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Note

On an Axiomatization of the Banzhaf Value without the Additivity Axiom

ANDRZEJ S. NOWAK

Institute of Mathematics, Technical University of Wrocław, 50-370 Wrocław, Poland

Abstract: We prove that the Banzhaf value is a unique symmetric solution having the dummy player property, the marginal contributions property introduced by Young (1985) and satisfying a very natural reduction axiom of Lehrer (1988).

This note is devoted to the value concept introduced by Banzhaf (1965). It is well-known that the Banzhaf value is not an efficient solution. Many authors tried to find a substitute of the efficiency axiom in the derivation of the Shapley value (Shapley, 1953) that would help to determine the Banzhaf value uniquely; see Roth (1977), Dubey and Shapley (1979) or Feltkamp (1995). The approach in Owen (1982) is based on a different idea but does not determine the Banzhaf value uniquely. A very interesting "reduction" property was introduced by Lehrer (1988) who gave two axiomatic characterizations of the Banzhaf value. His first characterization is based on a "transfer" property of the value restricted to simple games. Such a property was introduced by Dubey (1975) and then applied to studying the Banzhaf value by Dubey and Shapley (1979) and Feltkamp (1995). The second axiomatization of Lehrer (1988) is (in a very essential way) based on the linearity assumption. Haller (1994) characterized the Banzhaf value by using some collusion neutrality properties, but his approach is also heavily based on the linearity assumption.

In this paper, we employ equal treatment, dummy player axioms, a version of the reduction property due to Lehrer (1988) and the well-known postulate of Young (1985) saying that the value is (in some sense) determined by the marginal contributions of the players. Our axiomatic characterization of the Banzhaf value is a counterpart of the theorem of Young (1985) on the Shapley value.

Let N be a finite set of players. Subsets of N are called *coalitions*. To simplify the notation, one membered coalitions $\{i\}$ will sometimes be denoted by i. The cardinality of any coalition S is denoted by $|S|$. A *transferable utility game* (a TU-game) is a function v that assigns to each coalition S a real number $v(S)$ and, in particular, $v(\emptyset) = 0$. For each coalition S, $v(S)$ represents the *worth* or the *power* of S. For each nonempty coalition T, the *unanimity game* u_T of the coalition T is defined by $u_T(S) = 1$ if $T \subseteq S$, and $u_T(S) = 0$ otherwise.

A *value* is a mapping φ which associates with each n-person TU-game v a vector $\varphi(v) = (\varphi_1(v), \ldots, \varphi_n(v))$. The real number $\varphi_i(v)$ represents the *individual value* for player i in the game v.

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Following Lehrer (1988), for any n-person game v ($n \geq 2$) and different players $i, j \in N$, we put $p = \{i,j\}$ and consider the "reduced" game v_p (with $(N\backslash p) \cup \{p\}$ as the set of players) defined by:

$$v_p(S) = v(S) \text{ and } v_p(S \cup \{p\}) = v(S \cup p), \text{ for any } S \subseteq N\backslash p. \tag{1}$$

Clearly, v_p is an $(n-1)$-person game obtained by amalgamating the players i and j in the game v into one player p.

Let φ be a value. We are now ready to formulate the axioms.

Axiom ***EF*** *(2-Efficiency)*: Let i and j be two different players in a game v. Then

$$\varphi_i(v) + \varphi_j(v) = \varphi_p(v_p), \tag{2}$$

where $\varphi_p(v_p)$ is the payoff to player p in the game v_p.

Axiom ***D*** *(Dummy player)*: If $i \in N$ is a *dummy player* in a game v, i.e., $v(S \cup i) = v(S) + v(i)$ for all $S \subset N\backslash i$, then $\varphi_i(v) = v(i)$.

Axiom ***ET*** *(Equal Treatment)*: Let i and j be two different players in a game v. If $v(S \cup i) = v(S \cup j)$ for all $S \subset N\backslash\{i,j\}$, then $\varphi_i(v) = \varphi_j(v)$.

Axiom ***M*** *(Marginal Contributions)*: Let v, w be games. If for some player $i \in N$, we have

$$v(S \cup i) - v(S) = w(S \cup i) - w(S), \text{ for all } S \subset N\backslash i,$$

then $\varphi_i(v) = \varphi_i(w)$.

Axioms **D** and **ET** are standard. Axiom **M** was introduced by Young (1985). The "2-efficiency" axiom **EF** was originally discussed in Lehrer (1988). **EF** assumes a natural "reduction" property which has a natural interpretation.

The *Banzhaf value*, say β, for a given m-person TU-game v is defined by

$$\beta_i(v) = \sum_{S \subseteq N\backslash i} [v(S \cup i) - v(S)]/2^{m-1}, \quad i \in N.$$

Our characterization of the Banzhaf value is:

Theorem: A value φ considered on all TU-games satisfies Axioms **EF**, **ET**, **D**, and **M** if and only if φ is the Banzhaf value.

A similar result for the Shapley value was given by Young (1985). Before we give a proof of this result, we add some comments. Actually, Lehrer (1988) introduced a weaker version of **EF** formulated as follows:

Axiom SA (Super Additivity): Let i and j be two different players in a game v. Then

$$\varphi_i(v) + \varphi_j(v) \leq \varphi_p(v_p).$$

This property says that a unification of any two players is profitable and as such seems to have a more intuitive meaning than equality (2). A natural question arises as to whether Axiom **EF** can be replaced by the weaker assumption **SA** in our Theorem. We will give a negative answer to this question. At this moment, we would like to point out that, under assumption **SA**, the linearity property is used (in a very essential way) by Lehrer (1988), even in determining the Banzhaf value on the unanimity games: see pages 97–98 in Lehrer (1988).

Counterexample: Let N be a finite set of players and let

$$\varphi_i(v) = \min_{S \subseteq N \backslash i} [v(S \cup i) - v(S)],$$

for any player $i \in N$ and for any game v. Clearly, φ satisfies Axioms **ET**, **D** and **M**. Let $p = \{i,j\}$ where $i \neq j$. Note that, for any coalition $S \subseteq N \backslash \{i,j\}$, we have

$$\begin{aligned} v(S \cup p) - v(S) &= v(S \cup i \cup j) - v(S \cup j) + v(S \cup j) - v(S) \\ &\geq \min_{T \subseteq N \backslash i} [v(T \cup i) - v(T)] + \min_{T \subseteq N \backslash j} [v(T \cup j) - v(T)] \\ &= \varphi_i(v) + \varphi_j(v). \end{aligned}$$

Hence, it follows that $\varphi_p(v_p) \geq \varphi_i(v) + \varphi_j(v)$, that is, Axiom **SA** is also satisfied. Obviously, φ is not the Banzhaf value.

Proof of Theorem: ($\Leftarrow$) It is known that the Banzhaf value β satisfies **D** and **ET**. Clearly, β also has property **M**. The fact that β satisfies **EF** follows from Proposition 1 in Lehrer (1988).

($\Rightarrow$) Similarly as in Lehrer (1988) and Young (1985), we use an induction argument in the proof of the uniqueness of the Banzhaf value. Our proof consists of two steps:

Step 1: First, we show that φ coincides with the Banzhaf value β on unanimity games. We show that for any number m of the players and any nonempty coalition T, $|T| \leq m$, we have

$$\varphi_i(u_T) = (1/2)^{|T|-1} \text{ if } i \in T \text{ and } \varphi_i(u_T) = 0 \text{ if } i \notin T. \tag{3}$$

Note that (3) holds for any m and for $|T| = 1$. This follows from **D**, because $|T| = 1$ implies that every player in u_T is dummy. Now assume that (3) takes place for some m and for any coalition T such that $|T| = k \leq m$. Let w_T be an $(m+1)$-person unanimity game with $|T| = k+1$. Let $i \in T$. Amalgamate player i with any other player j in T, put $p = \{i,j\}$ and consider the game v_p defined by (1) where $v = w_T$. Clearly, v_p is the m-person unanimity game of the coalition $T' = (T \backslash p) \cup \{p\}$ and $|T'| = k$. By the induction hypothesis

$$\varphi_p(v_p) = (1/2)^{|T'|-1} = 1/2^{k-1}$$

and using **EF** we infer that

$$\varphi_i(w_T) + \varphi_j(w_T) = 1/2^{k-1}. \tag{4}$$

By **ET** and (4) we obtain that

$$\varphi_i(w_T) = 1/2^k = (1/2)^{|T|-1}, i \in T,$$

and by **D**, $\varphi_j(w_T) = 0$ when $j \notin T$. Thus, we have proved that φ agrees with β on unanimity games for any finite set of players. Similarly, we prove that φ and β coincide for every game cu_T where c is a real number.

Step 2: By Axiom **D**, for each one-person game u, we have $\varphi_i(u) = u(i)$ and we know that $\beta_i(u) = u(i)$. Suppose φ has been determined for all m-person games where $m \leq n$, for some positive integer n, and moreover, φ agrees with β on this set of games. We now exploit the fact (cf. Shapley, 1953) that any game u with a finite number of players can be expressed as a linear combination of unanimity games

$$u = \sum_{\emptyset \neq T \subset N} \gamma_T u_T,$$

where the constants γ_T are uniquely determined by the game u. Let $I(u)$ be the number of nonzero coefficients γ_T in the above representation for u. As in Young (1985), we will use the induction method on the index $I(u)$ to complete the proof of ($\Rightarrow$). From Step 1, we know that $\varphi(u) = \beta(u)$ for all games u with $I(u) = 1$ and any finite number of players. Assume that $\varphi(u) = \beta(u)$ for every $(n+1)$-person game u with $I(u) \leq k$, where k is some positive integer. Consider an $(n+1)$-person game v with $I(v) = k+1$. Then, we have $k+1$ different nonempty coalitions $T_1, \ldots, T_{k+1}$ such that

$$v = \sum_{r=1}^{k+1} \gamma_{T_r} u_{T_r}.$$

Let $T = T_1 \cap \cdots \cap T_{k+1}$. Since $k+1 \geq 2$, we have $N \backslash T \neq \emptyset$. Assume that $i \notin T$. Define a new game w by

$$w = \sum_{r: T_r \ni i} \gamma_{T_r} u_{T_r}.$$

Then $I(w) \leq k$ and $v(S \cup i) - v(S) = w(S \cup i) - w(S)$ for every coalition S not containing player i. From Axiom **M** and the latter induction hypothesis, it follows that $\varphi_i(v) = \varphi_i(w) = \beta_i(w)$. But the Banzhaf value β also satisfies condition **M**, and therefore, we have $\beta_i(v) = \beta_i(w)$. Thus, we get

$$\varphi_i(v) = \beta_i(v) \text{ for each } i \in N \setminus T. \tag{5}$$

It remains to show that $\varphi_j(v) = \beta_j(v)$ when $j \in T$. Amalgamate any player $j \in T$ with a fixed $i \in N \setminus T$, put $p = \{i,j\}$, and consider the n-person game v_p. By the former induction hypothesis,

$$\varphi_p(v_p) = \beta_p(v_p). \tag{6}$$

Applying Axiom **EF** to both values φ and β, we get

$$\varphi_p(v_p) = \varphi_i(v) + \varphi_j(v) \text{ and } \beta_p(v_p) = \beta_i(v) + \beta_j(v). \tag{7}$$

Combining (5), (6) and (7) we conclude that $\varphi_j(v) = \beta_j(v)$ for $j \in T$. Thus, we have shown that $\varphi(v) = \beta(v)$ for any $(n+1)$-person game v with $I(v) = k+1$. The proof is now complete.

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