3.1. Consider, therefore, points satisfying

$$f(x) = \bar{\alpha}. \tag{13}$$

Suppose first that x is a global minimum of f on C. Then, every sequence $\{\alpha^i\} \subset G$, $\{\alpha^i\} \to \bar{\alpha}$, must satisfy

$$\alpha^i \geqslant \bar{\alpha}, \qquad i = 1, 2,\dots . \tag{14}$$

Hence, we can construct the desired sequence $\{x^i\}$ by letting

$$x^i = x, \qquad i = 1, 2,\dots . \tag{15}$$

Suppose now that x is not a local minimum of f on C. Taking the sequence of open balls $B_{\delta(k)}(x)$ with radii

$$\delta(k) = 1/k, \qquad k = 1, 2,\dots, \tag{16}$$

it follows that there exists a sequence $\{\hat{x}^k\}$ converging to x such that

$$\hat{x}^k \in C \cap B_{\delta(k)}(x), \qquad k = 1, 2,\dots, \tag{17}$$

and

$$f(\hat{x}^k) < f(x) = \bar{\alpha}, \qquad k = 1, 2,\dots . \tag{18}$$

Now, from (18) and the convergence of $\{\alpha^i\}$ to $\bar{\alpha}$ follows the existence of a K_1 such that

$$f(\hat{x}^1) \leqslant \alpha^i, \qquad i = K_1, K_1 + 1,\dots . \tag{19}$$

For every α^i, $i = K_1 + 1,\dots$, of the sequence $\{\alpha^i\}$, let x^i be defined in the following way: Choose $\bar{k}(i)$ such that

$$\bar{k}(i) = \sup\{k: f(\hat{x}^k) \leqslant \alpha^i\} \tag{20}$$

if a finite supremum exists; otherwise, let

$$\bar{k}(i) = \bar{k}(i-1) + 1, \tag{21}$$

and let

$$x^i = \begin{cases} \hat{x}^1, & i = 1,\dots, K_1, \\ \hat{x}^{\bar{k}(i)}, & i = K_1 + 1,\dots . \end{cases} \tag{22}$$

Let us show now that $\{x^i\}$ converges to x; i.e., for any positive λ, there exists a natural number $K(\lambda)$ such that

$$x^i \in B_\lambda(x), \qquad i = K(\lambda), K(\lambda) + 1,\dots . \tag{23}$$

Define

$$\hat{k}(\lambda) = \begin{cases} \max\{k: \delta(k-1) = 1/(k-1) > \lambda\} & \text{if } \lambda < 1, \\ 1 \quad \text{if } \lambda \geqslant 1. \end{cases} \tag{24}$$

Then, $B_{\delta(\hat{k}(\lambda))}(x) \subset B_\lambda(x)$,

$$f(\hat{x}^{\hat{k}(\lambda)}) < \bar{\alpha}, \tag{25}$$

and we can find a $\bar{K}(\lambda) \geqslant K_1$ such that

$$f(\hat{x}^{\hat{k}(\lambda)}) \leqslant \alpha^i, \qquad i = \bar{K}(\lambda), \bar{K}(\lambda) + 1,\ldots . \tag{26}$$

Consequently, one of the following situations may occur.

(i) There exists an $\bar{i} \geqslant \bar{K}(\lambda)$ for which $\bar{k}(\bar{i})$ is obtained by (20). Then, for $i \geqslant \bar{i}$, we get from (20), (21), and (26) that $\bar{k}(i) \geqslant \hat{k}(\lambda)$ and

$$\hat{x}^{\bar{k}(i)} \in B_{\delta(\bar{k}(i))}(x) \subset B_{\delta(\hat{k}(\lambda))}(x) \subset B_\lambda(x). \tag{27}$$

Hence,

$$x^i \in B_\lambda(x), \qquad i = \bar{i}, \bar{i} + 1,\ldots . \tag{28}$$

(ii) For all $i \geqslant \bar{K}(\lambda)$, the elements of the sequence $\{x^i\}$ are chosen by (21) and (22). Clearly, for $i \geqslant \bar{K}(\lambda)$, the x^i are consecutively taken from the sequence $\{\hat{x}^k\}$ which converges to x.

In both situations, therefore, $\{x^i\}$ converges to x and

$$x^i \in L_f(\alpha^i), \qquad i = K_1, K_1 + 1,\ldots . \tag{29}$$

Let us illustrate now, by a simple example, a function having a local minimum which is not global and for which the above theorem fails to hold. The function appearing in Fig. 1 has a nonglobal local minimum at $\hat{x}$ and a global minimum at x^*. Also, note that $f(\hat{x}) = f(\bar{x}) = 0$. Let $\bar{\alpha} = 0$, and take the sequence $\{\alpha^i\}$ whose elements are

$$\alpha^i = (1/i)\, f(x^*), \qquad i = 1, 2,\ldots . \tag{30}$$

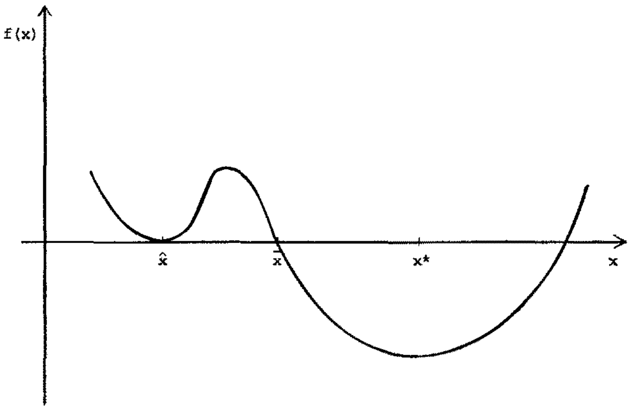

Fig. 1. Function having a nonglobal local minimum.

Clearly, $\{\alpha^i\} \subset G$ and $\{\alpha^i\} \to \bar{\alpha}$. The point $\hat{x}$ is contained in $L_f(0)$; but, in every sequence $\{x^i\}$ such that $x^i \in L_f(\alpha^i)$ for sufficiently large i, one must have $x^i > \bar{x}$, and there is no such sequence which would converge to $\hat{x}$. Hence $L_f(\alpha)$ is not lsc at $\bar{\alpha}$.

Corollary 3.2. Let f be a real function on $C \subset R^n$. If every local minimum of f is a global minimum of f on C, then $L_f(\alpha)$ is lsc on G.

From Theorems 3.1, 3.2 and Corollary 3.1, we immediately get the following characterization theorem of functions whose local minima are global.

Theorem 3.3. Let f be a real function on $C \subset R^n$. Every local minimum of f is a global minimum of f on C iff $L_f(\alpha)$ is lower semicontinuous on G.

The above results also suggest an alternative characterization theorem.

Theorem 3.4. Let f be a real function on $C \subset R^n$. Every local minimum of f is a global minimum of f on C iff, for any two points $\tilde{x} \in C$, $\bar{x} \in C$ such that $f(\tilde{x}) < f(\bar{x})$, there exists a sequence $\{x^i\} \subset C$, $\{x^i\} \to \bar{x}$ satisfying

$$f(x^i) \leqslant [(1/i)\,f(\tilde{x}) + (1 - 1/i)\,f(\bar{x})], \qquad i = 1, 2,\ldots . \tag{31}$$

Proof. Suppose that every local minimum of f is a global one on C. By Corollary 3.2, it follows that $L_f(\alpha)$ is lsc on G. Hence, for the sequence $\{\alpha^i\} \subset G$ given by

$$\alpha^i = [(1/i)\,f(\tilde{x}) + (1 - 1/i)\,f(\bar{x})], \qquad i = 1, 2,\ldots, \tag{32}$$

there exists a natural number K and a sequence $\{\hat{x}^i\} \subset C$ such that $f(\hat{x}^i) \leqslant \alpha^i$ for $i = K, K + 1,\ldots$. It follows that the sequence whose elements are $x^i = \tilde{x}$ for $i = 1,\ldots, K - 1$ and $x^i = \hat{x}^i$ for $i = K, K + 1,\ldots$ will satisfy (31). Conversely, let $f(\tilde{x}) < f(\bar{x})$, and suppose that a sequence $\{x^i\} \subset C$, converging to $\bar{x}$ and satisfying (31), exists. Assume that $\bar{x}$ is a local minimum of f which is not global. Hence, there exists a $\delta > 0$ such that

$$f(x) \geqslant f(\bar{x}) \tag{33}$$

for every $x \in C \cap B_\delta(\bar{x})$. The assumed sequence $\{x^i\} \subset C$ satisfying (31) also satisfies

$$f(x^i) < f(\bar{x}), \qquad i = 1, 2,\ldots . \tag{34}$$

But, for sufficiently large i, we must have $x^i \in C \cap B_\delta(\bar{x})$, contradicting (33). □

It is well known that every local minimum of a convex function defined on a convex subset $C \subset R^n$ is a global one. Many generalizations of convex functions have been proposed which have this property. One of them is the family of strictly quasiconvex functions (Ref. 5) defined as follows: A real function f on a convex set $C \subset R^n$ is said to be *strictly quasiconvex on* C if $\tilde{x} \in C$, $\bar{x} \in C$, $f(\tilde{x}) < f(\bar{x})$ and $0 < \lambda < 1$ imply that

$$f(\lambda\tilde{x} + (1 - \lambda)\bar{x}) < f(\bar{x}). \tag{35}$$

Let us outline now a proof that, for a strictly quasiconvex function, $L_f(\alpha)$ is lsc on G.

Theorem 3.5. Let f be a real strictly quasiconvex function on a convex set $C \subset R^n$. Then, $L_f(\alpha)$ is lower semicontinuous on G.

Proof. Let $\bar{\alpha} \in G$, $\{\alpha^i\} \subset G$, $\{\alpha^i\} \to \bar{\alpha}$. By Lemma 3.1, for every x such that $f(x) < \bar{\alpha}$, there exists a sequence $\{x^i\}$ with the required properties. Let us consider, therefore, only points $\bar{x}$ such that $f(\bar{x}) = \bar{\alpha}$. If $\bar{x}$ is a global minimum of f on C, then

$$\alpha^i \geqslant \bar{\alpha}, \qquad i = 1, 2, \ldots . \tag{36}$$

Hence, we can take $x^i = \bar{x}$ for all i. If $\bar{x}$ is not a global minimum, then there is at least one point $\tilde{x} \in C$ such that

$$f(\tilde{x}) < f(\bar{x}) = \bar{\alpha}. \tag{37}$$

Since f is strictly quasiconvex, we have

$$f(\lambda\tilde{x} + (1 - \lambda)\bar{x}) < f(\bar{x}) \tag{38}$$

for every $0 < \lambda < 1$. Since C is convex, we can take a sequence whose elements are

$$\hat{x}^k = [(1/k)\tilde{x} + (1 - 1/k)\bar{x}], \qquad k = 1, 2, \ldots . \tag{39}$$

Clearly, $\{\hat{x}^k\} \to \bar{x}$ and, by (38),

$$f[(1/k)\tilde{x} + (1 - 1/k)\bar{x}] = f(\hat{x}^k) < f(\bar{x}) = \bar{\alpha}, \qquad k = 1, 2, \ldots . \tag{40}$$

The rest of the proof is the same as in Theorem 3.2 and will not be repeated here. □

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