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Strain gradients and continuum modeling of size effect in metal matrix composites

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Summary. Constitutive modeling for the particle size effect on the strength of particulate-reinforced metal matrix composites is investigated. The approach is based on a gradient-dependent theory of plasticity that incorporates strain gradients into the expression of the flow stress of matrix materials, and a finite unit cell technique that is used to calculate the overall flow properties of composites. It is shown that the strain gradient term introduces a spatial length scale in the constitutive equations for composites, and the dependence of the flow stress on the particle size/spacing can be obtained. Moreover, a nondimensional analysis along with the numerical result yields an explicit relation for the strain gradient coefficient in terms of particle size, strain, and yield stress. Typical results for aluminum matrix composites with ellipsoidal particles are calculated and compare well with data measured experimentally.

1 Introduction

Experimental results have shown that the plastic behavior of particulate-reinforced metal matrix composites (MMCs) is influenced significantly by the particle size/spacing (for example, Kamat et al. [1], [2]). One way to model this size/spacing effect is to use dislocation theoris. For example, the Orowan model has been applied to small particles (Brown [3]) and predicted that the flow stress is proportional to the inverse of the particle spacing, while in the Ashby model (Ashby [4]) this dependence is the inverse of the square root of the particle spacing. Recently, a bowed-out tilt-wall model based on the dislocation pile-up was proposed (Rhee et al. [5], [6]) where the flow stress is proportional to $L^{-0.83}$ (L is the edge-edge particle spacing) together with a weak logarithmic dependence on L.

Alternatively, this issue may be addressed within the framework of continuum mechanics. We mention, for example, the recent contributions of Li and Weng [7], Ju and Tseng [8], Ponte Castaneda [9], Lee and Mear [10], Bao et al. [11], and Zbib and Zhi [12], who dealt with the effective nonlinear properties of MMCs. However, when applying classical phenomenological constitutive theories to MMCs, they have difficulties describing the size dependence. This is because these theories, originally developed for conventional metals or metallic alloys, assume that the deformation field is homogeneous, and thus do not include a deformation length scale in the constitutive equations.

In fact, classical continuum mechanics is local in character in that the stress at a material point is considered to be a functional of past deformation history of that point only. Eringen and co-workers have extended classical continuum mechanics (see, for example, Eringen, [13]-[15]) to include nonlocal effects. The basic assumption of nonlocal continuum mechanics is that the stress at a point is a functional of the past deformation histories of all materials points of the body, resulting in a spatial integral form of constitutive equations. Although difficult to use (especially when nonlinearities and finite domains are involved), the nonlocal theory is important in dealing with microscopic phenomena, because it incorporates length scales into the constitutive equations and can account for the heterogeneity and long-range interactions within the material.

Plastic deformation is known to arise from the accumulation of dislocations. From the study of dislocation motion it is clear that the state of the body at a point is influenced appreciably by the distortions that take place at neighboring points. Therefore plastic deformation is generally nonlocal. An alternative approach within the framework of continuum mechanics to describe the material heterogeneity and nonlocal interactions of plastic deformation is the strain gradient theory.

As suggested by Aifantis [16], [17] the strain gradient theory of plasticity introduces strain gradients into the constitutive equation for the flow stress, as opposed to the classical plasticity which usually assumes that the flow stress depends on the plastic strain alone. This is one way to account at a continuum level for presence of dislocations and their nonlocal interactions. In other words, the inclusion of strain gradient maybe viewed as modeling microscopic phenomena where the interactions between material points are not of the nearest neighbor (i.e. not contact forces) type. Dillon and Kratochvil [18] proposed a plasticity theory which included first and second strain gradients in the expression of free energy. Since the thermodynamics of plasticity is not well understood and, moreover, their expression of free energy had a special and complicated form, the results are probably limited in application.

In the work of Zbib and Aifantis [19], [20] the gradient effects are incorporated into the theory by modifying the expression of the flow stress. In this approach, they assume that the flow stress depends on higher order strain gradients. This leads to the inclusion of a length scale into plasticity theory, providing a framework for modeling phenomena which have fine length scales, such as shear bands and strengthening in MMCs.

Recently, a continuum model has been developed by the first two authors [12], [21] for the viscoplastic deformation in MMCs, utilizing a finite unit cell comprised of a rigid inclusion surrounded by a matrix metal. The model is capable of describing the influence of particle volume fraction, particle shape, and matrix properties on the overall strength of MMCs but no size effects. The main objective of the present work is to extend this model to account for the particle size effect on the flow strength of composites. Assuming a strain gradient-dependent flow stress in the matrix, the constitutive relation for MMCs is derived by a minimum principle. Then results are presented for MMCs with ellipsoidal particles, illustrating the dependence of flow stress on the strain gradient coefficient and other microstructural parameters of the composite. Finally, the yield stresses predicted by the present gradient-dependent continuum model for typical aluminum matrix composites are compared with a discrete dislocation model, as well as with relevant experimental observations.

2 Review of the unit cell model

As in [12], [21], we consider metal matrix composites reinforced by uniformly distributed particulates, represented by arrays of unit cells. Each unit cell contains one inclusion surrounded by a ductile matrix material. The matrix and the inclusions are assumed to be perfectly bonded.

The matrix is characterized as an incompressible viscoplastic material obeying the power law hardening relation

$$\bar{\sigma} = \sigma_0 \left(\frac{\dot{\bar{\varepsilon}}}{\dot{\varepsilon}_0} \right)^m, \tag{1}$$

and the von Mises flow rule

$$\dot{\varepsilon} = \frac{3}{2}\frac{\dot{\bar{\varepsilon}}}{\bar{\sigma}}\boldsymbol{\sigma}', \tag{2}$$

where σ_0, $\dot{\varepsilon}_0$, $\bar{\sigma}$, and $\dot{\bar{\varepsilon}}$ are the reference stress, references strain rate, effective stress and effective strain rate respectively, m is the strain rate sensitivity parameter, $\boldsymbol{\sigma}'$ is the deviatoric part of the microscopic Cauchy stress tensor $\boldsymbol{\sigma}$, and $\dot{\varepsilon}$ is the microscopic strain rate tensor defined by the symmetric part of the microscopic velocity.

The approach seeks an upper bound solution in which a kinematically admissible microscopic velocity field in the unit cell is assumed. Among all these admissible velocity fields that meet the incompressibility and appropriate boundary conditions, the actual one should minimize the total rate of plastic dissipation

$$\dot{W} = \frac{1}{V}\int_V \boldsymbol{\sigma}\cdot\dot{\varepsilon}\, dV, \tag{3}$$

where V is the volume of the representative unit cell. Furthermore, the macroscopic stress tensor $\mathbf{S}$ can be obtained as the conjugate of rate of plastic work

$$\dot{W} = \mathbf{S}\cdot\mathbf{D}, \tag{4}$$

where $\mathbf{D}$ is defined as the volume average of $\dot{\varepsilon}$, such that

$$\mathbf{D} = \frac{1}{V}\int_V \dot{\varepsilon}\, dV. \tag{5}$$

For the power law viscoplastic materials considered here, with $\dot{W}$ being homogeneous of a degree $m+1$ in $\mathbf{D}$, the stress tensor, $\mathbf{S}$, is obtained by

$$\mathbf{S} = \frac{1}{m+1}\frac{\partial \dot{W}}{\partial \mathbf{D}}. \tag{6}$$

A detailed derivation is given in [12], [21].

We consider two families of unit cells having the shapes of ellipsoids of revolution, which represent a wide range of inclusion shapes. One family is the prolate ellipsoids that extends vertically into whiskers. The other is oblate ellipsoids that extends horizontally into disc shapes.

For plastically incompressible matrix materials under axisymmetric loading conditions, the velocity vector $\mathbf{v}$ can be derived from a stream function ζ such that $\mathbf{v} = \nabla \times (0, 0, \zeta)$. In [12], [21], the following form of ζ was adopted:

$$\zeta(\alpha, \theta) = -\frac{\sqrt{g}}{2\alpha^2}\sum_{k=2,4\ldots}\sum_{i=-\infty}^{\infty} c_{ki}\alpha^i \sin k\theta \tag{7}$$

under an ellipsoidal coordinate system (α, θ, ϕ).

By using the velocity field derived from Eq. (7), applying the boundary conditions, and minimizing the total rate of plastic dissipation within the unit cell, the coefficient set $\{c_{ki}\}$ was determined and the overall constitutive relations for the composite were obtained. As demonstrated in [12], [21], this approach provides a model to assess the influence of particle volume fraction, particle shape and matrix properties on the flow strength of particulate-reinforced metal matrix composites. In the next Section, this unit cell model, combined with a gradient-dependent theory of plasticity, will be used to analyze the particle size effect on the constitutive behavior of composites.

3 Strain gradients and size effect

In the work of Zbib and Aifantis [19], [20], the classic expression for the flow stress has been modified by adding linear terms of strain gradients $\nabla^{(2k)}\bar{\varepsilon}\,(k = 1, 2, \ldots)$. Because of the isotropic assumption, the above linear gradient-dependence does not involve gradients of odd orders. It turns out that in many applications the $\nabla^2\bar{\varepsilon}$ term suffices to account for the heterogeneous evolution of deformation, yielding

$$\bar{\sigma} = \bar{\sigma}_H - c\nabla^2\bar{\varepsilon}, \tag{8}$$

where $\bar{\sigma}$ and $\bar{\sigma}_H$ are the total and the homogeneous part of the effective flow stress, respectively, $\bar{\varepsilon}$ is the effective plastic strain, and c is a force-like coefficient measuring the effect of strain gradients. This strain gradient plasticity has been applied to various problems where the size of the microstructure significantly affects the macroscopic mechanical properties of the material, such as shear banding problems in metals (see Zbib and Aifantis [22] and Zbib [23] as well as the recent articles by Aifantis [24], [25], [33] for an overview and other related applications in dislocation patterning, plastic flow, and failure). Now we extend the strain gradient plastic theory to the problem of size effect and plastic deformation in metal matrix composites.

Similar to the unit cell model [12], [21] for MMCs with a viscoplastic matrix, we first examine the plastic dissipation at a point within the plastic matrix. To account for the inhomogeneous plastic deformation, we divide the variation of local dissipation ω into two parts such that

$$\delta\omega = \delta\omega_H + \delta\omega_c, \tag{9}$$

where ω_H is the dissipation corresponding to the homogeneous response of the matrix and ω_c is the inhomogeneous part. In view of the second term in (8), ω_c has the following quadratic form:

$$\omega_c = \frac{c}{2}\nabla\bar{\varepsilon}\cdot\nabla\bar{\varepsilon}, \tag{10}$$

as shown below.

Moreover, an additional boundary condition arises from the higher order gradient, such that

$$\nabla\bar{\varepsilon}\cdot\mathbf{n} = 0, \text{ on outer cell boundary} \tag{11}$$

which is deduced from a variational principle similar to that given by Zbib [23], Muhlhaus and Aifantis [26], and Vardoulakis and Aifantis [27]. Under the familiar assumption of the universal stress-strain curve, we can write the plastic work in the incremental form

$$\delta\omega = \bar{\sigma}\,\delta\bar{\varepsilon} = \boldsymbol{\sigma}\cdot\delta\boldsymbol{\varepsilon}. \tag{12}$$

The proof of Eq. (10) can also be deduced from taking the volume average of $\delta\omega$ within a volume V, such that

$$\delta W = \frac{1}{V}\int_V \delta\omega \, dV = \frac{1}{V}\int_V \bar{\sigma}\,\delta\bar{\varepsilon}\, dV, \tag{13}$$

which, by using Eqs. (9) and (10), can also be written as

$$\delta W = \frac{1}{V}\int_V [\delta\omega_H + c\nabla\bar{\varepsilon}\cdot\nabla(\delta\bar{\varepsilon})]\, dV. \tag{14}$$

Since

$$\delta\omega_H = \sigma_H\,\delta\bar{\varepsilon}, \tag{15}$$

where $\bar{\sigma}_H$ is the homogeneous part of the flow stress, and using the divergence theorem

$$\int_V \nabla\bar{\varepsilon}\cdot\nabla(\delta\bar{\varepsilon})\, dV = \int_S \delta\bar{\varepsilon}\,\frac{\partial\bar{\varepsilon}}{\partial\mathbf{n}}\cdot d\mathbf{S} - \int_V \nabla^2\bar{\varepsilon}\delta\bar{\varepsilon}\, dV, \tag{16}$$

where the surface integral vanishes by the boundary condition (11), Eq. (14) becomes

$$\delta W = \frac{1}{V}\int_V (\bar{\sigma}_H - c\nabla^2\bar{\varepsilon})\,\delta\bar{\varepsilon}\, dV. \tag{17}$$

Comparing (17) with (13) yields the expression of the flow stress (8). Thus, the total strain energy for the gradient dependent material can be expressed as

$$W = \int_V \left(\omega_H + \frac{c}{2}\nabla\bar{\varepsilon}\cdot\nabla\bar{\varepsilon}\right) dV. \tag{18}$$

Within the J_2-deformation theory, Eq. (8) gives the following form of constitutive equation for the matrix material:

$$\boldsymbol{\sigma}' = \frac{2\bar{\sigma}(\bar{\varepsilon})}{3\bar{\varepsilon}}\,\varepsilon - \frac{2c\nabla^2\bar{\varepsilon}}{3\bar{\varepsilon}}\,\varepsilon. \tag{19}$$

For a power-law matrix

$$\bar{\sigma}_H = \varkappa\bar{\varepsilon}^n, \tag{20}$$

with the same averaging procedure and the upper bound solution as described by Zhu and Zbib [21], we obtain

$$\mathbf{S} = \frac{\partial W}{\partial\mathbf{E}} = \frac{\partial(W_H + W_c)}{\partial\mathbf{E}}, \tag{21}$$

where **S** is the macroscopic stress of the composite, **E** is the macroscopic strain, and W is the plastic dissipation which should be minimized for all kinematically admissible deformation fields in the composite. The total dissipation W consists of two parts. One is the homogeneous part, which is obtained by integrating (15) with $\bar{\sigma}_H$ given by (20),

$$W_H = \frac{1}{n+1}\left(\frac{2}{3}\right)^{\frac{n+1}{2}} \frac{\varkappa}{V} \int_V (\varepsilon \cdot \varepsilon)^{\frac{n+1}{2}}\, dV, \tag{22}$$

the other is the inhomogeneous part related to the strain gradient

$$W_c = -\frac{c}{2V} \int_V (\nabla^2 \bar{\varepsilon})\, \bar{\varepsilon}\, dV. \tag{23}$$

By applying the integral relation (16) and the boundary condition (11), we obtain another form of W_c,

$$W_c = \frac{c}{2V} \int_V \nabla\bar{\varepsilon} \cdot \nabla\bar{\varepsilon}\, dV, \tag{24}$$

which is the form used in the calculation. Compared to (23), expression (24) reduces the order of derivative which vastly improves the accuracy and stability of numerical computations.

After factorizing ε such that

$$\varepsilon = \frac{\bar{E}}{2(1-f^{1/3})^3}\, \varepsilon^*, \tag{25}$$

applying (21) and following the procedure described by Zhu and Zbib [21] for an axisymmetric loading of an ellipsoidal unit cell, we obtain

$$\bar{S} = \varkappa F_H(n,f)\, G_H(n,f,K)\, \bar{E}^n + \frac{c}{b^2} F_c(f)\, G_c(f,K)\, \bar{E}, \tag{26}$$

where $\bar{S}$ and $\bar{E}$ are the effective macroscopic stress and strain respectively, with

$$F_H(m,f) = \frac{1}{(1-f^{1/3})^{3(n+1)}}, \qquad G_H(m,f,K) = \frac{1}{V}\int_V \left(\frac{\varepsilon^* \cdot \varepsilon^*}{6}\right)^{\frac{n+1}{2}} dV, \tag{27}$$

and

$$F_c(f) = \frac{1}{2(1-f^{1/3})^6}, \qquad G_c(f,K) = \frac{1}{3V}\int_V |\nabla_* \sqrt{\varepsilon^* \cdot \varepsilon^*}|^2\, dV, \tag{28}$$

where ∇_* is the nondimensional gradient operator with the coordinate α replaced by α/b with b being the radius of the revolution of the ellipsoidal unit cell. f and K are particle volume fraction and aspect ratio, respectively.

Comparing Eq. (26) with the result without the strain gradient (i.e. $c = 0$) (Zbib and Zhu [12] and Zhu and Zbib [21]) we see that the strain gradient term in the flow stress of the matrix results in an additional term that further increases the flow stress of the composite. This stress increase is proportional to the strain gradient coefficient, and depends on the volume fraction and the shape of particles through a singular function F_c and a regular integral function G_c, which is similar to the homogeneous part. Most importantly, Eq. (26) shows that the strain gradient term naturally introduces a length scale into the constitutive equation for the composite, which allows us to analyze the size effect on the strength of the composite.

As in [12], [21], we assume a solution for the displacement field, impose the boundary conditions, minimize the dissipation energy, and solve for the effective stress $\bar{S}$ for a given set of parameters $\varkappa$, n, c, K, and f, as explained below.

4 Calculations for composites with ellipsoidal particles

We first look at the case of spherical particles where the simple geometry will enable us to examine the size effect more conveniently. By defining a characteristic length

$$l = \sqrt{\frac{c}{\varkappa}}, \tag{29}$$

the constitutive equation (26) can be written in the following form:

$$\bar{S} = \varkappa\left[F_H G_H \bar{E}^n + 4\left(\frac{l}{D}\right)^2 F_c G_c \bar{E}\right], \tag{30}$$

where D is the particle diameter. For a given volumen fraction f, upon converting D to the particle edge-edge spacing L by (Kamat et al. [1])

$$L = \left[\left(\frac{\pi}{6f}\right)^{1/2} - \frac{2}{\pi}\right] D, \tag{31}$$

we obtain another form of the constitutive equation,

$$\bar{S} = \varkappa\left[F_H G_H \bar{E}^n + 4\left(\frac{l}{L}\right)^2 \left(\sqrt{\frac{\pi}{6f}} - \frac{2}{\pi}\right)^2 F_c G_c \bar{E}\right]. \tag{32}$$

Notice that for common Al matrix materials, $\varkappa$ is of order 10^2 MPa and c is usually $1 \sim 10$ Newton (see [16], [17], [19], and [22]) which leads to values of l in the range of $10 \sim 10^2$ μm. This is of the same order of particle spacing as in many metal matrix composites.

By defining a nondimensionalized coefficient of strain gradient

$$\bar{c} = \frac{4c}{\varkappa D^2 \bar{E}^{n-1}}, \tag{33}$$

we can rewrite (30) as

$$\bar{S} = \bar{\sigma}_H (F_H G_H + \bar{c} F_c G_c), \tag{34}$$

where

$$\bar{\sigma}_H = \varkappa \bar{E}^n \tag{35}$$

is the flow stress of the matrix.

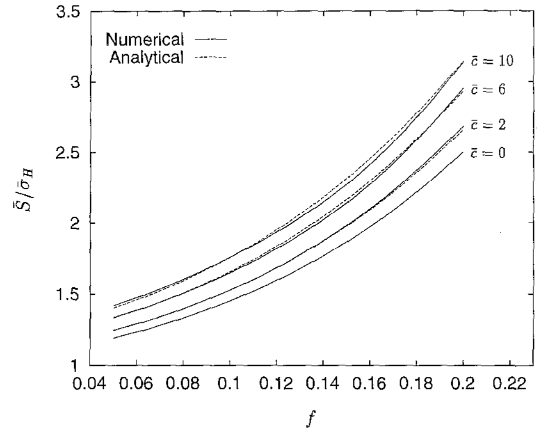

Fig. 1. Normalized flow stress of composites with linear matrix material and spherical particles

Figure 1 shows the normalized stress $\bar{S}/\bar{\sigma}_H$ as a function of volume fraction f values of $\bar{c}$ in the case of linear materials ($n = 1$). It can be seen from the figure that higher values of $\bar{c}$ result in a higher flow stress of the composite. For a given matrix property $\varkappa$, this may be either due to a smaller particle size D (or smaller inter-particle spacing L) or a higher gradient sensitivity c of the material. The former provides, at least qualitatively, an explanation for many experimental observations that composites with larger particles exhibit lower yield stress. Also plotted in Fig. 1 is an approximate expression representing the size effect predicted by the strain gradient model, which has the form

$$\frac{\bar{S} - \bar{S}_H}{\bar{\sigma}_H} = 0.018\,5\bar{c}(1 + 2f + 60f^2), \tag{36}$$

where $\bar{S}_H$ is the homogeneous part of flow stress (i.e. with $c = 0$), as discussed in [12], [21] and is given by

$$\bar{S}_H = \bar{\sigma}_H [1 + \beta(1 + 2.6m) f] \left(\frac{1 + a_1 f^{1/3} + a_2 f^{2/3} + a_3 f}{(1 - f^{1/3})^{3(m+1)/2}} \right), \tag{37}$$

where

$$\begin{aligned} \beta &= -3.01 + 1.69K + 1.32\frac{1}{K}, \\ a_1 &= -1.34 - 1.56m, \\ a_2 &= -0.32 + 2.37m, \\ a_3 &= -1.11 - 0.53m. \end{aligned} \tag{38}$$

It is seen that the simple expression (36) gives a very good approximation to the exact values predicted by the integral function (28). We remark that since $\bar{S}_H$ is derived from minimizing W with $c = 0$, the stress difference $\bar{S} - \bar{S}_H$ is not exactly linear in $\bar{c}$ as it appears in Eq. (30) or (32). This weak nonlinearity is shown in Fig. 2. A more accurate approximation is obtained by the following quadratic expression:

$$\frac{\bar{S} - \bar{S}_H}{\bar{\sigma}_H} = 0.022\bar{c}(1 - 0.023\bar{c})(1 + 2f + 60f^2). \tag{39}$$

For non-spherical particles, calculations showed that the flow stresses of composites are much more sensitive to the strain gradient coefficient. Figure 3 plots the normalized stress $\bar{S}/\bar{\sigma}_H$ versus the volume fraction f for various combinations of aspect ratio K and nondimensional gradient

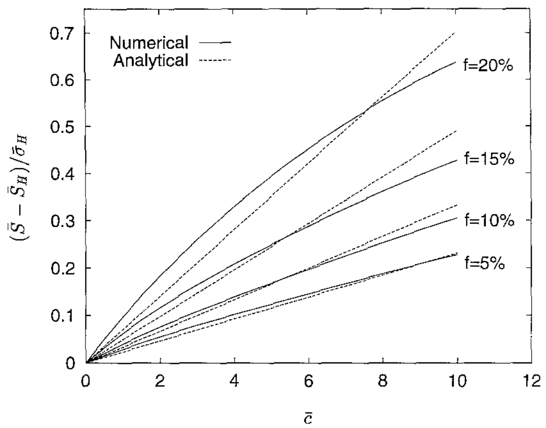

Fig. 2. Normalized strength enhancement of composites with linear matrix material and spherical particles

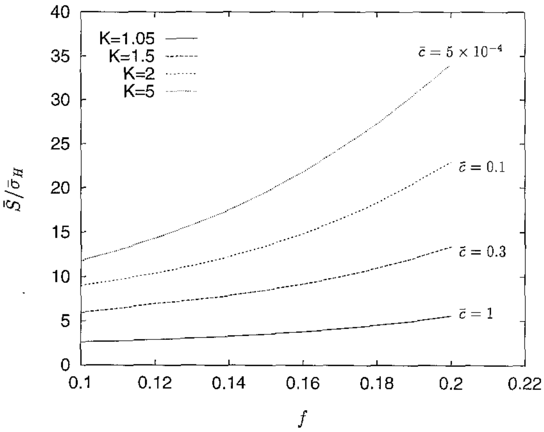

Fig. 3. Normalized stress of composites with linear matrix material and ellipsoidal particles

coefficient $\bar{c}$. The figure shows that as K increases, smaller values of $\bar{c}$ are needed to obtain the same level of strength enhancement over the matrix. For example, comparing the flow stress of composites with $K = 2, \bar{c} = 0.1$ and $K = 1.5, \bar{c} = 0.3$, the former has a lower value of $\bar{c}$ but exibits a higher level of stress. This is because as the particle shape deviates from the sphere, the deformation fields around particles develop more intense inhomogeneity and hence a higher strain gradient, which produces a higher flow stress by the present model.

5 Comparison with dislocation theory and experiment

In this Section, yield stresses predicted by the continuum model based on the strain gradient plasticity are compared with a dislocation model and experimental observations. For experimental data of MMCs with particle spacing in the micrometer range, dislocation models of Orowan type, which are suitable for particle spacing of nanometer orders, would not be expected to be applicable. Recently, Rhee et al. [5] developed a tilt-wall dislocation which appears to give good descriptions of yield stresses of MMCs with large particle spacing (Rhee et al. [6]), where continuum models should be applicable.

Figure 4 presents the yield strength (determined at 0.2% offset) obtained experimentally for the Al-Si-Mg composite model system described in Zhu et al. [28], and from theoretical models of Rhee et al. [5], and the present continuum model with $\varkappa = 285$ MPa, $n = 0.25$, and $K = 1$. While the basic parameter in the dislocation model is L, the present model has three parameters, c and any two of L, D, and f. Using the relation (31), the parameter L in the dislocation model can be converted to two parameters, D and f. Two curves from the dislocation model with $f = 9.7\%$, 19.7% are plotted. As D increases, these two curves converge to a value, which is the yield stress of the matrix. Also plotted in the figure are curves from the strain model with various values of c and f. The figure shows that curves produced by the strain gradient theory with different values of f converge to different stresses as D increases. One of the differences between these two types of model is that, while in the dislocation model particles infinitely apart denote the matrix material without particles, in the present continuum model this means that the strain gradient effect is ignored and only the homogeneous response is taken into

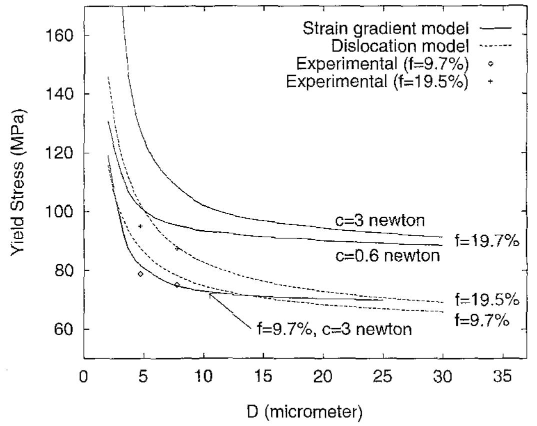

Fig. 4. Calculated yield stress (0.2% offset), with $\varkappa = 285$ MPA, $n = 0.25$, and $K = 1$, versus particle size compared to the experimental data from an Al-Si-Mg composite mode system [12] and dislocation model [6]

account. This homogeneous part depends on the volume fraction and, in general, the shape of particles, as described in [12], [21].

In Fig. 5 the yield stresses calculated by the strain gradient theory, using $\varkappa = 427$ MPa, $n = 0.25$, and $K = 1$, and dislocation theory are plotted against particle spacing L together with the experimental results of Kamat et al. [2]. The materials are composites comprising of 2024 O aluminum alloy matrices and alumina particulate. The particles are roughly equiaxed with an average diameter of 5 μm. Figure 6 shows similar calculations on another series of composites consisting of an Al alloy matrix reinforced by equiaxed TiB_2 particles 1.3 μm in diameter (Aikin and Christodoulou [29]). The theoretically predicted dependence of the yield stress on the particle spacing fits well with the observed data.

Table 1 shows the values of the gradient coefficient obtained by correlating c to flow stresses measured experimentally for some Al matrix composites. For these MMCs with Al matrix, the values of c are in the range $0.6 \sim 3$ Newton, which are of the same order as those calculated in shear banding problems [19], [22]. Based on these experimental data with 0.2% plastic strain and Eq. (33), the following relation between the value of the gradient coefficient and material parameters, namely, the matrix property $\varkappa$ and particle size D, can be postulated:

$$c = \alpha \varkappa D^2, \tag{40}$$

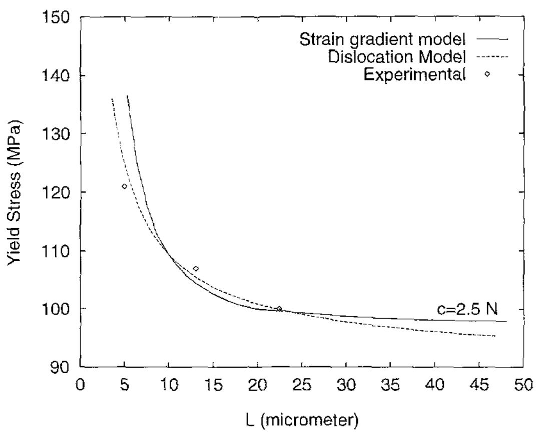

Fig. 5. Calculated yield stress (0.2% offset), with $\varkappa = 427$ MPA, $n = 0.25$ and $K = 1$, versus particle spacing for Al-Al_2O_3 composites compared to the experimental data [2] and dislocation model [6]

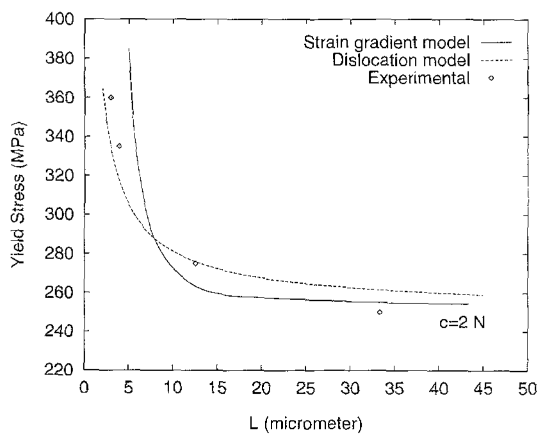

Fig. 6. Calculated yield stress (0.2% offset), with $\varkappa = 1135$ MPA, $n = 0.25$ and $K = 1$, versus particle spacing for Al-TiB_2 composites compared to the experimental data [23] and dislocation model [6]

Table 1. Values of the gradient coefficient c for particulate-reinforced aluminum-matrix composites

Material	$\varkappa$ (MPa)	n	c (Newton)	D_{avg} (μm)
Al-Si-Mg	285	0.25	0.6–3	4.7–7.8
Al-Al_2O_3	427	0.25	2.5	5
Al-TiB_2	1135	0.25	2.0	1.3

where the nondimensional coefficient α falls in the range of $1 \sim 6$. As demonstrated in our model, the strain gradient approach led to an evaluation of size effect and heterogeneity of deformation fields. An interesting topic is to identify the value of the gradient coefficient c for a given material. Equation (40) will help us to further investigate the nature of c and its relation to the mechanical properties and microstructure of a material. We mentioned that there are other models predicting a different dependence of the yield stress on the particle size, each supported by some experimental observations. Therefore, understanding and modeling the size effect on the constitutive relation of MMCs is an area requiring further investigation.

In concluding, we point out that a summary of the present results was included (by the authors) in a recent proceedings article [30]. In the same proceedings volume, the strain gradient theory was also used by Ning and Aifantis [31] to discuss size effects in short-fiber composites. A little earlier, it was shown by Aifantis [32] how the strain gradients can be used to account for size effects in twisted copper wires and statically deformed boreholes.

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