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HIGHLY LINKED GRAPHS

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A graph with at least $2k$ vertices is said to be *k-linked* if, for any choice $s_1,\dots,s_k$, $t_1,\dots,t_k$ of $2k$ distinct vertices there are vertex disjoint paths $P_1,\dots,P_k$ with P_i joining s_i to t_i, $1 \le i \le k$. Recently Robertson and Seymour [16] showed that a graph G is k-linked provided its vertex connectivity $\kappa(G)$ exceeds $10k\sqrt{\log_2 k}$. We show here that $\kappa(G) \ge 22k$ will do.

1. Introduction

According to Menger's theorem, if a graph is k-connected then, for any choice of $2k$ distinct vertices $s_1,\dots,s_k$, $t_1,\dots,t_k$ there are vertex disjoint paths $P_1,\dots,P_k$ joining $\{s_1,\dots,s_k\}$ to $\{t_1,\dots,t_k\}$. Menger's theorem gives no information as to which s_i is joined to which t_i; the graph is said to be *k-linked* if it has at least $2k$ vertices and for any choice of distinct $s_1,\dots,s_k,t_1,\dots,t_k$ we can always specify that P_i joins s_i to t_i, $1 \le i \le k$. Observe that any k-linked graph G has vertex connectivity $\kappa(G)$ at least $2k-1$, for if S is a set of $2k-2$ vertices separating a vertex s_1 from a vertex t_1, then with $\{s_2,\dots,s_k,t_2,\dots,t_k\}=S$ the paths P_i cannot be chosen.

We define the value of the function $f(k)$ to be the smallest value of the vertex connectivity $\kappa(G)$ of the graph G which ensures that G is k-linked. Of course, it is not immediately clear whether such a function does in fact exist. Indeed, Thomassen [20] showed that there is no such function for the analogous digraph problem. However, Larman and Mani [12] and Jung [7] noticed that if $\kappa(G) > 2k$ and if G contains a topological complete graph of order $3k$ then G is k-linked. Here a *topological complete graph* of order p, or TK_p, comprises p vertices $\{v_1,\dots,v_p\}$ and $\binom{p}{2}$ pairwise vertex disjoint paths $P_{i,j}$, $1 \le i < j,p$, such that $P_{i,j}$ joins v_i to v_j. Mader [13] was the first to prove that G contains a TK_{3k} if $\kappa(G)$ is sufficiently large, which implies the existence of $f(k)$. Later, in [15], he showed that if $e(G) \ge (3.2^{p-3}-p)|G|$ then G contains a TK_p, and therefore $f(k) \le 6\cdot 2^{3k-3}-6k$.

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The upper bound just cited for $f(k)$ is a far cry from the known lower bound. It is immediate that $f(1)=1$, and for the rest of the paper we will assume $k \geq 2$. Jung [7] characterized 2-linked graphs and established that $f(2)=6$. From this it follows that $f(k) \geq 2k+2$, since by joining two new vertices to every vertex of a graph which is not k-linked we obtain a graph which is not $(k+1)$-linked. Thomassen [19] extended Jung's characterization (the same extension is stated without proof in Seymour [17]) and conjectured that $f(k)=2k+2$, though later it was noticed that the graph K_{3k-1} with k independent edges removed shows that $f(k) \geq 3k-1$. Knowledge is much better of the analogous situation in which edge-disjoint rather than vertex-disjoint paths are required; Huck [6] proved an upper bound only one greater than the lower bound conjectured by Thomassen to be exact.

Recently the upper bound on $f(k)$ has been dramatically reduced. Komlós and Szemerédi [9] made a delicate investigation of the expansion properties of graphs and thereby proved that, for each $\eta > 14$, there exists a constant c_η such that if $e(G) > c_\eta p^2 (\log p)^\eta |G|$ then G contains a topological K_p; by the argument above this gives a much reduced bound for $f(k)$, namely $O(k^{2+\varepsilon})$. In a different vein, Robertson and Seymour [16] proved that there is a $O(|G|^3)$ algorithm for determining whether a graph G is k-linked (for fixed k; the problem becomes NP-complete if k be allowed as an input parameter to the problem, as was shown by Karp [8]). In the course of their proof they show, essentially, that the observation of Larman and Mani and of Jung remains true under the very much weaker condition that G have a complete minor of order $3k$. As usual, given a graph H with $V(H) = \{v_1, \ldots, v_h\}$, we say that H is a *minor* of G, denoted by $G \succ H$, if G contains disjoint subsets of vertices $V_1, \ldots, V_h$ such that $G[V_i]$ is connected for every i and G contains a $V_i - V_j$ edge whenever $v_i v_j \in E(H)$. Now Kostochka [11] and Thomason [18] determined, to within a constant factor, the number of edges in G needed to guarantee that $G \succ K_p$, and their result implies that $f(k) = O\left(k\sqrt{\log k}\right)$.

We prove here that $\kappa(G) \geq 22k$ is enough to ensure that G be k-linked. The proof is similar to that of Robertson and Seymour, except that we require only that G have a dense minor, rather than a complete one. The reason this modification is a useful one is that dense minors are considerably easier to come by; indeed, the number of edges needed is only linear in k, as shown by the next lemma, and thus the logarithmic factor in the previous estimates for $f(k)$ can be removed. In the lemma, the number β is a constant whose value slightly exceeds 0.37. As usual, $\delta(H)$ is the minimal degree of H.

Lemma 1. *Let $0 < \beta < 1$ be the root of the equation $1 = \beta(1 + \log(2/\beta))$ and let $k \geq 3$ be an integer. Let G be a graph of size $e(G) > k|G|$. Then $G \succ H$, where H is some graph satisfying $|H| \leq k+2$ and $2\delta(H) \geq |H| + \lfloor \beta k \rfloor - 1$.*

This lemma was proved originally by Thomason [18]. The statement of the lemma was slightly different in [18] to that given here. The lemma is restated, with proof, exactly as above by Bollobás and Thomason [2], in the course of a proof of a conjecture of Mader [13] and Erdős and Hajnal [5]. Their conjecture states that there is a constant c, such that every graph G with more than $cp^2 = |G|$ edges contains TK_p; in [2] it is shown that $256p^2|G|$ edges is enough. The conjecture has also been proved by Komlós and Szemerédi [10], following a refinement of their method mentioned above. Now it is easily verified that a $\binom{p+1}{2}$-linked graph contains a TK_p. Hence as a consequence of the present result about k-linked graphs (and a theorem of Mader [14] that a graph G with $e(G) \geq 2k|G|$ contains a k-connected subgraph) we obtain another proof of the conjecture, with a better constant, though the proof is less straightforward than that in [2].

2. Linking and dense minors

In this section we shall show that a graph is k-linked if it is well-connected and has a dense minor. We need first some terminology. Two subgraphs C and D in a graph G are said to meet if $V(C) \cap V(D) \neq \emptyset$. They are said to be adjacent if they do not meet but there is an edge between them, with one endvertex in C and the other in D. Given a subset $S \subseteq V(G)$, an *S-cut* is a pair (A, B) of subsets of $V(G)$ such that $V(G) = A \cup B$, $S \subseteq A$ and G has no edge joining $A - B$ to $B - A$. The order of the S-cut is $|A \cap B|$, and the S-cut is said to *avoid* the subgraph C if $A \cap V(C) = \emptyset$.

Lemma 2. *Let $d \geq 0$, $k \geq 2$ and $l \geq d + \lfloor 3k/2 \rfloor$ be integers. Let G be a graph containing vertex-disjoint non-empty connected subgraphs $C_1, \ldots, C_l$ such that each C_i is adjacent to all but at most d other C_j's. Suppose that $S = \{s_1, \ldots, s_k\}$ is a set of k vertices such that there is no S-cut of order less than k which avoids $d+1$ of the subgraphs $C_1, \ldots, C_l$. Then G contains vertex-disjoint non-empty connected subgraphs $D_1, \ldots, D_m$, where $m = l - \lfloor k/2 \rfloor$, such that for each i, $1 \leq i \leq k$, the subgraph D_i contains s_i and is adjacent to all but at most d of the subgraphs $D_{k+1}, \ldots, D_m$.*

Proof. The lemma would follow more or less immediately from Menger's theorem if all the subgraphs C_i were single vertices. It is therefore natural to attempt to prove the lemma in general by considering a counterexample which is minimal with respect to subcontraction (one of the standard proofs of Menger's theorem follows this route, see Dirac and Schuster [4]). However, in the familiar pattern of proof by induction, we are forced to strengthen slightly the statement of the lemma in order to make the induction step work. We therefore claim for the subgraphs C_i not that they are connected and each adjacent to all but at most d other C_j's, but that

(∗) each C_i is either connected or each of its components meets S; moreover each C_i is adjacent to all but at most d of those C_j's, $j \neq i$, which do not meet S.

Let us suppose, then, that the lemma as modified by (∗) is false, and that we have a graph G along with subgraphs $C_1, \ldots, C_l$ and a subset S forming a minimal counterexample. Notice that no vertex $v \in G$ is isolated, for otherwise if $v \notin S$ then the graph $G-v$ is a smaller counterexample, and if $v\in S$ then there is an S-cut of order $k-1$ avoiding $l-k>d+1$ of the C_i's, namely the S-cut $(S,V(G)-\{v\})$.

Suppose that (A,B) is an S-cut of order k avoiding $d+1$ of the C_i's. Let $S'=A\cap B$ let $G'=G[B]-E(S')$ and let $C_i'=C_i\cap G'$, $1\le i\le l$. Observe that if (A,B) avoids C_j then $C_j'=C_j$, and that there are at least $d+1$ subgraphs C_j for which this holds. From this observation it follows, first of all, that every other C_i is adjacent to at least one of these C_j's, and hence, in particular, C_i' is non-empty, $1\le i\le l$. Secondly, any component of a C_i' not meeting S' is a component of C_i, so each C_i' is either connected or each of its components meets S'. As any C_j' not meeting S' is equal to C_j, it follows that each C_i, and hence each C_i', is adjacent to all but at most d of those C_j', $j\neq i$, which do not meet S'. Therefore the statement $(*)$ holds true for the C_i''s and S'. Furthermore there can be no S'-cut (A',B') in G' of order less than k which avoids $d+1$ of the C_i''s; for otherwise $(A\cup A',B')$ would be an S-cut in G of order less than k avoiding $d+1$ of the C_i''s. Finally, notice that the graph $G[A]$ contains no S-cut (A'',B'') of order less than k with $S'\subseteq B''$, for otherwise $(A'',B''\cup B)$ would be an S-cut in G of the forbidden kind. Thus, by Menger's theorem, there are disjoint paths $P_1,\dots,P_k$ in $G[A]$ with P_i joining s_i to s_i', where $s_1',\dots,s_k'$ is some labelling of S'.

We now claim that if (A,B) is an S-cut of order k avoiding $d+1$ of the C_i's then $(A,B)=(S,V(G))$ and $E(S)=\emptyset$. For otherwise G' would be smaller than G, and so, since G', S' and the C_i''s satisfy the hypotheses of the lemma as modified by $(*)$, there would exist in G' subgraphs $D_1',\dots,D_m'$ satisfying the conclusions of the lemma. But then the subgraphs $D_1,\dots,D_m$ of G, defined by $D_i=D_i'\cup P_i$ for $1\le i\le k$ and $D_i=D_i'$ for $i>k$ would show that G is not in fact a counterexample to the lemma, contradicting our assumption.

The remainder of the proof is straightforward. The claim just proved implies at once that every edge e of G joins two of the C_i's, because otherwise the graph obtained by contracting e would be a counterexample (we already showed that e cannot lie inside S, and no forbidden S-cuts can arise by contracting e). Since no vertex is isolated we can conclude that $V(G)=\bigcup_{i=1}^{l}V(C_i)$, and that $|C_i|=1$ unless $V(C_i)\subseteq S$.

Let $C=\bigcup\{V(C_i):|C_i|>2\}$, so $C\subseteq S$. There are $|V(G)|-|C|$ subgraphs C_i with $|C_i|=1$ and at most $\lfloor |C|/2\rfloor$ with $|C_i|\ge 2$. Hence $l\le |V(G)|-|C|+\lfloor |C|/2\rfloor$, so $|V(G)|-|C|\ge l-\lfloor k/2\rfloor$. We assert that there is a set I of $|C|$ independent edges, each edge having one end in C and one in $V(G)-S$. Indeed, Hall's Theorem tells us that if I does not exist then there is a subset $X\subseteq C$ whose neighbour set $Y\subseteq V(G)-S$ satisfies $|Y|<|X|$. Let $A=S\cup Y$ and $B=V(G)-X$. Then (A,B) is an S-cut of order $|S\cup Y|-|X|<k$, and the number of C_i's avoided by this S-cut is

at least

$$|V(G)| - |Y| - |S| \geq |V(G)| - |X| + 1 - k \geq l - \lfloor 3k/2 \rfloor + 1 > d + 1,$$

contrary to our assumption.

But now we can construct appropriate subgraphs $D_1, \dots, D_m$, of G as follows: if $s_i \in C$ let D_i be formed by the edge of I meeting s_i, if $s_i \notin C$ let D_i consist of the single vertex s_i, and let $D_{k+1}, \dots, D_m$ be subgraphs of order one formed by $m-k$ of the vertices of $V(G) - S$ not incident with edges of I. Note that $|V(G)-S|-|I| = |V(G)-C|-k \geq l - \lfloor 3k/2 \rfloor \geq m-k$, so there are enough vertices to make this choice. Each of $D_{k+1}, \dots, D_m$ is identical with one of the C_i's not meeting S, so each D_i is adjacent to all but at most d of $D_{k+1}, \dots, D_m$, contradicting the fact that G is a counterexample to the (modified) lemma. ∎

The condition in the statement of Lemma 2 concerning forbidden S-cuts is somewhat strained; in applications it is reasonable to replace it with the stronger but much more natural condition that the vertex connectivity of the graph be at least k. It is in this form that we present the next theorem.

In order to state the theorem, we introduce a slight generalization of the definition of k-linked graphs. A graph G is said to be (k,n)-*knit* if $1 \leq n \leq k \leq |G|$ and, whenever S is a set of k vertices of G and $S_1, \dots, S_t$ is a partition of S into $t \geq n$ non-empty parts, then G contains vertex-disjoint connected subgraphs $F_1, \dots, F_t$ such that $S_i \subseteq V(F_i)$, $1 \leq i \leq t$. Clearly, a $(2k,k)$-knit graph is k-linked.

Theorem 3. *Let G be a graph with vertex connectivity $\kappa(G) \geq k$ such that $G \succ H$, where H is a graph with $2\delta(H) > |H| + \lfloor 5k/2 \rfloor - 2 - n$. Then G is (k,n)-knit. In particular, if $\kappa(G) \geq 2k$ and $2\delta(H) \geq |H| + 4k - 2$ then G is k-linked.*

Proof. Let $S = \{s_1, \dots, s_k\}$, let $l = |H|$ and let $d = |H| - 1 - \delta(H)$. Then $l = |H| = 2d + 2 - |H| + 2\delta(H) \geq 2d + \lfloor 5k/2 \rfloor - n$, so the conditions of Lemma 2 are satisfied, the subgraphs C_i being those subgraphs of G contracted in forming H. Therefore there are subgraphs $D_1, \dots, D_m$, as described therein. Observe that, for each pair D_i, D_j with $1 \leq i < j \leq k$, there are at least $m - k - 2d \geq l - \lfloor 3k/2 \rfloor - 2d \geq k - n$ subgraphs amongst $D_{k+1}, \dots, D_m$ that are adjacent to both D_i and D_j.

Let $S_1, \dots, S_t$ be a partition of S into $t \geq n$ nonempty parts; we may assume that there are numbers $r_1 = 0 < r_2 < \dots < r_{t+1} = k$ such that $S_i = \{s_r : r_i < r \leq r_{i+1}\}$, $1 \leq i \leq t$. Since for each r there are $k-n$ subgraphs among $D_{k+1}, \dots, D_n$, that are adjacent to both D_r and D_{r+1}, we may further assume that the notation has been chosen so that D_{k+r} is adjacent to D_r and D_{r+1} for the $k-t$ values of $r \in \{j : 1 \leq j \leq k,\ j \neq r_2, \dots, r_{t+1}\}$. Then the subgraphs

$$F_i = \bigcup \{D_r : r_i < r \leq r_{i+1}\} \ \cup \ \bigcup \{D_{k+r} : r_i < r < r_{i+1}\}$$

are connected, vertex-disjoint, and $S_i \subseteq V(F_i)$, $1 \leq i \leq t$. Therefore G is (k,n)-knit, as claimed. ∎

Theorem 3 implies, in particular, that if $\kappa(G) \geq 2k$ and $G \succ K_{4k}$ then G is k-linked. In this case, however, when $d = 0$, a more careful analysis is possible, and

Robertson and Seymour [16] thereby showed that $G \succ K_{3k}$ will do, as mentioned earlier.

We remark that the method we used to join $D_1,\dots,D_k$ in order to produce the subgraphs $F_1,\dots,F_t$ was quite profligate in its use of subgraphs from $D_{k+1},\dots,D_m$. The construction could be made much more efficient by making use of the fact that each D_j with $j > k$ is adjacent to many D_i's. However no amount of effort spent on these refinements would reduce the lower bound for $2\delta(H)$ by more than k.

3. Highly linked graphs and subgraphs

The possibility of applying Theorem 3 when H is not complete bears fruit when we make use of Lemma 1, which shows that such minors H will arise in graphs of only moderate average degree. We can thence show that quite a small value of vertex connectivity suffices to guarantee that a graph be k-linked. For each of the sufficient conditions given below for a graph to be k-linked there is a similar condition for it to be (k,n)-knit, though we shall not give these other conditions explicitly.

Theorem 4. *Let G be a graph with vertex connectivity $\kappa(G) \ge 22k$. Then G is k-linked.*

Proof. Since $\kappa(G) \ge 22k$, and so $e(G) \ge 11k|G|$, it follows from Lemma 1 that G has a minor H with $2\delta(H) \ge |H| + \lfloor 11\beta k \rfloor - 1$. Since β exceeds 0.37 in value, we have $2\delta(H) \ge |H| + 4k - 1$. It now follows immediately from Theorem 3 that G is k-linked. ∎

Mader [14] proved that a graph G of size greater than $(2k-3)(|G|-k-1)$ contains a k-connected subgraph. In particular if $e(G) > 44k|G|$ then G contains a $22k$-connected subgraph. In view of this, Theorem 4 has the following immediate consequence.

Corollary 5. *If G is a graph with $e(G) \ge 44k|G|$ then G contains a k-linked subgraph.*

We are now able to give a bound on the size of a graph ensuring that it will contain a topological K_p. In fact, we give a more general result concerning the existence of a topological H.

Theorem 6. *Let H be a graph with vertices $v_1,\dots,v_{|H|}$. Let G be a graph with $\kappa(G) \ge 22e(H) + |H|$ and let $u_1,\dots,u_{|H|}$ be distinct vertices of G. Then G contains $e(H)$ pairwise vertex disjoint paths $P_{i,j}$ with $P_{i,j}$ joining u_i to u_j whenever $v_iv_j \in E(H)$.*

Proof. Let $U = \{u_1,\dots,u_{|H|}\}$. Since $\delta(G - E[U]) > 2e(H)$, in the graph $G - U$ we may select disjoint subsets $Y_1,\dots,Y_{|H|}$ of vertices such that $|Y_i| = d_H(v_i)$ and every vertex of Y_i is joined to u_i, $1 \le i \le |H|$. Now $\kappa(G-U) \ge 22e(H)$ and so by Theorem

4 the graph $G-U$ is $e(H)$-linked. Therefore it is possible to find $e(H)$ vertex disjoint paths linking the sets $Y_1,\dots,Y_{|H|}$ so as to form the paths $P_{i,j}$ claimed in the theorem. ∎

A special case of Theorem 6 can be considered to be an extension of a theorem of Dirac. In 1960 Dirac [3] proved that any $k \geq 2$ vertices of a k-connected graph are contained in some cycle. Theorem 6 implies that if our graph is $23k$-connected then we may specify the order in which our vertices appear on the cycle. It would be of interest to determine the smallest constant that would do instead of 23.

The next corollary follows at once from Theorem 6 and Mader's theorem [14].

Corollary 7. *If* $\kappa(G) \geq 11p^2$ *or if* $e(G) \geq 22p^2|G|$ *then* $G \supset TK_p$.

This corollary improves upon a recent result in [2] where it was shown that $G \supset TK_p$ provided $e(G) \geq 256p^2|G|$.

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