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Exceptional Sets and Wavelet Packets Orthonormal Bases

Sandra Saliani

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ABSTRACT. We give a partial positive answer to a problem posed by Coifman et al. in [1]. Indeed, starting from the transfer function m_0 arising from the Meyer wavelet and assuming $m_0 = 1$ only on $[-\pi/3, \pi/3]$, we provide an example of pairwise disjoint dyadic intervals of the form

$$I(n,q) = \left[2^q n, 2^q(n+1)\right), \quad (n,q) \in E \subset \mathbf{N} \times \mathbf{Z},$$

which cover $[0, +\infty)$ except for a set A of Hausdorff dimension equal to 1/2, and such that the corresponding wavelet packets

$$2^{q/2} w_n \left(2^q x - k\right), \quad k \in \mathbf{Z}, \ (n,q) \in E \subset \mathbf{N} \times \mathbf{Z}$$

form an orthonormal basis of $L^2(\mathbf{R})$.

1. Introduction

Wavelet packets provide a large class of orthonormal bases of $L^2(\mathbf{R})$, each one corresponding to a different splitting of $L^2(\mathbf{R})$ into a direct sum of its closed subspaces.

The definition of wavelet packets is due to the work of Coifman et al. [1]. Starting with a pair of QMFs with transfer functions $m_0(\theta)$ and $m_1(\theta) = e^{i\theta}\overline{m_0(\theta+\pi)}$ associated to a multiresolution analysis (MRA) with wavelet ψ and scaling function ϕ, one defines first the basic wavelet packets, defined recursively by the formulas (for the Fourier transform):

$$\begin{aligned} \hat{w}_0(\theta) &= \hat{\phi}(\theta), \quad \hat{w}_1(\theta) = \hat{\psi}(\theta), \\ \hat{w}_{2n}(\theta) &= m_0\left(\frac{\theta}{2}\right)\hat{w}_n\left(\frac{\theta}{2}\right), \\ \hat{w}_{2n+1}(\theta) &= m_1\left(\frac{\theta}{2}\right)\hat{w}_n\left(\frac{\theta}{2}\right). \end{aligned}$$

Then the general wavelet packets are given by taking some of the dilation and translation of the basic ones, i.e.,

$$2^{q/2} w_n\left(2^q x - k\right), \qquad k \in \mathbf{Z}, \quad (n,q) \in E \subset \mathbf{N} \times \mathbf{Z}. \tag{1.1}$$

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In the above-mentioned paper, the authors prove that, under the following conditions on m_0,

1) $m_0 \in C^\infty([-\pi, \pi])$ is even and 2π periodic,
2) $m_0(\theta) = 1$ for $\theta \in [-\frac{\pi}{3}, \frac{\pi}{3}]$,
3) $0 \le m_0(\theta) \le 1$,
4) $m_0^2(\theta) + m_0^2(\theta + \pi) = 1$,
5) m_0 is decreasing on $[0, \pi]$,

(1.1) is an orthonormal basis of $L^2(\mathbf{R})$ provided the set E satisfies the following assumption: the dyadic intervals

$$I(n, q) = \left[2^q n, 2^q(n+1)\right), \qquad (n, q) \in E, \tag{1.2}$$

form a disjoint covering of $[0, +\infty)$ except for a denumerable set A (A is called, here and in the sequel, the "exceptional" set).

Each choice of E corresponds to a different splitting of $L^2(\mathbf{R})$ and so to a different orthonormal basis: $E = \{1\} \times \mathbf{Z}$ leads to the wavelet basis, $E = \mathbf{N} \times \{0\}$ to the basis $w_n(x-k)$, $k \in \mathbf{Z}$, $n \in \mathbf{N}$. In the first case $A = \{0\}$, in the second case A is the empty set.

However, there are choices of E where the intervals $I(n, q)$ form a disjoint covering of $[0, +\infty)$ and the exceptional set A is not denumerable: think of A as a Cantor-like set. It is shown (e.g., [1]) that, with additional hypotheses on m_0, we obtain wavelet packets orthonormal bases corresponding to some of these particular choices.

Therefore, Coifman et al. have posed the question if the above result could be generalized to exceptional sets A with zero Lebesgue measure.

In [2] we gave a positive answer to this question in the case the Hausdorff dimension of A is strictly less then $1/2$ with no additional hypothesis on m_0.

In this note we go a little step further, by showing that, without any other hypothesis on m_0, we can find a wavelet packet orthonormal basis where the Hausdorff dimension of the exceptional set A is exactly $1/2$.

As always [1], statements about wavelet packets derive from statements about a general Hilbert space that can be decomposed as a direct sum of an infinite number of closed subspaces usually denoted by H_I, I being a dyadic subinterval of $[0, 1)$. The "splitting rule" depends on the MRA. (How this works will be explained in Section 2.)

Therefore, it will be sufficient to prove the following:

Theorem 1.

Let H be a Hilbert space. Then there is a collection $(I_n)_{n\in\mathbf{N}}$ of dyadic subintervals of $[0, 1)$ which forms a disjoint covering of $[0, 1)$, except for a set A of Hausdorff dimension equal to $1/2$, and such that

$$H = \overline{\bigoplus_{n\in\mathbf{N}}} H_{I_n}$$

where the sum is orthogonal.

To prove Theorem 1, we shall use the measure σ on $[0, 1)$ introduced by Séré in [3]; the definition of σ will be given in Section 2 and in Section 3 we prove a fundamental estimate for it. As for now, we recall that σ is a continuous measure verifying the following:

Theorem 2.

Let H be a Hilbert space. Let $(I_n)_{n\in\mathbf{N}}$ be a pairwise disjoint dyadic subintervals of $[0, 1)$, then the following are equivalent:

a) $H = \overline{\bigoplus}_{n\in\mathbf{N}} H_{I_n}$, *and the sum is orthogonal,*
b) $\sigma\left([0, 1)/\bigcup_{n\in\mathbf{N}} I_n\right) = 0$.

From Theorem 1 and Theorem 2 it follows that is then sufficient to prove the following:

Theorem 3.

Let H be a Hilbert space. Then we can find a collection $(I_n)_{n\in\mathbf{N}}$ of dyadic subintervals of $[0, 1)$ which forms a disjoint covering of $[0, 1)$, except for a set A of Hausdorff dimension equal to $1/2$, and such that

$$\sigma(A) = \sigma\left([0, 1)\setminus \bigcup_{n\in\mathbf{N}} I_n\right) = 0 .$$

Theorem 3 will be proved in Section 4.

2. Notation and Assumptions

We assume that the reader is familiar with the contents of [1]. To fix notation, recall that for

$$m_0(\theta) = \frac{1}{\sqrt{2}}\sum_k u_k\, e^{ik\theta}, \quad m_1(\theta) = \frac{1}{\sqrt{2}}\sum_k v_k\, e^{ik\theta}, \tag{2.1}$$

coming from an MRA, the matrix

$$\begin{pmatrix} m_0(\theta) & m_1(\theta) \\ m_0(\theta+\pi) & m_1(\theta+\pi) \end{pmatrix}$$

is unitary. It is well known [1] that, in this case, any Hilbert space H, equipped with an orthonormal basis $(e_k)_{k\in\mathbf{Z}}$, $e_k \in H$, can be decomposed into a direct sum

$$H = H_0 \oplus H_1$$

where the orthogonal closed subspaces H_0 and H_1 have the following elements as orthonormal basis, respectively,

$$f_{2k} = \sum_h u_{2k-h} e_h, \qquad k \in \mathbf{Z}, \tag{2.2}$$

and

$$f_{2k+1} = \sum_h v_{2k-h} e_h, \qquad k \in \mathbf{Z}. \tag{2.3}$$

The same recipe, applied each time to any subspace, gives a further decomposition of the given Hilbert space as finite orthogonal sum of closed subspaces.

For the sake of notation, we shall associate a dyadic interval $I \subset [0, 1)$ to any subspace so obtained. The choice of I reflects the occurrence of either one of the two sequences, corresponding to an orthonormal basis obtained either by (2.2) or by (2.3). Namely, we set

$$\begin{aligned} H_{[0,1)} &= H, \\ H_{[0,\frac{1}{2})} &= H_0, \ H_{[\frac{1}{2},1)} = H_1, \\ H_I &= H_{(\varepsilon_1,\dots,\varepsilon_j)}, \quad \varepsilon_i = 0, 1, \end{aligned}$$

where

$$I = \left[\frac{\varepsilon_1}{2} + \frac{\varepsilon_2}{4} + \dots + \frac{\varepsilon_j}{2^j}, \frac{\varepsilon_1}{2} + \frac{\varepsilon_2}{4} + \dots + \frac{\varepsilon_j}{2^j} + \frac{1}{2^j}\right).$$

We shall denote by $\pi_I : H \to H_I$ the orthogonal projection onto H_I.

We shall assume, as in [1], that the following properties hold for m_0 and m_1:

1) $m_0 \in C^\infty([-\pi,\pi])$ is even and 2π periodic,
2) $m_0(\theta) = 1$ for $\theta \in [-\frac{\pi}{3}, \frac{\pi}{3}]$,
3) $0 \le m_0(\theta) \le 1$,
4) $m_0^2(\theta) + m_0^2(\theta+\pi) = 1$,
5) m_0 is decreasing on $[0,\pi]$.

Then it follows that, given any unit vector $x \in H$, the set function $\mu_x(I) = \|\pi_I(x)\|^2$, defined for any dyadic set $I \subset [0,1)$, extends to a continuous measure on Borel sets of $[0,1)$. The same is true for (see [3]):

$$\sigma = \frac{1}{3}\sum_{k\in\mathbf{Z}} \frac{\mu_{e_k}}{2^{|k|}} .$$

We shall identify the Hilbert space H with the space $L^2[0,2\pi]$ of square summable 2π-periodic functions equipped with the Lebesgue measure. With this identification, each basis element e_k of H corresponds to the function $e^{ik\theta} \in L^2[0,2\pi]$ and each basis element $e_{k,I}$ of H_I corresponds to the function $2^{j/2}e^{i2^jk\theta}m_{\varepsilon_1}(\theta)m_{\varepsilon_2}(2\theta)\dots m_{\varepsilon_j}(2^{j-1}\theta)$.

Hence, for any $k \in \mathbf{Z}$, we have

$$\mu_{e_k}(I) = \|\pi_I(e_k)\|^2 = 2^j \sum_{p\in\mathbf{Z}} \left| \int_0^{2\pi} m_{\varepsilon_1}(\theta)\dots m_{\varepsilon_j}(2^{j-1}\theta)\, e^{2^j p\theta i}\, e^{-k\theta i}\, \frac{d\theta}{2\pi} \right|^2 . \tag{2.4}$$

We shall study in detail the support of

$$g_n(\theta) = m_{\varepsilon_1}(\theta)m_{\varepsilon_2}(2\theta)\dots m_{\varepsilon_j}(2^{j-1}\theta) ,$$

(where $n = \varepsilon_1 + 2\varepsilon_2 + \dots + 2^{j-1}\varepsilon_j$), i.e., the closure of the set of $\theta \in [0,2\pi]$ where $g_n(\theta) \neq 0$ (denoted by $\operatorname{supp} g_n$). By construction, it consists of a finite union of closed intervals. Note that if we define $M_\varepsilon(\theta) = | m_\varepsilon(\theta) |$, for $\varepsilon = 0, 1$, and

$$W_n(\theta) = M_{\varepsilon_1}(\theta)M_{\varepsilon_2}(2\theta)\dots M_{\varepsilon_j}\left(2^{j-1}\theta\right) ,$$

then $M_0(\theta) = m_0(\theta)$, $M_1(\theta) = m_0(\theta+\pi)$, and for all $n \in \mathbf{N}$, $\operatorname{supp} W_n = \operatorname{supp} g_n$. Also $\|W_n\|_2^2 = \|g_n\|_2^2 = \frac{1}{2^j}$.

For a Lebesgue measurable set I, we shall denote by $| I |$ the Lebesgue measure of I.

3. An Estimate for σ

We start this section by noting that, for any f in $L^2[0,2\pi]$, $f(\theta) = \sum_{-\infty}^{\infty} c_k\, e^{ik\theta}$, which is zero outside a fixed interval $I \subset [0,2\pi]$, with $|I| = 2\pi/m$, $m \ge 2$, we have, for any $\theta \in I$

$$f(\theta) = f(\theta) + f\left(\theta + \frac{2\pi}{m}\right) + \dots + f\left(\theta + \frac{2(m-1)\pi}{m}\right) = m \sum_{-\infty}^{\infty} c_{mk}\, e^{imk\theta} ,$$

hence $m\sum_{-\infty}^{\infty} | c_{mk} |^2 = \sum_{-\infty}^{\infty} | c_k |^2 = \|f\|_2^2$.

Note that this reasoning is the starting point for subband coding schemes.

In the following lemma we generalize the above equality in the case the support of f consists of more intervals, as in the case of W_n. We take $m = 2^j$, $j \ge 1$, since this will be useful in the sequel.

Lemma 1.

Let f be in $L^2[0, 2\pi]$. Let us assume that there exists $j \geq 1$ such that $\sup f$ is contained in a finite union of pairwise disjoint subintervals of $[0, 2\pi]$, say I_k, $k \in \mathcal{I}_j$, and $|I_k| \leq \frac{\pi}{2^{j-1}}$.
Let us denote by α_j the minimum number of these intervals.
Let us denote by c_p, $p \in \mathbf{Z}$, the pth Fourier coefficient of f, i.e.,

$$c_p = \int_0^{2\pi} f(\theta) e^{-pi\theta} \frac{d\theta}{2\pi} .$$

Then:

$$2^j \sum_{k \in \mathbf{Z}} \left| c_{2^j k} \right|^2 \leq 2\alpha_j \|f\|_2^2 . \tag{3.1}$$

Proof. For any $m \geq 2$ and $\theta \in [0, 2\pi]$:

$$\begin{aligned} & f(\theta) + f\left(\theta + \frac{2\pi}{m}\right) + \ldots + f\left(\theta + \frac{2\pi(m-1)}{m}\right) \\ &= \sum_{k \in \mathbf{Z}} c_k e^{ik\theta} + \sum_{k \in \mathbf{Z}} c_k e^{ik(\theta + \frac{2\pi}{m})} + \ldots + \sum_{k \in \mathbf{Z}} c_k e^{ik(\theta + \frac{2\pi(m-1)}{m})} \\ &= \sum_{k \in \mathbf{Z}} c_k e^{ik\theta} \left(\sum_{p=0}^{m-1} e^{ip\frac{2\pi}{m}k} \right) = m \sum_{k \in \mathbf{Z}} c_{km} e^{ikm\theta} . \end{aligned} \tag{3.2}$$

Now, by hypotheses, $\operatorname{supp} f \subset \bigcup_{k \in \mathcal{I}_j} I_k$, $|I_k| \leq \frac{\pi}{2^{j-1}}$ and $|\mathcal{I}_j| = \alpha_j$. Let $J = [0, \frac{2\pi}{2^j})$, and consider

$$\mathcal{B} = \left\{ h \in \mathbf{N} : 0 \leq h \leq 2^j - 1, \left(J + \frac{2\pi h}{2^j} \right) \cap \bigcup_{k \in \mathcal{I}_j} I_k \neq \emptyset \right\} .$$

For $h \in \mathbf{N}$, $h \leq 2^j - 1$, we have that $h \notin \mathcal{B}$ implies $f(\theta + \frac{2\pi h}{2^j}) = 0$ for all $\theta \in J$. Indeed, if for some $\theta \in J$, $f(\theta + \frac{2\pi h}{2^j}) \neq 0$, then $\theta + \frac{2\pi h}{2^j} \in \operatorname{supp} f$ and so $h \in \mathcal{B}$.

We show now that $|\mathcal{B}| \leq 2\alpha_j$. Indeed, the intervals $J + \frac{2\pi h}{2^j}$ form a disjoint covering of $[0, 2\pi]$. Since each interval has exactly the measure of J, at most two of them can intersect any I_p for a fixed p; being that the intervals I_p are disjoint, we have that $|\mathcal{B}|$ is at most $2|\mathcal{I}_j| = 2\alpha_j$.

Now, let us compute:

$$\begin{aligned} \int_J \left| 2^j \sum_h c_{h2^j} e^{ih2^j\theta} \right|^2 \frac{d\theta}{2\pi} &= 2^j \int_0^{2\pi} \left| \sum_h c_{h2^j} e^{ih\theta} \right|^2 \frac{d\theta}{2\pi} \\ &= 2^j \sum_h \left| c_{h2^j} \right|^2 . \end{aligned}$$

From (3.2) and the discussion above:

$$\begin{aligned} 2^j \sum_h \left| c_{h2^j} \right|^2 &= \int_J \left| \sum_{k=0}^{2^j-1} f\left(\theta + \frac{2\pi k}{2^j}\right) \right|^2 \frac{d\theta}{2\pi} = \int_J \left| \sum_{k \in \mathcal{B}} f\left(\theta + \frac{2\pi k}{2^j}\right) \right|^2 \frac{d\theta}{2\pi} \\ &\leq \int_J |\mathcal{B}| \sum_{k \in \mathcal{B}} \left| f\left(\theta + \frac{2\pi k}{2^j}\right) \right|^2 \frac{d\theta}{2\pi} = |\mathcal{B}| \, \|f\|_2^2 . \end{aligned}$$

Remark 1. When the support of f consists exactly of one interval I, we can take J next to I and so $|\mathcal{B}| = \alpha_j = 1$.

We can now apply Lemma 1 to any

$$g_n(\theta)e^{-ik\theta} = m_{\varepsilon_1}(\theta)m_{\varepsilon_2}(2\theta)\dots m_{\varepsilon_j}\left(2^{j-1}\theta\right)e^{-ik\theta}\ . \tag{3.3}$$

(where $n = \varepsilon_1 + 2\varepsilon_2 + \dots + 2^{j-1}\varepsilon_j$ and $k \in \mathbf{Z}$). We get the following result:

Corollary 1.
For

$$G_j = \left[\frac{\varepsilon_1}{2} + \frac{\varepsilon_2}{4} + \dots + \frac{\varepsilon_j}{2^j}, \frac{\varepsilon_1}{2} + \frac{\varepsilon_2}{4} + \dots + \frac{\varepsilon_j}{2^j} + \frac{1}{2^j}\right),$$

we have

$$\sigma\left(G_j\right) \le \frac{\alpha_j^n}{2^{j-1}},$$

where, with the same notation as in Lemma 1, α_j^n corresponds to $W_n(\theta) = M_{\varepsilon_1}(\theta)M_{\varepsilon_2}(2\theta)\dots M_{\varepsilon_j}(2^{j-1}\theta)$.

Proof. Let us recall that

$$\sigma\left(G_j\right) = \frac{1}{3}\sum_{k\in\mathbf{Z}}\frac{\mu_{e_k}\left(G_j\right)}{2^{|k|}} = \frac{1}{3}\sum_{k\in\mathbf{Z}}\frac{\left\|\pi_{G_j}(e_k)\right\|_2^2}{2^{|k|}}\ .$$

On the other hand, from (2.4),

$$\|\pi_{G_j}(e_k)\|_2^2 = 2^j\sum_{p\in\mathbf{Z}}\left|\int_0^{2\pi} g_n(\theta)e^{-ik\theta}e^{i2^jp\theta}\frac{d\theta}{2\pi}\right|^2 = 2^j\sum_{p\in\mathbf{Z}}\left|c_{2^jp}^k\right|^2$$

where c_p^k denotes the pth Fourier coefficient of the function $g_n(\theta)\,e^{-ik\theta}$.

From the discussion at the end of Section 2 and Lemma 1 we get for all $k \in \mathbf{Z}$:

$$\|\pi_{G_j}(e_k)\|_2^2 \le \frac{\alpha_j^n}{2^{j-1}}$$

and everything is proved. □

4. Proof of Theorem 3

For any j even we consider the set

$$\mathcal{P}_j = \left\{n = \sum_{h=1}^{j}\varepsilon_h 2^{h-1} : \varepsilon_{2i} = 0, i = 1, \dots, \frac{j}{2}\right\}.$$

Also for any $n \ge 0$, $n = \sum_{h=1}^{j}\varepsilon_h 2^{h-1}$ we denote the associate interval by G_j^n. In this way, corresponding to any even $j \in N$, we get a decomposition of $[0, 1)$ in terms of dyadic intervals of length exactly $1/2^j$, namely:

$$[0, 1) = \bigcup_{n\in\mathcal{P}_j} G_j^n \cup \bigcup_{n\notin\mathcal{P}_j} G_j^n\ .$$

Theorem 3 will be proved once we show the following:

1. $\sum_{n\in\mathcal{P}_j}\alpha_j^n = 4\cdot 3^{j/2-1}$ (α_j^n is defined as in Corollary 1),
2. $\lim_j \mid \bigcup_{n\in\mathcal{P}_{2j}} G_{2j}^n \mid = 0$,
3. $\lim_j \sigma(\bigcup_{n\in\mathcal{P}_{2j}} G_{2j}^n) = 0$,
4. $[0,1) = \bigcup_{j\in\mathbf{N}}\bigcup_{n\notin\mathcal{P}_{2j}} G_{2j}^n \cup A$, where the set

$$A = \bigcap_{j\in\mathbf{N}}\bigcup_{n\in\mathcal{P}_{2j}} G_{2j}^n ,$$

has Lebesgue measure zero, Hausdorff dimension equal to $1/2$ and $\sigma(A) = 0$.

Proof of 1. The proof will be done in several steps.

First step.

We note that for p even:

$$\begin{aligned}
&\operatorname{supp}\left(M_1\left(2^{p-2}\theta\right)M_0\left(2^{p-1}\theta\right) + M_0\left(2^{p-2}\theta\right)M_0\left(2^{p-1}\theta\right)\right)\\
&= \operatorname{supp}\left(M_1\left(2^{p-2}\theta\right)M_0\left(2^{p-1}\theta\right)\right) \cup \operatorname{supp}\left(M_0\left(2^{p-2}\theta\right)M_0\left(2^{p-1}\theta\right)\right)\\
&= \operatorname{supp}\left(M_0\left(2^{p-1}\theta\right)\right)
\end{aligned} \tag{4.1}$$

Indeed the first equality in (4.1) holds since the functions are positive. For the second one we have (cfg. [4]):

$$\begin{aligned}
M_0\left(2^{p-2}\theta\right)M_0\left(2^{p-1}\theta\right) &= M_0\left(2^{p-2}\theta\right)M_0\left(2\left(2^{p-2}\theta\right)\right)\\
&= \begin{cases} M_0\left(2\left(2^{p-2}\theta\right)\right) & \text{for } 2^{p-2}\theta \in \left[-\frac{\pi}{3},\frac{\pi}{3}\right] + 2\pi\mathbf{Z}\\ 0 & \text{otherwise}\end{cases}\\
&= \begin{cases} M_0\left(2^{p-1}\theta\right) & \text{for } \theta \in \left[-\frac{\pi}{3\cdot 2^{p-2}},\frac{\pi}{3\cdot 2^{p-2}}\right] + \frac{2\pi\mathbf{Z}}{2^{p-2}}\\ 0 & \text{otherwise}\end{cases}
\end{aligned} \tag{4.2}$$

Also

$$\begin{aligned}
M_1\left(2^{p-2}\theta\right)M_0\left(2^{p-1}\theta\right) &= M_1\left(2^{p-2}\theta\right)M_0\left(2\left(2^{p-2}\theta\right)\right)\\
&= \begin{cases} M_0\left(2\left(2^{p-2}\theta\right)\right) & \text{for } 2^{p-2}\theta \in \left[\frac{2\pi}{3},\frac{4\pi}{3}\right] + 2\pi\mathbf{Z}\\ 0 & \text{otherwise}\end{cases}\\
&= \begin{cases} M_0\left(2^{p-1}\theta\right) & \text{for } \theta \in \left[\frac{2\pi}{3\cdot 2^{p-2}},\frac{4\pi}{3\cdot 2^{p-2}}\right] + \frac{2\pi\mathbf{Z}}{2^{p-2}}\\ 0 & \text{otherwise}\end{cases}
\end{aligned} \tag{4.3}$$

Now it is easy to verify that the intervals in (4.2) and (4.3) are disjoint. Indeed they all have length equal to $\frac{\pi}{3\cdot 2^{p-3}}$ and the distance between two next intervals is $\frac{\pi}{6\cdot 2^{p-3}}$ so that a step of $\frac{\pi}{2^{p-3}}$ brings us out.
Also

$$\operatorname{supp} M_0\left(2^{p-1}\theta\right) \subset \left[-\frac{\pi}{3\cdot 2^{p-2}},\frac{\pi}{3\cdot 2^{p-2}}\right] + \frac{2\pi\mathbf{Z}}{2^{p-1}},$$

i.e., the union of all intervals in (4.2) and (4.3) and so the second equality in (4.1) holds.

Second step.

By repeated use of (4.1) we get:

$$\begin{aligned}
&\operatorname{supp}\left(\sum_{\varepsilon_i=0,1} M_{\varepsilon_1}(\theta)M_0(2\theta)M_{\varepsilon_3}\left(2^2\theta\right)\dots M_{\varepsilon_{j-1}}\left(2^{j-2}\theta\right)M_0\left(2^{j-1}\theta\right)\right)\\
&= \operatorname{supp}\left((M_1(\theta)M_0(2\theta)+M_0(\theta)M_0(2\theta))\left(\sum_{\varepsilon_i=0,1} M_{\varepsilon_3}\left(2^2\theta\right)\dots M_0\left(2^{j-1}\theta\right)\right)\right)\\
&= \operatorname{supp}(M_1(\theta)M_0(2\theta)+M_0(\theta)M_0(2\theta))\\
&\quad\cap \operatorname{supp}\left(\sum_{\varepsilon_i=0,1} M_{\varepsilon_3}(2^2\theta)\dots M_0\left(2^{j-1}\theta\right)\right)\\
&= \dots\\
&= \operatorname{supp}(M_1(\theta)M_0(2\theta)+M_0(\theta)M_0(2\theta))\cap\\
&\quad \dots\cap \operatorname{supp}\left(M_1\left(2^{j-2}\theta\right)M_0\left(2^{j-1}\theta\right)+M_0\left(2^{j-2}\theta\right)M_0\left(2^{j-1}\theta\right)\right)\\
&= \operatorname{supp}(M_0(2\theta))\cap \operatorname{supp}\left(M_0\left(2^3\theta\right)\right)\cap\dots\cap \operatorname{supp}\left(M_0\left(2^{j-1}\theta\right)\right)\\
&= \operatorname{supp}\left(M_0(2\theta)M_0\left(2^3\theta\right)\dots M_0\left(2^{j-1}\theta\right)\right)
\end{aligned}$$

Third step.

We claim that, for $j \geq 4$ even, $\operatorname{supp}(M_0(2\theta)M_0(2^3\theta)\dots M_0(2^{j-1}\theta))$ is contained in $2\cdot 3^{j/2-1}-1$ disjoint intervals of length $\frac{4\pi}{3\cdot 2^{j-1}}$ plus two disjoint intervals of length $\frac{2\pi}{3\cdot 2^{j-1}}$.

Indeed we already know that the support of $M_0(2\theta)$ is contained in one interval of length $\frac{2\pi}{3}$, namely $[-\frac{\pi}{3},\frac{\pi}{3}]$, plus two intervals of length $\frac{\pi}{3}$, namely $[-\pi,-\frac{2\pi}{3}]$ and $[\frac{2\pi}{3},\pi]$.

Passing from $M_0(2\theta)$ to $M_0(2\theta)M_0(2^3\theta)$ it is not hard to see, by direct calculations, that only three intervals of the support of $M_0(2^3\theta)$ fall in the "big" interval of $M_0(2\theta)$, and only one and a half in the "small" ones. Hence, we get $3\cdot 1+1\cdot 2$ disjoint intervals of length $\frac{4\pi}{3\cdot 2^3}$ plus two disjoint intervals of length $\frac{2\pi}{3\cdot 2^3}$.

Hence, by a recurrence argument, $\operatorname{supp}(M_0(2\theta)M_0(2^3\theta)\dots M_0(2^{j-1}\theta))$ is contained in $3(2\cdot 3^{(j-2)/2-1}-1)+2 = 2\cdot 3^{j/2-1}-1$ disjoint intervals of length $\frac{4\pi}{3\cdot 2^{j-1}}$ plus two disjoint intervals of length $\frac{2\pi}{3\cdot 2^{j-1}}$.

Fourth step.

Let us recall that α_j^n, $n=\varepsilon_1+2\varepsilon_2+\dots+2^{j-1}\varepsilon_j$, is defined as the minimum number of intervals in the support of $W_n(\theta)=M_{\varepsilon_1}(\theta)M_{\varepsilon_2}(2\theta)\dots M_{\varepsilon_j}(2^{j-1}\theta)$ of length less then $\frac{\pi}{2^{j-1}}$. Also, from (4.2) and (4.3) we deduce that for $n, m\in\mathcal{P}_j$, $n\neq m$ the support of W_n and W_m are disjoint. Hence, $\sum_{n\in\mathcal{P}_j}\alpha_j^n$ is the minimum number of intervals in the support of

$$\sum_{\varepsilon_i=0,1} M_{\varepsilon_1}(\theta)M_0(2\theta)M_{\varepsilon_3}(2^2\theta)\dots M_{\varepsilon_{j-1}}(2^{j-2}\theta)M_0\left(2^{j-1}\theta\right)$$

of length less then $\frac{\pi}{2^{j-1}}$. By the second and third step we get that this number is exactly $2(2\cdot 3^{j/2-1}-1)+2 = 4\cdot 3^{j/2-1}$ and 1 is proved. □

Proof of 2. The set $\mathcal{P}_{2j}$ contains 2^j elements; hence,

$$\left|\bigcup_{n\in\mathcal{P}_{2j}} G_{2j}^n\right| = \frac{|\mathcal{P}_{2j}|}{2^{2j}} = \frac{1}{2^j}$$

and so 2 is proved. □

Proof of 3. By Corollary 1 and 1:

$$\sigma\left(\bigcup_{n\in\mathcal{P}_{2j}} G_{2j}^n\right) \le \sum_{n\in\mathcal{P}_{2j}} \alpha_{2j}^n \frac{1}{2^{2j-1}} = \frac{4\cdot 3^{j-1}}{2^{2j-1}} = \frac{8}{3}\left[\frac{3}{4}\right]^j$$

and so 3 is proved. □

Proof of 4. For any $j \ge 1$,

$$[0,1)\setminus \bigcup_{n\notin\mathcal{P}_{2j}} G_{2j}^n = \bigcup_{n\in\mathcal{P}_{2j}} G_{2j}^n\,;$$

hence,

$$\begin{aligned}
[0,1) &= \left[\bigcup_{j\in\mathbf{N}}\bigcup_{n\notin\mathcal{P}_{2j}} G_{2j}^n\right] \cup [0,1)\setminus\left[\bigcup_{j\in\mathbf{N}}\bigcup_{n\notin\mathcal{P}_{2j}} G_{2j}^n\right]\\
&= \left[\bigcup_{j\in\mathbf{N}}\bigcup_{n\notin\mathcal{P}_{2j}} G_{2j}^n\right] \cup \left[\bigcap_{j\in\mathbf{N}}\bigcup_{n\in\mathcal{P}_{2j}} G_{2j}^n\right]\\
&= \left[\bigcup_{j\in\mathbf{N}}\bigcup_{n\notin\mathcal{P}_{2j}} G_{2j}^n\right] \cup A\,.
\end{aligned}$$

Now, for any $j \ge 1$, we have $A \subset \bigcup_{n\in\mathcal{P}_{2j}} G_{2j}^n$, and so, by 2 we get $|A| = 0$, while by 3 we get $\sigma(A) = 0$. Also, we deduce that, for any $j \ge 1$, A can be covered by 2^j disjoint intervals of Lebesgue measure equal to $1/2^{2j}$; hence, A has Hausdorff dimension equal to $1/2$ and everything is proved. □

Remark 2. Unfortunately there are examples of other exceptional sets of Hausdorff dimension equal to $1/2$ for which Lemma 1 does not give the desired estimate. One such example is obtained if we naturally start from the set (j even),

$$\mathcal{S}_j = \left\{ n = \sum_{h=1}^{j} \varepsilon_h 2^{h-1} : \varepsilon_{2i} = 1, i = 1,\ldots,\frac{j}{2}\right\}.$$

In this case one can easily prove, as done above, that $\sum_{n\in\mathcal{S}_j} \alpha_j^n = C5^{j/2}$, where C is an absolute constant independent of j. Therefore, we obtain the estimate

$$\sigma\left(\bigcup_{n\in\mathcal{S}_j} G_j^n\right) \le \sum_{n\in\mathcal{S}_j} \alpha_j^n \frac{1}{2^{j-1}} = 2C\left[\frac{5}{4}\right]^{j/2}$$

and we cannot conclude that $\sigma(A) = 0$.

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Dipartimento di Matematica, Università degli Studi della Basilicata, 85100 Potenza, Italia
e-mail: saliani@unibas.it