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Localization and Convergence of Eigenfunction Expansions

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ABSTRACT. We state a localization principle for expansions in eigenfunctions of a self-adjoint second order elliptic operator and we prove an equiconvergence result between eigenfunction expansions and trigonometric expansions. We then study the Gibbs phenomenon for eigenfunction expansions of piecewise smooth functions on two-dimensional manifolds.

Two of the most elementary but basic results in harmonic analysis are the theorem of Dirichlet on the convergence of Fourier series and the localization principle of Riemann. The Riemann localization principle states that the behavior of the partial sums of a Fourier series at a given point depends only on the behavior of the function in an arbitrary small neighborhood of this point. The Dirichlet theorem, originally stated for functions with a finite number of maxima and minima, when applied to piecewise smooth functions guarantees the convergence at every point of the partial sums of Fourier series to the function expanded. At a discontinuity the convergence is to the midpoint of the jump but, as it was observed by Wilbraham and later by Michelson and Gibbs, in a neighborhood of the discontinuities, the partial sums have wild oscillations and overshoot the target by about 9% of the value of the jump.

The classical trigonometric series is a quite faithful model for more general one-dimensional expansions. After suitable changes of variables, a regular Sturm-Liouville problem can be put in a canonical form:

$$\begin{cases} -\frac{d^2}{dx^2}y(x) + \left(Q(x) - \lambda^2\right)y(x) = 0, \text{ if } 0 \le x \le 1\,, \\ \alpha\frac{d}{dx}y(0) + \beta y(0) = \gamma\frac{d}{dx}y(1) + \delta y(1) = 0\,. \end{cases}$$

Moreover, if $\alpha\gamma \neq 0$ and if $n \to +\infty$, the sequences of eigenvalues $\{\lambda_n^2\}_{n=0}^{+\infty}$ and normalized eigenfunctions $\{\phi_n(x)\}_{n=0}^{+\infty}$ admit asymptotic expansions,

$$\begin{cases} \lambda_n = \pi n + an^{-1} + O(n^{-2})\,, \\ \phi_n(x) = \sqrt{2}\cos(\pi nx) + n^{-1}A(x)\sin(\pi nx) + O(n^{-2}) \end{cases}$$

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for a suitable constant a and function $A(x)$ independent of n. Using these formulas, Haar [8] proved that Fourier series with respect to the system of eigenfunctions $\{\phi_n(x)\}_{n=0}^{+\infty}$ are equiconvergent with Fourier series with respect to the trigonometric system $\left\{\sqrt{2}\cos(\pi n x)\right\}_{n=0}^{+\infty}$, that is for every integrable function $f(x)$ on $[0,1]$ one has

$$\lim_{n\to+\infty}\left|\sum_{k=0}^{n}\left(\int_0^1 f(t)\phi_n(t)dt\right)\phi_n(x)-\sum_{k=0}^{n}\left(\int_0^1 f(t)\sqrt{2}\cos(\pi n t)dt\right)\sqrt{2}\cos(\pi n x)\right|=0\,.$$

The idea of the proof is indeed very simple. By the asymptotic formula for eigenfunctions, the difference between the Dirichlet kernels $\sum_{k=0}^{n}\phi_n(t)\phi_n(x)$ and $2\sum_{k=0}^{n}\cos(\pi n t)\cos(\pi n x)$ is uniformly bounded. It then follows that

$$\begin{aligned}&\left|\sum_{k=0}^{n}\left(\int_0^1 f(t)\phi_n(t)dt\right)\phi_n(x)-\sum_{k=0}^{n}\left(\int_0^1 f(t)\sqrt{2}\cos(\pi n t)dt\right)\sqrt{2}\cos(\pi n x)\right|\\&=\left|\int_0^1\left(\sum_{k=0}^{n}\phi_n(t)\phi_n(x)-2\sum_{k=0}^{n}\cos(\pi n t)\cos(\pi n x)\right)f(t)dt\right|\\&\le c\int_0^1|f(t)|\,dt\,.\end{aligned}$$

Norm boundedness and convergence to zero for a dense class of functions imply the convergence to zero for every integrable function. In particular, for these Sturm–Liouville expansions we have an exact analogue of the Dirichlet convergence theorem and of the Riemann localization principle.

We shall try to prove something similar for expansions in eigenfunctions of a second-order self-adjoint elliptic Dirichlet problem on an N-dimensional domain; however, since an asymptotic expansion of eigenfunctions is not available, as a tool we shall use the wave equation. In particular, we shall synthesize the Dirichlet kernel, or spectral function, using the fundamental solution of the wave equation and we shall control the errors using estimates on restriction of Fourier transform to spheres and approximation results of Jackson–Bernstein type. For technical reasons, the most precise results will be obtained for expansions on two-dimensional manifolds without boundary. In this case, we shall prove the pointwise convergence for eigenfunction expansions of piecewise smooth functions and the associated Gibbs phenomenon in a neighborhood of the discontinuities. This is related to some results obtained by Weyl [25], who studied the Gibbs phenomenon for spherical harmonic expansions on the two-dimensional sphere $\left\{x^2+y^2+z^2=1\right\}$. Another two-dimensional manifold is the torus $\{0\le x, y<1\}$ and in this case the expansions in eigenfunctions of the Laplacian are the classical two-dimensional trigonometric Fourier series. When applied to this case, our results give a new proof of an identity stated by Voronoi and proved by Hardy [9] for the number of integer points in a disk of the plane.

We want to mention that expansions in eigenfunctions of elliptic operators have been extensively studied (see, e.g., the survey by Alimov-Il'in-Nikishin [1] and the references there). In particular, these authors proved definitive results for localization and pointwise convergence of expansions of functions in Sobolev spaces. However, the piecewise smooth functions considered in this paper are just in the critical spaces and, as far as we know, the results we have found in the literature do not immediately imply ours. As we said, the method of proof is based on the wave equation, restriction theorems for Fourier transforms, and approximation results of Jackson–Bernstein type. A curious feature is that in order to prove a pointwise result for piecewise smooth functions, we use $\mathbb{L}^p$ norms with $1<p<2$ since $p=1,2$ are not enough.

The index of our exposition is the following. In order to illustrate the methods used in the paper on simple model cases, in Section 1 we state an extension of the classical Riemann localization principle to multiple Fourier integrals and in Section 2 we study localization and convergence of expansions in eigenfunctions of the Dirichlet problem in bounded open sets in $\mathbb{R}^N$ with smooth boundary. It turns out that trigonometric and eigenfunction expansions are related via an equiconvergence theorem that is an analog of Haar's result for Sturm–Liouville expansions. In Section 3, which is the motivation of our work and perhaps the most original part of the paper, we restrict our attention to smooth two-dimensional manifolds without boundary and we study the convergence of eigenfunction expansions of piecewise smooth functions and the associated Gibbs phenomenon. In Section 4 we give a new proof of the identity of Hardy for the number of integer points in a disc of the plane.

1. Localization for Fourier Integrals in $\mathbb{R}^N$

For functions in $\mathbb{L}^1 + \mathbb{L}^2(\mathbb{R}^N)$ we consider the Fourier transform and the associated Fourier expansion

$$\mathbb{F}f(\xi) = \int_{\mathbb{R}^N} f(y)\exp(-2\pi i\xi \cdot y)dy,\ f(x) = \int_{\mathbb{R}^N} \mathbb{F}f(\xi)\exp(2\pi i x \cdot \xi)d\xi\ .$$

The classical Riemann localization principle states that the behavior of the partial sums of a one-dimensional Fourier series or integral at a given point depends only on the behavior of the function in an arbitrary small neighborhood of this point. It was shown by Tonelli [23] for multiple Fourier series and by Bochner [4] for multiple Fourier integrals that localization may not hold when $N > 1$. See also [1]. The idea is that the spherical partial sum operators can be expressed as convolutions against kernels which are unbounded even outside the origin. Indeed, see Stein–Weiss [21], IV,

$$\int_{\{|\xi|\le\Lambda\}} \mathbb{F}f(\xi)\exp(2\pi i x\cdot\xi)d\xi = \int_{\mathbb{R}^N} f(x-y)\Lambda^N |\Lambda y|^{-N/2} J_{N/2}(2\pi|\Lambda y|)dy\ .$$

Let us fix x and consider the space of continuous functions on $\mathbb{R}^N$ with supports $\{\varepsilon \le |y-x| \le \delta\}$. The above integrals are linear functionals on this space, but since Bessel functions have asymptotic expansion, for $z \to +\infty$,

$$J_\alpha(z) = \sqrt{\frac{2}{\pi z}}\cos\left(z - \frac{\alpha\pi}{2} - \frac{\pi}{4}\right) + O(z^{-3/2})\ ,$$

these functionals have norms of the order of $\Lambda^{(N-1)/2}$. Hence, they are not uniformly bounded and, by the Banach–Steinhaus theorem, there exists a continuous function with support in $\{\varepsilon \le |y-x| \le \delta\}$ for which the above integrals do not converge at x. A simple explicit example of this lack of localization is given by the spherical partial sums of a characteristic function of a ball in $\mathbb{R}^3$ at its center,

$$\begin{aligned}\int_{\{|\xi|\le\Lambda\}} \mathbb{F}\chi_{\{|x|\le 1\}}(\xi)d\xi &= \int_{\{|x|\le 1\}} \Lambda^3 |\Lambda x|^{-3/2} J_{3/2}(2\pi|\Lambda x|)dx \\ &= 4\Lambda\int_0^1 \left(\frac{\sin(2\pi\Lambda t)}{2\pi\Lambda t} - \cos(2\pi\Lambda t)\right)dt \approx 1 - \frac{2}{\pi}\sin(2\pi\Lambda)\ .\end{aligned}$$

In order to recover the localization, one can introduce suitable means of the Fourier integrals, or one can require appropriate Tauberian conditions on the function expanded. In particular, we shall see that localization holds when the Fourier transform is suitably small and spread out.

Let $m(s)$ be a bounded even function on $-\infty < s < +\infty$ and for $\Lambda > 0$ let $m_\Lambda(s) = m(s/\Lambda)$. For functions in $\mathbb{L}^1 + \mathbb{L}^2(\mathbb{R}^N)$ define the operator $\mathbb{M}_\Lambda$ by

$$\mathbb{M}_\Lambda f(x) = \int_{\mathbb{R}^N} m_\Lambda(2\pi|\xi|)\mathbb{F}f(\xi)\exp(2\pi i x\cdot\xi)d\xi\ .$$

Since the multiplier $m(2\pi\,|\xi|)$ is bounded, the means $\mathbb{M}_\Lambda f(x)$ are tempered distributions. If $m_\Lambda(2\pi\,|\xi|)\mathbb{F}f(\xi)$ is square integrable, then these means are defined at least almost everywhere as functions in $\mathbb{L}^2(\mathbb{R}^N)$. However, here we are interested in pointwise localization and convergence, so that in order to have means which are also defined pointwise, in the sequel we shall require that $m(s)$ has compact support, or sufficiently rapid decay at infinity. Under this assumption, the operators $\mathbb{M}_\Lambda$ map the space $\mathbb{L}^1+\mathbb{L}^2(\mathbb{R}^N)$ into a suitable space of test functions. For example, the characteristic functions $m_\Lambda(s)=\chi_{[-\Lambda,\Lambda]}(s)$, which define the partial sums operators, give rise to absolutely convergent integrals and $\mathbb{M}_\Lambda f(x)$ are entire functions. Other classical examples are the Bochner–Riesz means of order δ defined by $m_\Lambda(s)=\left(1-(s/\Lambda)^2\right)_+^\delta$ and one is interested in the behavior of these means as $\Lambda\to+\infty$.

Let $\psi(s)$ be an even test function on $-\infty<s<+\infty$ with cosine Fourier transform $\widehat{\psi}(t)=\frac{2}{\pi}\int_0^{+\infty}\psi(s)\cos(ts)ds$ vanishing for $|t|\geq\varepsilon$ and with $\int_{-\infty}^{+\infty}\psi(s)ds=1$. For the application we have in mind, it is convenient to choose ε small and also to assume that $\int_{-\infty}^{+\infty}\psi(s)s^jds=0$ for $j=1,2,\ldots$, even if these assumptions are not always necessary. The convolution $m_\Lambda*\psi(s)=\int_{-\infty}^{+\infty}m_\Lambda(s-t)\psi(t)dt$ can be seen as an approximation of $m_\Lambda(s)$ and $\int_{\mathbb{R}^N}m_\Lambda*\psi(2\pi\,|\xi|)\mathbb{F}f(\xi)\exp(2\pi ix\cdot\xi)d\xi$ as an approximation of $\mathbb{M}_\Lambda f(x)$. Under these assumptions we have the following localization results.

Theorem 1.

1) *The value of the means*

$$\int_{\mathbb{R}^N}m_\Lambda*\psi(2\pi\,|\xi|)\,\mathbb{F}f(\xi)\exp(2\pi ix\cdot\xi)d\xi$$

at a point x depends only on the values of $f(y)$ at points $|y-x|\leq\varepsilon$.

2) *Assume that*

$$\lim_{\Lambda\to+\infty}\left|\int_{\mathbb{R}^N}(m_\Lambda(2\pi\,|\xi|)-m_\Lambda*\psi(2\pi\,|\xi|))\,\mathbb{F}f(\xi)\exp(2\pi ix\cdot\xi)d\xi\right|=0\,.$$

Then the behavior as $\Lambda\to+\infty$ of the means

$$\int_{\mathbb{R}^N}m_\Lambda(2\pi\,|\xi|)\mathbb{F}f(\xi)\exp(2\pi ix\cdot\xi)d\xi$$

at x depends only on the values of $f(y)$ at points $|y-x|\leq\varepsilon$. In particular if $f(y)=g(y)$ for $|y-x|\leq\varepsilon$ and both $f(y)$ and $g(y)$ satisfy the above Tauberian condition, then the means of $f(y)$ and $g(y)$ are equiconvergent at the point x,

$$\lim_{\Lambda\to+\infty}\left|\int_{\mathbb{R}^N}m_\Lambda(2\pi\,|\xi|)\mathbb{F}f(\xi)\exp(2\pi ix\cdot\xi)d\xi-\int_{\mathbb{R}^N}m_\Lambda(2\pi\,|\xi|)\mathbb{F}g(\xi)\exp(2\pi ix\cdot\xi)d\xi\right|=0\,.$$

3) *If*

$$\limsup_{\Lambda\to+\infty}\left|\int_{\mathbb{R}^N}(m_\Lambda(2\pi\,|\xi|)-m_\Lambda*\psi(2\pi\,|\xi|))\,\mathbb{F}f(\xi)\exp(2\pi ix\cdot\xi)d\xi\right|>0\,,$$

then the behavior of the means $\int_{\mathbb{R}^N}m_\Lambda(2\pi\,|\xi|)\mathbb{F}f(\xi)\exp(2\pi ix\cdot\xi)d\xi$ as $\Lambda\to+\infty$ depends also on points $|y-x|\geq\varepsilon$. In this case there is no localization.

Proof. The idea, by now quite classical, is to synthesize the operators $\mathbb{M}_\Lambda$ by means of the fundamental solution of the wave equation $\cos(t\sqrt{\Delta})$.

Lemma 1.

Let $\cos(t\sqrt{\Delta})f(x)$ *be the solution of the Cauchy problem for the wave equation in* $\mathbb{R}\times\mathbb{R}^N$,

$$\begin{cases} \frac{\partial^2}{\partial t^2}u(t,x)-\sum_{j=1}^{N}\frac{\partial^2}{\partial x_j^2}u(t,x)=0\,, \\ u(0,x)=f(x)\,, \\ \frac{\partial}{\partial t}u(0,x)=0\,. \end{cases}$$

Then in the distribution sense we have the equalities

$$\int_{\mathbb{R}^N} m_\Lambda(2\pi\,|\xi|)\,\mathbb{F}f(\xi)\exp(2\pi i\xi\cdot x)d\xi=\int_0^{+\infty}\widehat{m_\Lambda}(t)\cos(t\sqrt{\Delta})f(x)dt,$$

$$\int_{\mathbb{R}^N} m_\Lambda*\psi(2\pi\,|\xi|)\,\mathbb{F}f(\xi)\exp(2\pi i\xi\cdot x)d\xi=\pi\int_0^{+\infty}\widehat{m_\Lambda}(t)\widehat{\psi}(t)\cos(t\sqrt{\Delta})f(x)dt\,.$$

If $m(s)$ *has compact support, or sufficiently rapid decay at infinity, then for every* $f(x)$ *in* $\mathbb{L}^1+\mathbb{L}^2(\mathbb{R}^N)$ *the above distributions can be identified with smooth functions.*

Proof. In the distribution sense $\mathbb{F}\left(\cos(t\sqrt{\Delta})f\right)(\xi)=\cos(2\pi\,|\xi|\,t)\mathbb{F}f(\xi)$ and we have

$$\begin{aligned}&\int_{\mathbb{R}^N} m_\Lambda(2\pi\,|\xi|)\,\mathbb{F}f(\xi)\exp(2\pi i\xi\cdot x)d\xi\\ &=\int_{\mathbb{R}^N}\left(\int_0^{+\infty}\widehat{m_\Lambda}(t)\cos(2\pi\,|\xi|\,t)dt\right)\mathbb{F}f(\xi)\exp(2\pi i\xi\cdot x)d\xi\\ &=\int_0^{+\infty}\widehat{m_\Lambda}(t)\left(\int_{\mathbb{R}^N}\cos(2\pi\,|\xi|\,t)\mathbb{F}f(\xi)\exp(2\pi i\xi\cdot x)d\xi\right)dt\\ &=\int_0^{+\infty}\widehat{m_\Lambda}(t)\cos(t\sqrt{\Delta})f(x)dt\,.\end{aligned}$$

The interchange in the order of integration can be justified as in the proof of Fourier inversion formula. This proves the first equality and the proof of the second is similar. It only uses the formula for the Fourier transform of a convolution $(m_\Lambda*\psi)\hat{}(t)=\pi\widehat{m_\Lambda}(t)\widehat{\psi}(t)$. If $f(x)$ is in $\mathbb{L}^1+\mathbb{L}^2(\mathbb{R}^N)$, then $\mathbb{F}f(\xi)$ is in $\mathbb{L}^2+\mathbb{L}^\infty(\mathbb{R}^N)$ and if $m(s)$ has a rapid decay at infinity, then the integral on the left-hand side of the equality is absolutely convergent and defines a smooth function. □

Now recall that waves propagate with finite speed. In particular, the distribution kernel of $\cos(t\sqrt{\Delta})$ has support in $\{|y-x|\leq t\}$ and $\cos(t\sqrt{\Delta})f(x)$ depends only on the values of $f(y)$ at points $|y-x|\leq t$. Since $\widehat{\psi}(t)=0$ if $t\geq\varepsilon$, then 1) follows.

2) and 3) are immediate consequences of 1). □

If the function $m(s)$ is smooth with compact support or, more generally, if $m(s)$ is smooth with rapidly decaying derivatives and if the function $\psi(s)$ is suitably chosen, then for every positive integer h and k,

$$\begin{aligned}|m_\Lambda*\psi(s)-m_\Lambda(s)| &= \left|\int_{-\infty}^{+\infty}(m_\Lambda(s-t)-m_\Lambda(s))\,\psi(t)dt\right|\\ &= \left|\int_{-\infty}^{+\infty}\left(m_\Lambda(s-t)-\sum_{j=0}^{h-1}\frac{d^j}{ds^j}m_\Lambda(s)\frac{(-t)^j}{j!}\right)\psi(t)dt\right|\end{aligned}$$

$$\leq \frac{\Lambda^{-h}}{h!}\int_{-\infty}^{+\infty}\left|\frac{d^h}{ds^h}m((s-\vartheta t)/\Lambda)\right|\left|t^h\psi(t)\right|dt$$
$$\leq c\Lambda^{-h}\left(1+(s/\Lambda)^2\right)^{-k}.$$

Hence, $|m_\Lambda(s)-m_\Lambda*\psi(s)|$ vanishes so fast as $\Lambda\to+\infty$ that for every function in $\mathbb{L}^1+\mathbb{L}^2(\mathbb{R}^N)$,

$$\lim_{\Lambda\to+\infty}\int_{\mathbb{R}^N}|m_\Lambda(2\pi|\xi|)-m_\Lambda*\psi(2\pi|\xi|)|\,|\mathbb{F}f(\xi)|\,d\xi=0\,.$$

Therefore, for these means localization holds. Of course, this result is well known and immediately follows from the fact that smooth multipliers have fast decaying kernels.

Let us now consider the case of the spherical partial sums which are associated to the discontinuous multipliers $m_\Lambda(s)=\chi_{[-\Lambda,\Lambda]}(s)$. In this case, localization does not hold for every function, but we have the following.

Corollary 1.

Let $f(x)$ be in $\mathbb{L}^1+\mathbb{L}^2(\mathbb{R}^N)$ and assume that

$$\lim_{\Lambda\to+\infty}\int_{\{\Lambda\leq2\pi|\xi|\leq\Lambda+1\}}|\mathbb{F}f(\xi)|\,d\xi=0\,.$$

Then for the spherical partial sums of $f(x)$ localization holds. In particular, if this function is smooth in an open set, then for all points x in this open set we have

$$\lim_{\Lambda\to+\infty}\int_{\{2\pi|\xi|\leq\Lambda\}}\mathbb{F}f(\xi)\exp(2\pi ix\cdot\xi)d\xi=f(x)\,.$$

Proof. If $m_\Lambda(s)=\chi_{[-\Lambda,\Lambda]}(s)$, it is not difficult to see that $|m_\Lambda(s)-m_\Lambda*\psi(s)|$ is essentially a bump around Λ of height and width one. Indeed, for every k,

$$|m_\Lambda(s)-m_\Lambda*\psi(s)|\leq c\,(1+|\Lambda-|s||)^{-k}\,.$$

Because of the assumptions on the Fourier transform, as $\Lambda\to+\infty$ we have

$$\left|\int_{\mathbb{R}^N}(m_\Lambda(2\pi|\xi|)-m_\Lambda*\psi(2\pi|\xi|))\,\mathbb{F}f(\xi)\exp(2\pi ix\cdot\xi)d\xi\right|$$
$$\leq c\int_{\mathbb{R}^N}(1+|\Lambda-2\pi|\xi||)^{-k}\,|\mathbb{F}f(\xi)|\,d\xi\to0\,.\qquad\square$$

It follows from the Riemann–Lebesgue lemma and the Plancherel equality that for $N=1$ and $f(x)$ in $\mathbb{L}^1+\mathbb{L}^2(\mathbb{R})$ the assumptions in the corollary are automatically satisfied. When $N>1$ the hypotheses of the corollary are not automatic and we want to present some classes of functions for which these hypotheses hold.

A first natural candidate is the Sobolev space $\mathbb{W}^2_\alpha(\mathbb{R}^N)$ defined by the norm

$$\left\{\int_{\mathbb{R}^N}\left(1+|\xi|^2\right)^\alpha|\mathbb{F}f(\xi)|^2\,d\xi\right\}^{1/2}<+\infty\,.$$

The following result, also contained in [1], easily follows from the previous arguments.

Corollary 2.

For spherical partial sums of functions in $\mathbb{W}^2_\alpha(\mathbb{R}^N)$ localization holds when $\alpha\geq(N-1)/2$ and this index is best possible.

Proof. If $f(x)$ is in $\mathbb{W}^2_\alpha(\mathbb{R}^N)$, then

$$\int_{\{\Lambda\leq 2\pi|\xi|\leq\Lambda+1\}} |\mathbb{F}f(\xi)|\,d\xi \leq c(1+\Lambda)^{-\alpha+(N-1)/2}\left\{\int_{\{\Lambda\leq 2\pi|\xi|\leq\Lambda+1\}}\left(1+|\xi|^2\right)^\alpha |\mathbb{F}f(\xi)|^2\,d\xi\right\}^{1/2}.$$

When $\alpha \geq (N-1)/2$ the above quantity tends to zero as $\Lambda \to +\infty$. In order to prove that the index $(N-1)/2$ is best possible, it suffices to construct a function in $\mathbb{W}^2_\alpha(\mathbb{R}^N)$ with $\alpha < (N-1)/2$ and such that

$$\limsup_{\Lambda\to+\infty}\left|\int_{\mathbb{R}^N}(m_\Lambda(2\pi|\xi|) - m_\Lambda * \psi(2\pi|\xi|))\,\mathbb{F}f(\xi)\exp(2\pi i x\cdot\xi)d\xi\right| = +\infty\,.$$

Since $|m_\Lambda(s) - m_\Lambda * \psi(s)|$ is essentially a bump around Λ of height and width one and since $\mathbb{W}^2_\alpha(\mathbb{R}^N)$ is defined in terms of the modulus of the Fourier transform, for a fixed x and every Λ there exists a function $f_\Lambda(y)$ of norm one in $\mathbb{W}^2_\alpha(\mathbb{R}^N)$, with $\mathbb{F}f_\Lambda(\xi)$ supported in an annulus $\{\Lambda-1 < 2\pi|\xi| < \Lambda+1\}$ and

$$\int_{\mathbb{R}^N}(m_\Lambda(2\pi|\xi|) - m_\Lambda * \psi(2\pi|\xi|))\,\mathbb{F}f_\Lambda(\xi)\exp(2\pi i x\cdot\xi)d\xi \geq c\Lambda^{(N-1)/2-\alpha}\,.$$

Then $f(y) = \sum_{k=1}^{+\infty} k^{-2} f_{2^k}(y)$ has the required properties. □

In order to apply the above results to other classes of functions, we introduce a class of piecewise smooth functions in several variables. Let K be a compact subset of $\mathbb{R}^N$ with piecewise smooth boundary and let $f(x)$ be a real function which is smooth in K and vanishes outside K. It is convenient to normalize this function on the boundary ∂K by dividing its values by two, or by $\alpha/2\pi$ if at x the boundary has an angle of α. Linear combinations of such functions generate the space of piecewise smooth functions.

The example of characteristic functions of balls shows that for spherical partial sums of piecewise smooth functions, the localization may fail at least in dimension $N \geq 3$. However, in dimension two we have not only localization, but also convergence. This result is already contained in [15, 16, 5], but the following proof is different and the technique, which combines the wave equation with restriction for Fourier transforms and approximation with functions of exponential type, will be used throughout the paper.

Corollary 3.

For spherical partial sums of piecewise smooth functions of two variables localization holds. Moreover, at every point of continuity,

$$\lim_{\Lambda\to+\infty}\int_{\{2\pi|\xi|\leq\Lambda\}}\mathbb{F}f(\xi)\exp(2\pi i x\cdot\xi)d\xi = f(x)\,.$$

Proof. It is enough to show that for piecewise smooth functions of two variables,

$$\lim_{\Lambda\to+\infty}\int_{\{\Lambda\leq 2\pi|\xi|\leq\Lambda+1\}}|\mathbb{F}f(\xi)|\,d\xi = 0\,.$$

If this is true, by the previous corollary localization holds and from localization one can deduce convergence in every open set where the function is smooth.

By the restriction theorem for the Fourier transform, see Stein [20], VIII, if $1 \leq p \leq \frac{2N+2}{N+3}$,

$$\left\{\int_{\{|\xi|=1\}}|\mathbb{F}f(\xi)|^2\,d\xi\right\}^{1/2} \leq c\left\{\int_{\mathbb{R}^N}|f(x)|^p\,dx\right\}^{1/p}.$$

Hence, integrating in polar coordinates and rescaling,

$$\begin{aligned}
&\int_{\{\Lambda\le 2\pi|\xi|\le\Lambda+1\}} |\mathbb{F}f(\xi)|\,d\xi \\
&\le \left\{\int_{\{\Lambda\le 2\pi|\xi|\le\Lambda+1\}} d\xi\right\}^{1/2} \left\{\int_{\Lambda/2\pi}^{(\Lambda+1)/2\pi} \left(\int_{\{|\xi|=1\}} |\mathbb{F}f(r\xi)|^2\,d\xi\right) r^{N-1}dr\right\}^{1/2} \\
&\le c\left\{\int_{\{\Lambda\le 2\pi|\xi|\le\Lambda+1\}} d\xi\right\}^{1/2} \left\{\int_{\Lambda/2\pi}^{(\Lambda+1)/2\pi} \left(\int_{\mathbb{R}^N} \left|r^{-N}f(r^{-1}x)\right|^p dx\right)^{2/p} r^{N-1}dr\right\}^{1/2} \\
&\le c(1+\Lambda)^{N/p-1}\left\{\int_{\mathbb{R}^N} |f(x)|^p\,dx\right\}^{1/p} .
\end{aligned}$$

These estimates can be improved by replacing $f(x)$ with an appropriate function whose Fourier transform agrees with $\mathbb{F}f(\xi)$ when $\Lambda \le |\xi| \le \Lambda+1$ but with smaller norm. In particular, subtracting from $f(x)$ any function $h(x)$ with $\mathbb{F}h(\xi) = 0$ for $|\xi| > \Lambda$ one obtains

$$\int_{\{\Lambda\le 2\pi|\xi|\le\Lambda+1\}} |\mathbb{F}f(\xi)|\,d\xi \le c(1+\Lambda)^{N/p-1}\left\{\int_{\mathbb{R}^N} |f(x)-h(x)|^p\,dx\right\}^{1/p} ,$$

and one has the freedom to choose an $h(x)$ that makes the right-hand side of this inequality small. Let

$$\omega(s) = \inf\left\{\int_{\mathbb{R}^N} |f(x)-h(x)|^p\,dx\right\}^{1/p} ,$$

where the infimum is taken over all $h(x)$ of exponential type s^{-1}. By the theorems of Jackson and Bernstein, see Nikol'skiĭ [14], $\omega(s)$ is a measure of the smoothness of $f(x)$ in $\mathbb{L}^p(\mathbb{R}^N)$. In particular, for a piecewise smooth function,

$$\omega(s) \le c\sup_{|y|<s}\left\{\int_{\mathbb{R}^N} |f(x+y)-f(x)|^p\,dx\right\}^{1/p} \le cs^{1/p} .$$

When $N = 2$ and $p = 6/5$, we thus obtain

$$\int_{\{\Lambda\le 2\pi|\xi|\le\Lambda+1\}} |\mathbb{F}f(\xi)|\,d\xi \le c(1+\Lambda)^{-1/6} . \qquad \square$$

Observe that piecewise smooth functions are in $\mathbb{W}^2_\alpha(\mathbb{R}^N)$ for $\alpha < 1/2$ but not in $\mathbb{W}^2_{1/2}(\mathbb{R}^N)$, so that in the above arguments we cannot use $p = 2$. Similarly, we cannot use $p = 1$. Also observe that the above corollary holds not only for piecewise smooth functions, but also for any function in $\mathbb{L}^p(\mathbb{R}^N)$ with modulus of continuity $\omega(s) \le cs^{1/p}$. It is not too difficult to see that convergence holds even at points on the lines of discontinuity. A proof of this fact with a discussion of Gibbs phenomenon is contained in Colzani–Vignati [5].

We want to mention that a different proof, based on a result in Varchenko [24], suggests for the Fourier transform of a piecewise smooth function an average decay

$$\int_{\{\Lambda\le 2\pi|\xi|\le\Lambda+1\}} |\mathbb{F}f(\xi)|\,d\xi \le c(1+\Lambda)^{(N-3)/2} .$$

When $N = 2$ this quantity tends to zero and when $N = 3$ it stays bounded, so that, even if localization may fail in dimension three, at least the spherical partial sums are not unbounded. See also Pinsky [15] and Kahane [12].

2. Localization and Equiconvergence of Eigenfunction Expansions

Let Ω be a bounded open set in $\mathbb{R}^N$ with smooth boundary and let $\{\lambda^2\}$ and $\{\varphi_\lambda(x)\}$ be the eigenvalues and an orthonormal system of eigenfunctions of the Dirichlet problem for the Laplace operator $\Delta = -\sum_{j=1}^N \partial^2/\partial x_j^2$,

$$\begin{cases} -\sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2}\varphi_\lambda(x) = \lambda^2\varphi_\lambda(x) & \text{if } x \in \Omega\,, \\ \varphi_\lambda(x) = 0 & \text{if } x \in \partial\Omega\,. \end{cases}$$

To every function $f(x)$ in $\mathbb{L}^2(\Omega)$ we can associate a Fourier transform and an eigenfunction expansion

$$\mathcal{F}f(\lambda) = \int_\Omega f(y)\overline{\varphi_\lambda(y)}dy,\ f(x) = \sum_\lambda \mathcal{F}f(\lambda)\varphi_\lambda(x)\,.$$

As in the previous section, the operator $\mathcal{M}_\Lambda$ is defined by

$$\mathcal{M}_\Lambda f(x) = \sum_\lambda m_\Lambda(\lambda)\mathcal{F}f(\lambda)\varphi_\lambda(x)\,.$$

If we extend $f(x)$ to all $\mathbb{R}^N$ by putting $f(x) = 0$ outside Ω, we can also consider the classical trigonometric Fourier integral expansion. We want to compare $\mathcal{M}_\Lambda f(x)$ with $\mathbb{M}_\Lambda f(x)$. In particular, we want to show that under appropriate Tauberian conditions these two means are equiconvergent. This allows us to transfer results for Fourier integrals to eigenfunction expansions.

If we approximate $\mathcal{M}_\Lambda f(x)$ with $\sum_\lambda m_\Lambda * \psi(\lambda)\mathcal{F}f(\lambda)\varphi_\lambda(x)$, we have the following.

Theorem 2.

1) *If the point x varies in a compact set K in Ω and if the diameter of the support of $\widehat{\psi}(t)$ is smaller than the distance of K from the boundary $\partial\Omega$, then*

$$\sum_\lambda m_\Lambda * \psi(\lambda)\mathcal{F}f(\lambda)\varphi_\lambda(x) = \int_{\mathbb{R}^N} m_\Lambda * \psi(2\pi\,|\xi|)\mathbb{F}f(\xi)\exp(2\pi i x\cdot\xi)d\xi\,.$$

In particular, the value of the means $\sum_\lambda m_\Lambda * \psi(\lambda)\mathcal{F}f(\lambda)\varphi_\lambda(x)$ at a point x depends only on the values of $f(y)$ at points $|y - x| \le \varepsilon$.

2) *Assume that*

$$\lim_{\Lambda\to+\infty}\left|\sum_\lambda \left(m_\Lambda(\lambda) - m_\Lambda * \psi(\lambda)\right)\mathcal{F}f(\lambda)\varphi_\lambda(x)\right| = 0\,,$$

$$\lim_{\Lambda\to+\infty}\left|\int_{\mathbb{R}^N}\left(m_\Lambda(2\pi\,|\xi|) - m_\Lambda * \psi(2\pi\,|\xi|)\right)\mathbb{F}f(\xi)\exp(2\pi i x\cdot\xi)d\xi\right| = 0\,.$$

Then the eigenfunction expansions and the trigonometric Fourier integrals are equiconvergent,

$$\lim_{\Lambda\to+\infty}\left|\sum_\lambda m_\Lambda(\lambda)\mathcal{F}f(\lambda)\varphi_\lambda(x) - \int_{\mathbb{R}^N} m_\Lambda(2\pi\,|\xi|)\mathbb{F}f(\xi)\exp(2\pi i x\cdot\xi)d\xi\right| = 0\,.$$

Moreover, the behavior as $\Lambda \to +\infty$ of the means $\sum_\lambda m_\Lambda(\lambda)\mathcal{F}f(\lambda)\varphi_\lambda(x)$ at x depends only on the values of $f(y)$ at points $|y - x| \le \varepsilon$.

Proof. We have

$$\begin{aligned}
&\sum_\lambda m_\Lambda * \psi(\lambda)\mathcal{F}f(\lambda)\varphi_\lambda(x) \\
&= \sum_\lambda \left(\int_0^{+\infty} (m_\Lambda * \psi)^\wedge(t)\cos(\lambda t)dt\right)\mathcal{F}f(\lambda)\varphi_\lambda(x) \\
&= \int_0^{+\infty} \left(\pi\widehat{m_\Lambda}(t)\widehat{\psi}(t)\right)\left(\sum_\lambda \cos(\lambda t)\mathcal{F}f(\lambda)\varphi_\lambda(x)\right)dt \\
&= \pi\int_0^{+\infty} \widehat{m_\Lambda}(t)\widehat{\psi}(t)\cos(t\sqrt{\Delta})f(x)dt\,,
\end{aligned}$$

where $\cos(t\sqrt{\Delta})f(x)$ solves the wave equation in $\mathbb{R}\times\Omega$,

$$\begin{cases}
\frac{\partial^2}{\partial t^2}u(t,x) - \sum_{j=1}^{N}\frac{\partial^2}{\partial x_j^2}u(t,x) = 0 & \text{if } t\in\mathbb{R} \text{ and } x\in\Omega\,, \\
u(t,x) = 0 & \text{if } t\in\mathbb{R} \text{ and } x\in\partial\Omega\,, \\
u(0,x) = f(x) & \text{if } x\in\Omega\,, \\
\frac{\partial}{\partial t}u(0,x) = 0 & \text{if } x\in\Omega\,.
\end{cases}$$

Similarly,

$$\int_{\mathbb{R}^N} m_\Lambda * \psi(2\pi\,|\xi|)\mathbb{F}f(\xi)\exp(2\pi i x\cdot\xi)d\xi = \pi\int_0^{+\infty}\widehat{m_\Lambda}(t)\widehat{\psi}(t)\cos(t\sqrt{\Delta})f(x)dt\,,$$

where this time $\cos(t\sqrt{\Delta})f(x)$ solves the wave equation in $\mathbb{R}\times\mathbb{R}^N$. Since waves propagate with finite speed, for small times t the solutions of the wave equation in $\mathbb{R}^N$ and in a compact subset of Ω are the same. Since $\widehat{\psi}(t)\neq 0$ only for small t, 1) follows.

2) is an immediate consequence of 1). □

As in the previous section, it is not difficult to see that when the multiplier $m(s)$ is smooth, the Tauberian conditions in the theorem are automatically satisfied. To see this, recall that if the function $m(s)$ has rapidly decaying derivatives, then for every h and k, $|m_\Lambda(s) - m_\Lambda * \psi(s)| \leq c\Lambda^{-h}\left(1+(s/\Lambda)^2\right)^{-k}$. Moreover, by Sobolev imbedding theorems, the size of the eigenfunctions $|\varphi_\lambda(x)|$ is at most of polynomial growth in λ and by Weyl estimates on the eigenvalues of a Dirichlet problem the number of eigenvalues $\lambda\leq\Lambda$ grows at a polynomial rate in Λ. Hence, for every integrable function $f(x)$ also the Fourier transform $|\mathcal{F}f(\lambda)|$ has at most polynomial growth in λ and for some p,

$$\sum_{n-1\leq\lambda\leq n} |\mathcal{F}f(\lambda)|\,|\varphi_\lambda(x)| \leq cn^p\,.$$

Of course it is possible to be more precise, but these estimates already imply that if $m(s)$ is suitably smooth, then

$$\lim_{\Lambda\to+\infty}\sum_\lambda |m_\Lambda(\lambda) - m_\Lambda * \psi(\lambda)|\,|\mathcal{F}f(\lambda)|\,|\varphi_\lambda(x)| = 0\,.$$

Hence, the assumptions in the above theorem are satisfied and this implies that the means of the eigenfunction expansions are equiconvergent with the classical trigonometric expansions. Indeed, for a smooth multiplier it is also possible to compare the size of $|\mathcal{M}_\Lambda f(x) - f(x)|$ with the one of $|\mathbb{M}_\Lambda f(x) - f(x)|$ and the result is that, for every q,

$$|\mathcal{M}_\Lambda f(x) - f(x)| \le |\mathbb{M}_\Lambda f(x) - f(x)| + c\Lambda^{-q} .$$

In particular, in the next section we shall need an approximation theorem of Jackson type for eigenfunction expansions and this can be obtained from the Euclidean case using this method. See also Taylor [22], XII.

Finally, let us briefly consider the spherical partial sums which are associated to the multipliers $m_\Lambda(s) = \chi_{[-\Lambda,\Lambda]}(s)$. In this case,

$$\begin{aligned}&\left|\sum_\lambda (m_\Lambda(\lambda) - m_\Lambda * \psi(\lambda))\,\mathcal{F}f(\lambda)\varphi_\lambda(x)\right| \\ &\le \left\{\sum_\lambda |m_\Lambda(\lambda) - m_\Lambda * \psi(\lambda)|\,|\varphi_\lambda(x)|^2\right\}^{1/2} \left\{\sum_\lambda |m_\Lambda(\lambda) - m_\Lambda * \psi(\lambda)|\,|\mathcal{F}f(\lambda)|^2\right\}^{1/2} .\end{aligned}$$

Recall that $|m_\Lambda(\lambda) - m_\Lambda * \psi(\lambda)|$ is essentially a bump of height and width one around Λ. Also, it follows from the asymptotic behavior of the spectral function that, for x in Ω and $n = 1, 2, \ldots$,

$$\left\{\sum_{n-1\le\lambda\le n} |\varphi_\lambda(x)|^2\right\}^{1/2} \le cn^{(N-1)/2} ,$$

where the constant c may depend on the distance of x from $\partial\Omega$. See, for example, Hörmander [10] or Theorem 17.5.7 in [11]. Hence, $\left|\sum_\lambda (m_\Lambda(\lambda) - m_\Lambda * \psi(\lambda))\,\mathcal{F}f(\lambda)\varphi_\lambda(x)\right|$ is roughly speaking controlled by $\Lambda^{(N-1)/2}\left\{\sum_{\Lambda-1\le\lambda\le\Lambda+1} |\mathcal{F}f(\lambda)|^2\right\}^{1/2}$. If we define the Sobolev spaces $\mathbb{W}^2_\alpha(\Omega)$ by means of the norm

$$\left\{\sum_\lambda \left(1+\lambda^2\right)^\alpha |\mathcal{F}f(\lambda)|^2\right\}^{1/2} < +\infty ,$$

then for functions in $\mathbb{W}^2_\alpha(\Omega) \bigcap \mathbb{W}^2_\alpha(\mathbb{R}^N)$ with $\alpha \ge (N-1)/2$ the Tauberian conditions hold and we can conclude that for functions in these spaces we have localization.

We conclude this section by observing that the results stated for expansions in eigenfunctions of a Dirichlet problem are of local nature and the boundary plays no particular role; hence, there are analogous results for expansions in eigenfunctions of other boundary value problems.

3. Partial Sums of Eigenfunction Expansions on Two-Dimensional Manifolds

In a series of papers on the Gibbs phenomenon, Weyl [25] studied the convergence of spherical harmonic expansions on the two-dimensional sphere $\{x^2 + y^2 + z^2 = 1\}$. Motivated by these results, Colzani and Vignati studied the summability of Fourier integrals of piecewise smooth functions and the associated Gibbs phenomenon in a neighborhood of the discontinuities, see [5]. Similar problems have also been the subject of a series of papers by Pinsky, Gray, Stanton, Trapa, Taylor, Kahane, De Michele, and Roux, see [15, 16, 7, 17, 18, 12, 6]. Here we want to study the convergence of expansions in eigenfunctions on a manifold and we start with an explicit example.

The spherical harmonic expansions of radial functions on a three-dimensional sphere reduce to expansions into Tchebichef polynomials of second kind

$$\left\{\frac{\sin((n+1)\vartheta)}{\sin(\vartheta)}\right\}_{n=0}^{+\infty} ,$$

which are a complete orthonormal system in $\mathbb{L}^2\left([0,\pi], \frac{2\sin^2(\vartheta)}{\pi}d\vartheta\right)$. In particular, for the characteristic function of the interval $0 \le \vartheta \le a < \pi$ one has

$$\chi_{\{0\le\vartheta\le a\}}(\vartheta) = \sum_{n=0}^{+\infty}\left(\frac{\sin(na) - \sin(na+2a)}{\pi(n+2)} + \frac{2\sin(na)}{\pi n\,(n+2)}\right)\frac{\sin((n+1)\vartheta)}{\sin(\vartheta)}\,.$$

It is clear that this series converges for every $0 < \vartheta < \pi$, but not for $\vartheta = 0$ and $\vartheta = \pi$. When translated to the three-dimensional sphere, this result implies that the spherical harmonic expansion of the characteristic function of a spherical cap diverges at the center of the cap and at the antipodal point.

In general, when the manifold has dimension greater than two, the spherical partial sums of eigenfunction expansions may exhibit a lack of localization. This fact and other technical difficulties force us to consider only the case of dimension two.

In this section we switch from a domain in $\mathbb{R}^N$ to a two-dimensional manifold $\mathbb{M}$, smooth, compact, without boundary. Let Δ be a second order positive elliptic operator on $\mathbb{M}$, with smooth real coefficients and self adjoint with respect to some positive smooth density $d\mu$. As before, let $\{\lambda^2\}$ and $\{\varphi_\lambda(x)\}$ be the eigenvalues and a system of eigenfunctions of Δ orthonormal in $\mathbb{L}^2(\mathbb{M}, d\mu)$. To every function $f(x)$ in $\mathbb{L}^2(\mathbb{M}, d\mu)$ we can associate a Fourier transform $\{\mathcal{F}f(\lambda)\}$ with respect to the system $\{\varphi_\lambda(x)\}$ and an eigenfunction expansion $\sum_\lambda \mathcal{F}f(\lambda)\varphi_\lambda(x)$. We want to study the pointwise convergence, as $\Lambda \to +\infty$, of the partial sums $\sum_{\lambda\le\Lambda}\mathcal{F}f(\lambda)\varphi_\lambda(x)$ when the function $f(x)$ is piecewise smooth.

Let K be a compact subset of $\mathbb{M}$ with smooth boundary and let $f(x)$ be a real function which is smooth in K and vanishes outside K. It is convenient to normalize this function on the boundary ∂K by dividing its values by two. Linear combinations of such functions generate the space of piecewise smooth functions.

Theorem 3.

1) *The partial sums $\{S_\Lambda f(x)\}_{\Lambda>0}$ of the eigenfunction expansion of a piecewise smooth function converge to $f(x)$ at every point.*

2) *Assume that the point x is on a line of discontinuity and at x the function has a jump of $2J$. Then, if $\Lambda \to +\infty$ and if $y \to x$ along lines not tangent to the discontinuities,*

$$\limsup_{\substack{\Lambda\to+\infty\\ y\to x}}\left|f(x) - \sum_{\lambda\le\Lambda}\mathcal{F}f(\lambda)\varphi_\lambda(y)\right| = J\frac{2}{\pi}\int_0^\pi \frac{\sin(t)}{t}dt\,.$$

We recall that $\frac{2}{\pi}\int_0^\pi \frac{\sin(t)}{t}dt \approx 1.179$. In particular, exactly as for the classical Fourier series there is a Gibbs phenomenon: in a neighborhood of the lines of discontinuity, the partial sums overshoot the target by about 9% the value of the jump. For two-dimensional spherical harmonic expansions, this result is due to Weyl and the analogous result in three dimensions may fail.

Proof. As in the previous section, the idea is to show that the eigenfunction expansion is equiconvergent with a trigonometric Fourier integral expansion. Let $m_\Lambda(s) = \chi_{[-\Lambda,\Lambda]}(s)$ and write

$$\sum_{\lambda\le\Lambda}\mathcal{F}f(\lambda)\varphi_\lambda(x) = \sum_\lambda m_\Lambda * \psi(\lambda)\mathcal{F}f(\lambda)\varphi_\lambda(x) + \sum_\lambda\left(m_\Lambda(\lambda) - m_\Lambda * \psi(\lambda)\right)\mathcal{F}f(\lambda)\varphi_\lambda(x)\,.$$

We first consider the remainder $\sum_\lambda\left(m_\Lambda(\lambda) - m_\Lambda * \psi(\lambda)\right)\mathcal{F}f(\lambda)\varphi_\lambda(x)$ and we prove that this is uniformly small.

Lemma 2.

If $f(x)$ is piecewise smooth, then for every x we have

$$\lim_{\Lambda\to+\infty}\left|\sum_{\lambda}(m_\Lambda(\lambda)-m_\Lambda*\psi(\lambda))\mathcal{F}f(\lambda)\varphi_\lambda(x)\right|=0\,.$$

Proof.

$$\left|\sum_{\lambda}(m_\Lambda(\lambda)-m_\Lambda*\psi(\lambda))\mathcal{F}f(\lambda)\varphi_\lambda(x)\right|$$
$$\le\left\{\sum_{\lambda}|m_\Lambda(\lambda)-m_\Lambda*\psi(\lambda)|\,|\mathcal{F}f(\lambda)|^2\right\}^{1/2}\left\{\sum_{\lambda}|m_\Lambda(\lambda)-m_\Lambda*\psi(\lambda)|\,|\varphi_\lambda(x)|^2\right\}^{1/2}.$$

For every k we have $|m_\Lambda(\lambda)-m_\Lambda*\psi(\lambda)|\le c\,(1+|\Lambda-|\lambda||)^{-k}$ and, by the sharp form for the remainder of the spectral function, see Hörmander [10], for every $n=1,2,\ldots$, $\left\{\sum_{n-1\le\lambda\le n}|\varphi_\lambda(x)|^2\right\}^{1/2}\le cn^{1/2}$. Hence,

$$\left\{\sum_{\lambda}|m_\Lambda(\lambda)-m_\Lambda*\psi(\lambda)|\,|\varphi_\lambda(x)|^2\right\}^{1/2}\le c\Lambda^{1/2}\,.$$

By the restriction theorem for eigenfunction expansions, see Sogge [19],

$$\left\{\sum_{n-1\le\lambda\le n}|\mathcal{F}f(\lambda)|^2\right\}^{1/2}$$
$$\le\begin{cases} cn^{2/p-3/2}\left\{\int_\Omega|f(x)|^p\,d\mu(x)\right\}^{1/p} & \text{if } 1\le p\le 6/5\,,\\ cn^{(2-p)/4p}\left\{\int_\Omega|f(x)|^p\,d\mu(x)\right\}^{1/p} & \text{if } 6/5\le p\le 2\,,\end{cases}$$

but these estimates can be improved using the smoothness of the function. Let

$$\omega(s)=\inf\left\{\int_\Omega|f(x)-h(x)|^p\,d\mu(x)\right\}^{1/p},$$

where the infimum is taken over all $h(x)$ of the form $\sum_{\lambda\le 1/s}c_\lambda\varphi_\lambda(x)$. By the theorems of Jackson and Bernstein, $\omega(s)$ is a measure of the smoothness of $f(x)$. See Nikol'skiĭ [14] for the Euclidean case, the extension to a manifold can be achieved using the techniques in the preceding section or in Taylor [22], XII. Introducing $f(x)-h(x)$ in the restriction theorem we have

$$\left\{\sum_{n-1\le\lambda\le n}|\mathcal{F}f(\lambda)|^2\right\}^{1/2}\le cn^\gamma\omega(n^{-1})\,,$$

with γ the appropriate exponent in the restriction theorem. In particular, for a piecewise smooth function, $\omega(s)\le cs^{1/p}$ and with $p=6/5$ we obtain

$$\left\{\sum_{\lambda}|m_\Lambda(\lambda)-m_\Lambda*\psi(\lambda)|\,|\mathcal{F}f(\lambda)|^2\right\}^{1/2}\le c\Lambda^{-2/3}\,.\quad\square$$

In order to consider the main term $\sum_\lambda m_\Lambda * \psi(\lambda)\mathcal{F}f(\lambda)\varphi_\lambda(x)$, we recall the Hadamard construction of a parametrix of the Cauchy problem for the wave equation, see Hörmander [11] 17.4.3, or [2] and [3].

We recall that by taking the trigonometric Fourier integral transform of $\cos(2\pi t\,|\xi|)$ one obtains the fundamental solution of the wave equation on $\mathbb{R}^N$,

$$\pi^{(1-N)/2} t \frac{\left(t^2-|x-y|^2\right)_+^{-(N+1)/2}}{\Gamma((1-N)/2)} ,$$

where the distributions $t_+^{-\alpha}/\Gamma(1-\alpha)$ are defined for every α recursively by

$$\int_0^{+\infty} \frac{t^{-\alpha}}{\Gamma(1-\alpha)} f(t)dt = -\int_0^{+\infty} \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} \frac{d}{dt} f(t)dt .$$

It is natural to conjecture a relation between the fundamental solutions of the wave equations on the manifold $\mathbb{M}$ and on the Euclidean space $\mathbb{R}^2$, at least for small times. Indeed, for t small, $-\varepsilon < t < \varepsilon$, one has

$$\begin{aligned}\cos(t\sqrt{\Delta})(x,y) &= \sum_\lambda \cos(t\lambda)\varphi_\lambda(x)\overline{\varphi_\lambda(y)} \\ &= -\frac{1}{2\pi} t \left(t^2-d(x,y)^2\right)_+^{-3/2} U(x,y) + tV(t,x,y) .\end{aligned}$$

The series $\sum_\lambda \cos(t\lambda)\varphi_\lambda(x)\overline{\varphi_\lambda(y)}$ converges in the topology of distributions, $d(x,y)$ denotes the distance between x and y with respect to the Riemannian metric associated to the principal part of the differential operator Δ, the function $U(x,y)$ is smooth and $V(t,x,y)$ has at most a singularity of type $\left(t^2-d(x,y)^2\right)_+^{-1/2}$. In particular, for small times the fundamental solutions of the wave equations on $\mathbb{M}$ and on $\mathbb{R}^2$ are not equal but similar, and

$$\begin{aligned}&\sum_\lambda m_\Lambda * \psi(\lambda)\mathcal{F}f(\lambda)\varphi_\lambda(x) = \int_0^{+\infty} (m_\Lambda * \psi)\hat{}(t)\cos(t\sqrt{\Delta})f(x)dt \\ &= \int_{\mathbb{M}} \left(-\frac{1}{2\pi}\int_0^{+\infty} (m_\Lambda * \psi)\hat{}(t) t \left(t^2-d(x,y)^2\right)_+^{-3/2} dt\right) U(x,y)f(y)d\mu(y) \\ &+ \int_{\mathbb{M}} \left(\int_0^{+\infty} (m_\Lambda * \psi)\hat{}(t) t V(t,x,y)dt\right) f(y)d\mu(y) .\end{aligned}$$

Lemma 3.

If $f(x)$ is piecewise smooth, then

$$\lim_{\Lambda\to+\infty} \int_{\mathbb{M}} \left(\int_0^{+\infty} (m_\Lambda * \psi)\hat{}(t) t V(t,x,y)dt\right) f(y)d\mu(y) = 0 .$$

Proof. We have $(m_\Lambda * \psi)\hat{}(t) = \frac{2\sin(\Lambda t)}{t}\widehat{\psi}(t)$ and

$$\int_0^{+\infty} (m_\Lambda * \psi)\hat{}(t) t V(t,x,y)dt = 2\int_0^{+\infty} \widehat{\psi}(t)V(t,x,y)\sin(\Lambda t)dt .$$

Since the function $\widehat{\psi}(t)V(t,x,y)$ is integrable in t, by the Riemann–Lebesgue lemma, the integral converges to zero as $\Lambda \to +\infty$. □

We come back for a moment to the Euclidean space $\mathbb{R}^2$. The following lemma is just a restatement of some results in the first section.

Lemma 4.

If $f(x)$ is piecewise smooth on $\mathbb{R}^2$, then

$$\lim_{\Lambda\to+\infty}\int_{\mathbb{R}^2}\left(-\frac{1}{2\pi}\int_0^{+\infty}(m_\Lambda*\psi)\hat{}(t)t\left(t^2-|x-y|^2\right)_+^{-3/2}dt\right)f(y)dy=f(x)\,.$$

Proof. Observe that

$$\int_{\mathbb{R}^2}\left(-\frac{1}{2\pi}\int_0^{+\infty}(m_\Lambda*\psi)\hat{}(t)t\left(t^2-|x-y|^2\right)_+^{-3/2}dt\right)f(y)dy$$
$$=\int_{\mathbb{R}^2}m_\Lambda*\psi(2\pi\,|\xi|)\mathbb{F}f(\xi)\exp(2\pi ix\cdot\xi)d\xi\,.$$

We proved in the first section that

$$\lim_{\Lambda\to+\infty}\int_{\mathbb{R}^2}|m_\Lambda(2\pi\,|\xi|)-m_\Lambda*\psi(2\pi\,|\xi|)|\,|\mathbb{F}f(\xi)|\,d\xi=0\,,$$
$$\lim_{\Lambda\to+\infty}\int_{\mathbb{R}^2}m_\Lambda(2\pi\,|\xi|)\mathbb{F}f(\xi)\exp(2\pi ix\cdot\xi)d\xi=f(x)\,.$$

Hence, we also have

$$\lim_{\Lambda\to+\infty}\int_{\mathbb{R}^2}m_\Lambda*\psi(2\pi\,|\xi|)\mathbb{F}f(\xi)\exp(2\pi ix\cdot\xi)d\xi=f(x)\,.\qquad\square$$

Integrating in polar coordinates, we may restate the above lemma as follows:

$$\lim_{\Lambda\to+\infty}\int_0^{+\infty}\left(-\frac{1}{2\pi}\int_0^{+\infty}(m_\Lambda*\psi)\hat{}(t)t\left(t^2-s^2\right)_+^{-3/2}dt\right)g(s)sds=\frac{1}{2\pi}g(0)\,.$$

where $g(s)=\int_{\{|\sigma|=1\}}f(x-s\sigma)d\sigma$.

We can introduce polar coordinates also on a manifold using the exponential map. For fixed x and y close to x we write $y=Exp_x(s\sigma)$ with $s=d(x,y)$ and σ unit vector in the tangent space at x. Moreover, $d\mu(Exp_x(s\sigma))=W(x,s\sigma)sdsd\sigma$ for some smooth density $W(x,s\sigma)$. The following lemma concludes the proof of the first part of the theorem.

Lemma 5.

If $f(x)$ is piecewise smooth, then

$$\lim_{\Lambda\to+\infty}\int_{\mathbb{M}}\left(-\frac{1}{2\pi}\int_0^{+\infty}(m_\Lambda*\psi)\hat{}(t)t\left(t^2-d(x,y)^2\right)_+^{-3/2}dt\right)U(x,y)f(y)d\mu(y)=f(x)\,.$$

Proof. Observe that $(m_\Lambda*\psi)\hat{}(t)t\left(t^2-d(x,y)^2\right)_+^{-3/2}$ is different from zero only if $d(x,y)<|t|<\varepsilon$. It is then legitimate to introduce polar coordinates centered at x and write

$$\int_{\mathbb{M}}\left(-\frac{1}{2\pi}\int_0^{+\infty}(m_\Lambda*\psi)\hat{}(t)t\left(t^2-d(x,y)^2\right)_+^{-3/2}dt\right)U(x,y)f(y)d\mu(y)$$
$$=\int_0^{+\infty}\left(-\frac{1}{2\pi}\int_0^{+\infty}(m_\Lambda*\psi)\hat{}(t)t\left(t^2-s^2\right)_+^{-3/2}dt\right)g(s)sds$$

where $g(s)$ is a spherical mean of $f\left(Exp_x(s\sigma)\right)$,

$$g(s)=\int_{\{|\sigma|=1\}}f\left(Exp_x(s\sigma)\right)U\left(x,Exp_x(s\sigma)\right)W(x,s\sigma)d\sigma\,.$$

By the remark that follows the previous lemma,

$$\begin{aligned}\lim_{\Lambda\to+\infty}\int_0^{+\infty}\left(-\frac{1}{2\pi}\int_0^{+\infty}(m_\Lambda*\psi)\hat{}(t)t\left(t^2-s^2\right)_+^{-3/2}dt\right)g(s)sds\\ =\frac{1}{2\pi}g(0)\\ =Cf(x)\,,\end{aligned}$$

where $C=\frac{U(x,x)W(x,0)}{2\pi}$ is independent of $f(x)$. If we apply the whole process to a test function, for which we already know that convergence takes place, we deduce that $C=1$. □

The proof of part 1) of the theorem is then complete. To prove part 2) it is enough to observe that in 1) we have actually proved a uniform equiconvergence result between eigenfunction and trigonometric expansions. Then we can refer to the discussion of Gibbs phenomenon for trigonometric expansions in Colzani–Vignati [5]. □

4. An Application

The Gauss circle problem is the estimate of the number of integer points in a large disk in the plane. Let $r(n)=\#\left\{k\in\mathbb{Z}^2:k_1^2+k_2^2=n\right\}$ and $R(t)=\sum_{n\le t^2}r(n)$. Then

$$\frac{R(t-)+R(t+)}{2}=\pi t^2+t\sum_{n=1}^{\infty}\frac{r(n)}{\sqrt{n}}J_1(2\pi\sqrt{n}t)\,.$$

This identity was first stated by Voronoi and then proved by Hardy in [9]. The connection between this problem in analytic number theory and Fourier series is clearly stated by Kendall in [13]. Instead of considering lattice points in a disk centered at the origin, we let the disk move in the plane. The number of integer points in a disk $B(x,t)$ of center x and radius t then becomes a periodic function of x and, by Poisson summation formula, one has the Fourier expansion

$$\sum_{k\in\mathbb{Z}^2}\chi_{B(x,t)}(k)=\sum_{k\in\mathbb{Z}^2}\chi_{B(0,1)}(t^{-1}(k-x))=\sum_{k\in\mathbb{Z}^2}t^2\widehat{\chi}_{B(0,1)}(tk)\exp\left(2\pi ik\cdot x\right)\,.$$

Since the Fourier transform, a disk can be expressed in terms of Bessel functions, $\widehat{\chi}_{B(0,1)}(\xi)=|\xi|^{-1}J_1(2\pi|\xi|)$, when $x=0$ the Fourier series gives formally Hardy's identity, but of course one has to show that the series is pointwise convergent. The point is that the function $\sum_{k\in\mathbb{Z}^2}\chi_{B(x,t)}(k)$ is piecewise constant; therefore, we can apply the result in the previous section and conclude that the Fourier series converges at every point. Observe that since the torus is locally Euclidean, for small times we have an exact expression for the fundamental solution of the wave equation; hence, it is not necessary to introduce a parametrix. Also observe that since we have an explicit expression for the Fourier transform of a disk, we can avoid the use of restriction theorems for Fourier transforms and Jackson and Bernstein approximation theorems.

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Added in Proof

Recently we have obtained some results on the pointwise convergence of Bochner–Riesz means of eigenfunction expansions on N-dimensional manifolds. This generalizes some of the results in this paper.

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