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Disjointness preserving operators on C^*-algebras

By

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1. Introduction. In [1] the author characterized the so-called Lamperti operators on the Banach lattice $C(X)$ of all continuous functions on the compact Hausdorff space X as a main tool in his theory of Lamperti operators on general Banach lattices. More precisely he proved the following result:

1.1. Proposition. *Let T be a bounded linear operator on $C(X)$. Assume in addition that T has the disjointness preserving property*

(DP) $\inf(|f|, |g|) = 0 \Rightarrow \inf(|Tf|, |Tg|) = 0 \quad (f, g \in C(X))$.

Then there exists an element $h \in C(X)$ and a mapping $\varphi: X \to X$ being continuous on $\{x: h(x) \neq 0\} =: U$ such that $(Tf)(x) = h(x) f(\varphi(x))$ holds for all $f \in C(X)$, $x \in X$.

Thus an operator satisfying (DP) is a weighted composition operator, i.e. of the form $T = hS$ where S is a lattice homomorphism of $C(X)$ into the space $C_b(U)$ of all bounded continuous functions on U. In general S does not map $C(X)$ into $C(X)$, as the following simple example on $C([-1, 1])$ shows:

$$Tf(x) = \begin{cases} x f(\sin(2\pi/x)) : x \neq 0 \\ 0 \qquad\qquad\qquad : x = 0 \end{cases}$$

Now in $C(X)$ $\inf(|f|, |g|) = 0$ is equivalent to $fg = 0$. In fact in [3] the authors made already use of this fact in order to generalize the results to so-called disjointness preserving operators on $C(X)$-moduls.

Our aim is to prove an analogue of 1.1 within the framework of arbitrary C^*-algebras. As we will show we have to replace $C_b(U)$ above by the multiplyer algebra of the principal ideal generated by $T1 = h$ in the commutant of h. We apply our characterization to uniformly continuous semigroups $(T_t)_{t \geqq 0}$ of such operators.

Our main results are to be found in Section 2, their proofs in Section 3.

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2. The main results. Let us start with the basic definition.

2.1. Definition. Let $\mathscr{A}$, $\mathscr{B}$ denote two C^*-algebras. A bounded linear operator T from $\mathscr{A}$ to $\mathscr{B}$ is called *disjointness preserving* if the following two conditions are satisfied:

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(1) T is symmetric, i.e. $(T(x))^* = T(x^*)$ holds for all x in $\mathscr{A}$.
(2) $a, b \in \mathscr{A}_{sa}$ and $ab = 0$ always implies $T(a)\,T(b) = 0$.

Here $\mathscr{A}_{sa} = \{x \in \mathscr{A} : x^* = x\}$ is the selfadjoint part of $\mathscr{A}$. For notions concerning C^*-algebras and not mentioned here see e.g. [4, 7, 8].

2.2. Examples. (1) Every *-homomorphism and more generelly every Jordan *-homomorphism is disjointness preserving. (2) Let $h \in \mathscr{B}_{sa}$ be in the center of B and let $S : \mathscr{A} \to \mathscr{B}$ be a Jordan *-homomorphism. Then $T : a \to hS(a)$ is disjointness preserving.

Our main result says that every disjointness preserving operator looks almost like the second example. More precisely let us denote by M' the commutant of the subset M of a given C^*-algebra $\mathscr{C}$. In addition let us denote by $\mathscr{M}(\mathscr{C})$ the multiplier algebra of $\mathscr{C}$. Our main result then is the following:

2.3. Theorem. *Let $\mathscr{A}$, $\mathscr{B}$ denote two C^*-algebras and assume that $\mathscr{A}$ is unital. Let T be a disjointness preserving operator from $\mathscr{A}$ to $\mathscr{B}$ mapping $1_{\mathscr{A}}$ onto h. Then the following two assertions hold:*

(1) *$T(\mathscr{A})$ is contained in $\overline{h\{h\}'} =: \mathscr{C}$.*
(2) *There exists a Jordan *-homomorphism S from $\mathscr{A}$ into $\mathscr{M}(\mathscr{C})$ satisfying $S 1_{\mathscr{A}} = 1_{\mathscr{M}(\mathscr{C})}$ such that $Tf = hS(f)$ holds for all $f \in \mathscr{A}$.*

2.4. Corollary. *Assume in addition to the hypotheses of 2.3 that also $\mathscr{B}$ is unital and that moreover $T1_{\mathscr{A}} = h$ is invertible. Then T maps $\mathscr{A}$ into $\{h\}'$ and there exists a Jordan *-homomorphism S from $\mathscr{A}$ into $\{h\}'$ such that $Tf = hS(f)$ for all f in $\mathscr{A}$.*

Surprizingly the proof of our main result shows that the following uniquenes result for *-homomorphisms holds also:

2.5. Proposition. *Let U, V be unital *-homomorphisms from the unital C^*-algebra $\mathscr{A}$ into the unital C^*-algebra $\mathscr{B}$. If $ab = 0$ implies $(Ua)(Vb) = 0$ for all $a, b \in \mathscr{A}_{sa}$ then $U = V$.*

Coming back to our considerations on disjointness preserving operators let us now apply our results to the characterization of norm-continuous one-parameter semigroups of such operators. Let from now on $\mathscr{A}$ be a fixed unital C^*-algebra. Assume first that $\mathscr{A}$ is commutative, i.e. $\mathscr{A} \cong C(X)$ where X is compact and Hausdorff. Let $T = (T_t)_{t \geqq 0}$ be a norm-continuous semigroup of disjointness preserving operators on $\mathscr{A}$. Then it follows from Theorem 3.6 on p. 146 of [6] that $(T_t f)(x) = \exp(t\,h(x)) f(x)$ holds where $h = (T_t 1_X)^{\cdot}|_{t=0}$.

In the non-commutative case we get a much more interesting answer. In fact the result above will follow from our generalization.

2.6. Theorem. *Let $T = (T_t)_{t \geqq 0}$ be a uniformly continuous semigroup of disjointnes preserving operators on the unital C^*-algebra $\mathscr{A}$. Then there exists a uniquely determined element h in the center $Z(\mathscr{A})$, and also a uniquely determined uniformly continuous group $S = (S_t)_{t \in \mathbb{R}}$ of *-automorphisms on $\mathscr{A}$ such that $T_t(a) = \exp(th)\,S_t(a)$ holds for all $a \in \mathscr{A}$, $t \in \mathbb{R}_+$.*

2.7. Corollary. *Under the assumptions of the theorem $S_t|_{Z(\mathscr{A})} = I_{Z(\mathscr{A})}$ holds, or in other words $T_t a = \exp(th) \cdot a$ for all $a \in Z(\mathscr{A})$.*

3. The proofs. Before we can give the proofs we have to establish several lemmas.

3.1. Lemma. *Let X be a compact subset of the unit intervall $[0, 1]$ and let $T: C(X) \to \mathscr{A}$ be disjointness preserving. Let $a < b \leqq c < d \leqq 1$ be real numbers and set $Y =]a, b] \cap X$, $Z =]c, d] \cap X$. Then $T''(1_Y)\, T''(1_Z) = 0$.*

Here T'' denotes the bi-adjoint of T and the algebra of all bounded Borel functions on X is canonically identified with a subalgebra of the second dual $C(X)''$ of $C(X)$.

Proof. Set $$\tilde{e}_n(x) = \begin{cases} 0 & x \leqq a + 2^{-n-1}(b-a) \text{ or } x > b + 2^{-n} \\ 1 & a + 2^{-n}(b-a) \leqq x \leqq b \\ \text{linear and continuous elsewhere}. \end{cases}$$

In an analogous manner $\tilde{f}_n$ is defined on $[0, 1]$ for $]c, d]$ in place of $]a, b]$. Now set $e_n = \tilde{e}_n|_X$, $f_n = \tilde{f}_n|_X$. By Lebesgue's dominated convergence theorem $(e_n) \to 1_Y$, $(f_n) \to 1_Z$ with respect to the weak*-topology on $C(X)''$. Moreover, to every n there exists $p(n)$ satisfying $e_{n+p} f_n = 0$ for all $p \geqq p(n)$. This implies $0 = T(e_{n+p})\, T(f_n)$ by hypothesis. Hence $0 = T''(1_Y)\, T(f_n)$, thus $0 = T''(1_Y)\, T''(1_Z)$, since multiplication is separately continuous for $\sigma(C(X)'', C(X)')$ and T'' is $\sigma(C(X)'', C(X)') - \sigma(\mathscr{A}'', \mathscr{A}')$ continuous (here $\sigma(E, F)$ denotes the weak topology on E with respect to the dual pairing (E, F)).

By the same methods we show the following lemma which is the basic tool for the proof of 2.5.

3.2. Lemma. *Let X be as before, and let $U, V: C(X) \to \mathscr{A}$ be unital *-homomorphisms. If $ab = 0$ implies $(Ua)(Vb) = 0$ for all $a, b \in C(X)_{sa}$ then $U = V$.*

Proof. Similarly as above one shows $U''(1_Y)\, V''(1_Z) = V''(1_Y)\, U''(1_Z) = 0$ (we adhere to the notation of 3.1). This implies $U''(1_Y)\, V''(1_X - 1_Y) = 0$, hence

$$U''(1_Y) = U''(1_Y)\, V''(1_X) = U''(1_Y)\, V''(1_Y) = U''(1_X)\, V''(1_Y) = V''(1_Y). \tag{1}$$

Now for every $n \in \mathbb{N}$ consider the partition $(X_{k,n})_{0 \leqq k \leqq n-1}$ of X given by

$$X_{k,n} = \begin{cases} [0, 1/n] \cap X & \text{if } k = 0 \\ ]k/n, (k+1)/n] \cap X & \text{if } 1 \leqq k \leqq n-1. \end{cases}$$

For each k, n let $x_{k,n} \in X_{k,n}$ be arbitrary, and let $f \in C(X)$ be arbitrary. Then

$$f = \lim_{n \to \infty} \sum_{k=0}^{n} f(x_{k,n})\, 1_{X_{k,n}},$$

hence $Uf = Vf$ by (1).

Now we can prove our first characterization of disjointness preserving operators:

3.3. Lemma. *Let $\mathscr{A}$, $\mathscr{B}$ be unital C^*-algebras and let T be a disjointness preserving operator from $\mathscr{A}$ to $\mathscr{B}$ satisfying $T1_{\mathscr{A}} = 1_{\mathscr{B}}$. Then T is a Jordan *-homomorphism.*

Proof. Recall that the Jordan product on $\mathscr{A}_{sa}$ is defined by

$$a \circ b = (ab + ba)/2 = ((a+b)^2 - (a-b)^2)/4 .$$

Thus we have only to show $(Tf)^2 = T(f^2)$ for all f satisfying $0 \leqq f \leqq 1_{\mathscr{A}}$. For such an f the C^*-algebra generated by f and $1_{\mathscr{A}}$ is isomorphic to $C(X)$, where the spectrum $\sigma(f) = X$ of f is a compact subset of $[0, 1]$.

Using the partitions $(X_{k,n})$ of the proof of 3.2 we obtain

$$Tg = \lim_{n \to \infty} \sum_{k=0}^{n} g(x_{k,n})\, T''(1_{X_{k,n}})$$

for all $g \in C(X)$.

Lemma 1 implies $T''(1_{X_{k,n}})\, T''(1_{X_{l,n}}) = T''(1_{X_{k,n}})\, \delta_{kl}$ (δ_{kl}: Kronecker symbol). This gives $Tf^2 = (Tf)^2$ by an easy calculation where f is identified with the identity $x \to x$ on X by the construction of X.

3.4. Proof of 2.5. As in the previous proof we choose $f \in \mathscr{A}_{sa}$ between 0 and $1_{\mathscr{A}}$, then we identify the algebra generated by f and $1_{\mathscr{A}}$ with $C(X)$ and apply 3.2. The remainder is clear.

3.5. Proof of 2.3. (I) As in the previous proofs we choose an arbitrary $f \in \mathscr{A}_{sa}$ between 0 and $1_{\mathscr{A}}$, and we identify the C^*-algebra generated by f and $1_{\mathscr{A}}$ with $C(X)$, where $X = \sigma(f) \subset [0, 1]$.

Moreover we consider also the partitions $(X_{k,n})$ of X and arbitrary points $x_{k,n} \in X_{k,n}$ as in the proof of 3.2.

From

$$1 = \sum_{0}^{n} 1_{X_{k,n}}$$

we get

$$h := T 1_{\mathscr{A}} = \sum_{0}^{n} T'' 1_{X_{k,n}} ,$$

hence by 3.1

$$h \cdot T''(1_{X_{l,n}}) = \Sigma\, T''(1_{X_{k,n}}) \cdot T''(1_{X_{l,n}}) = (T''(1_{X_{l,n}}))^2$$

and similarly $T''(1_{X_{l,n}})\, h = (T''(1_{X_{l,n}}))^2$ for all l and n. This implies that h commutes with all $T''(1_{X_{l,n}})$, hence with Tf (see the proof of 3.3). Since f was arbitrary this gives $T(\mathscr{A}) \subset \{h\}'$.

(II) For $g = \Sigma\, \alpha_k 1_{X_{k,n}}$ we obtain by 3.1

$$(T''(g))^2 = \sum_{k} \alpha_k^2 (T'' 1_{X_{k,n}})^2 = h\, T''(g^2) .$$

Now using $f = \lim_{n} \sum_{k} f(x_{k,n})\, 1_{X_{k,n}}$ we get

$$(Tf)^2 = h\, T(f^2)$$

which in turn implies $T(\mathscr{A}) \subset \{h\}' \overline{h \{h\}'}$. So assertion (1) follows.

(III) Set $\mathscr{C} = \overline{h\{h\}'}$, and for each $n \geqq 1/\|h\|$ in $\mathbb{N}$ denote by p_n the central projection in $\mathscr{C}''$ (given by the spectral decomposition of h) corresponding to the subset $\{\xi \in \sigma(h) : |\xi| \geqq \frac{1}{n}\}$.

Then $h_n := p_n h$ is invertible in $p_n \mathscr{C}'' =: \mathscr{C}_n$ and T_n given by $T_n(x) = p_n T(x)$ is disjointness preserving satisfying $T_n 1_{\mathscr{A}} = h_n$. Thus by 3.3 S_n, defined by $S_n x = h_n^{-1} T_n x$, is a Jordan *-homomorphism from $\mathscr{A}$ into $\mathscr{C}_n$. Now P_n, given by $P_n y = p_n y$ defines a sequence of contractions from $\mathscr{C}$ into $\mathscr{C}''$, which converges strongly to the identity on $\mathscr{C}$. But then (T_n) converges strongly to T on $\mathscr{A}$.

(IV) Let $\mathscr{U}$ be a free ultrafilter on $\mathbb{N}$. Since bounded sets in $\mathscr{C}''$ are $\sigma(\mathscr{C}'', \mathscr{C}')$ – relatively compact, by $Sx = \sigma(\mathscr{C}'', \mathscr{C}') - \lim_{\mathscr{U}} (S_n x)$ there is defined a bounded linear operator S from $\mathscr{A}$ into $\mathscr{C}''$. S has the following properties:

(1) $S(\mathscr{A}_{sa}) \subset \mathscr{C}''_{sa}$.
For C''_{sa} is $\sigma(\mathscr{C}'', \mathscr{C}')$ – closed by [8], 1.7.1.

(2) $S(1_{\mathscr{A}}) = 1_{\mathscr{M}(\mathscr{C})}$.
For $\mathscr{C} = \overline{h\mathscr{C}}$ implies $\lim_n p_n x = x$ for all x in $\mathscr{C}$.

(3) $T(x) = hS(x)$ for all $x \in \mathscr{A}$.
For $hS_n(x) = T_n(x)$ by definition. So the assertion holds by step III.

(4) $S(\mathscr{A}) \subset \mathscr{M}(\mathscr{C})$.
For let $z \in \mathscr{C}$ arbitrary. Then $S(x)hz = T(x)z \in \mathscr{C}$, hence $S(x)h\mathscr{C} \subset \mathscr{C}$ and (4) follows by [7], 3.12.1.

(5) *S is disjointness preserving.*
For let $a, b \in \mathscr{A}_{sa}$ satisfy $ab = 0$. Then $h^2 S(a) S(b) = T(a) T(b) = 0$ by hypothesis. But since $\mathscr{C} = \overline{h\mathscr{C}}$ we obtain that $y \in \mathscr{M}(\mathscr{C}) \to h^2 y$ is injective.

(V) The theorem follows by IV, (1)–(5) above.

3.6. Proof of 2.6 and 2.7. Since by hypothesis $t \to T_t$ is continuous with respect to the operator norm there exists a bounded linear operator A on $\mathscr{A}$, such that $T_t = \exp(tA)$. So we may extend this mapping to all of $\mathbb{R}$.

Now by 2.3 for every $t \geqq 0$ there exists a Jordan *-homomorphism S_t such that $T_t = M_t S_t$, where M_t denotes multiplication by $h_t = T_t 1$. Since T_t is bijective so is M_t (i.e. h_t is invertible) as well as S_t. Since $t \to T_t$ and $t \to M_t$ are (obviously) continuously differentiable (with respect to the operator norm), so is $t \to S_t = M_t^{-1} T_t$. In particular each S_t lies in the connected component of the identity hence each S_t is even a *-automorphism (apply [2], 7.4.9 on p. 163 to the bi-adjoint of S_t). Finally $(S_t)_{t \geqq 0}$ is a semigroup. This follows easily from the following set of equations:

$$(1) \qquad h_{u+v} = T_{u+v}(1_{\mathscr{A}}) = T_u(T_v 1_{\mathscr{A}}) = h_u S_u(h_v)$$

$$(2) \qquad T_{u+v}(a) = h_{u+v} S_{u+v}(a) = h_u S_u(h_v) S_{u+v}(a)$$

$$(3) \qquad T_{u+v}(a) = T_u(T_v(a)) = h_u S_u(h_v S_v(a)) = h_u S_u(h_v) S_u S_v(a).$$

The generator B of (S_t) is a bounded derivation hence it vanishes on $Z(\mathscr{A})$ by [8], 4.1.2). Moreover $h_t \in Z(\mathscr{A})$ (see above) and thus $Z(\mathscr{A})$ is invariant under T. (1) above shows that (h_t) is a cocycle with respect to $S_t|_{Z(\mathscr{A})} = I_{Z(\mathscr{A})}$. Hence $h_t = \exp(th)$ where $h = (h_t)^{\cdot}|_{t=0} = A 1_{\mathscr{A}}$.

3.7. Final remark. The proof of Theorem 2.3 (cf. step III of 3.5) enables us in principal to characterize those disjointness preserving operators which are compact. For if $T1_{\mathscr{A}}$ is invertible then S has to be compact and since Jordan *-homomorphisms are open onto their range S has to be of finite rank. Unfortunately it is cumbersome to characterize such an operator in the non-commutative case contrary to the commutative case. So we are not able to generalize the results of Kamowitz [5] in any reasonable manner.

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