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BIPARTITE SUBGRAPHS

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It is shown that there exists a positive c so that for any large integer m, any graph with $2m^2$ edges contains a bipartite subgraph with at least $m^2+m/2+c\sqrt{m}$ edges. This is tight up to the constant c and settles a problem of Erdős. It is also proved that any triangle-free graph with $e>1$ edges contains a bipartite subgraph with at least $\frac{e}{2}+c'e^{4/5}$ edges for some absolute positive constant c'. This is tight up to the constant c'.

1. Introduction

For a graph G, let $f(G)$ denote the maximum number of edges in a bipartite subgraph of G. For a positive integer e let $f(e)$ denote the minimum value of $f(G)$, as G ranges over all graphs with e edges. Thus, $f(e)$ is the largest integer f such that *any* graph with e edges contains a bipartite subgraph with at least $f(e)$ edges. Edwards [4], [5] proved that for every e

$$(1) \qquad f(e) \geq \frac{e}{2} + \frac{-1+\sqrt{8e+1}}{8},$$

and noticed that this is tight in infinitely many cases. In [7] Erdős conjectured that the limsup of the difference

$$f(e) - \left(\frac{e}{2} + \frac{-1+\sqrt{8e+1}}{8}\right)$$

tends to infinity as e tends to infinity, and offered 50 dollars for a proof or disproof. In the present paper we prove this is indeed the case, in the following more precise form.

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Theorem 1.1. *There exists a positive constant c and an integer n_0 so that for every even integer $n>n_0$, if $e=n^2/2$ then*

$$f(e) \geq \frac{e}{2} + \sqrt{e/8} + ce^{1/4}.$$

We also observe that this estimate is tight in the sense that there is a positive constant C so that

$$f(e) \leq \frac{e}{2} + \sqrt{e/8} + Ce^{1/4} \tag{2}$$

for every e.

Erdős and Lovász (see [6]) showed that if G is a *triangle-free* graph with e edges, then

$$f(G) \geq e/2 + \Omega\left(e^{2/3}\left(\frac{\log e}{\log\log e}\right)^{1/3}\right).$$

(Here and in what follows all logarithms are in the natural basis 2.71828..) This has been improved by a logarithmic factor by Poljak and Tuza [14], and further improved by Shearer [16], who proved that if G is a triangle-free graph with e edges then

$$f(G) \geq \frac{e}{2} + \Omega(e^{3/4}). \tag{3}$$

In the next theorem we improve the exponent $3/4$ to $4/5$ and show that this is tight.

Theorem 1.2. *There exists a constant $c'>0$ such that for every triangle-free graph G with $e>1$ edges*

$$f(G) \geq \frac{e}{2} + c'e^{4/5}. \tag{4}$$

This is tight up to the multiplicative constant in the sense that there exists a constant $C'>0$ so that for every e there exists a triangle-free graph G with e edges satisfying

$$f(G) \leq \frac{e}{2} + C'e^{4/5}.$$

The proof of Theorem 1.1 is presented, together with some related remarks, in the next section. Theorem 1.2 is proved in Section 3 by combining the arguments of Shearer with some additional combinatorial ideas together with a simple eigenvalue technique and a known construction of triangle-free graphs with extremal spectral properties given in [1]. The final section contains some concluding remarks and open problems.

2. The proof of Theorem 1.1

For any two disjoint subsets of vertices U and W in a graph G let $e(U,W)$ denote the number of edges between U and W, and let $e(U)$ denote the number of edges in the induced subgraph of G on U. For $m \geq r$, let $t(m,r)$ denote the number of edges in the complete r-partite graph on m vertices with nearly equal color classes, that is

$$t(m,r) = \sum_{1 \leq i < j \leq r} \left\lfloor \frac{m+i-1}{r} \right\rfloor \left\lfloor \frac{m+j-1}{r} \right\rfloor .$$

We need (a special case of) the following simple lemma, proved by various researchers, including Locke. Since the proof is very short we reproduce it here, for completeness.

Lemma 2.1. ([11, Corollary 1], see also [3], [10].) *Let $G=(V,E)$ be an m-colorable graph with e edges. Then G contains an r-colorable subgraph with at least $e \cdot t(m,r)/\binom{m}{2}$ edges. In particular, if G is $2s$-colorable then $f(G) \geq \frac{s}{2s-1} e = e/2 + e/(4s-2)$.*

Proof. Fix a partition of V into m independent sets $V_1, \dots, V_m$, and let us partition these sets into r classes randomly, where the class number i contains precisely $\lfloor (m+i-1)/r \rfloor$ sets V_j. For each fixed edge of G, the probability its ends lie in distinct classes is precisely $t(m,r)/\binom{m}{2}$ and hence, by linearity of expectation, the expected number of edges in the r-partite subgraph of G whose color classes are the classes above is $e \cdot t(m,r)/\binom{m}{2}$, completing the proof. ∎

Remark. A special case of the above lemma implies that any graph with $e = \binom{rn}{2}$ edges contains an r-partite subgraph with at least $\binom{r}{2} n^2$ edges. Indeed, the chromatic number of such a graph cannot exceed $m = rn$, since in any proper coloring with a minimum number of colors there is an edge between any two distinct color classes, and the desired result follows. This (for $r=3$) settles another problem mentioned in [7]. The same argument yields a short proof of (1). This has also been observed, independently, by Hofmeister and Lefmann [9]. A different short proof of (1) has recently been given in [8].

Proof of Theorem 1.1. To simplify the presentation, we make no attempt to optimize the value of the constant $c>0$ in our proof. Let n be a (large) even integer, and let G be a graph with $e=n^2/2$ edges. Fix a positive small ϵ (e.g., $\epsilon = 0.01$), and consider two possible cases.

Case 1. G is $2s$-colorable for some integer s, with $2s \leq n - \epsilon\sqrt{n} + 1$.

In this case, by Lemma 2.1,

$$f(G) \geq \frac{e}{2} + \frac{e}{4s-2} \geq \frac{e}{2} + \frac{n^2}{4n - 4\epsilon\sqrt{n}} \geq \frac{e}{2} + \sqrt{e/8} + \frac{\epsilon 2^{1/4}}{4} e^{1/4},$$

implying the desired result.

Case 2. The chromatic number of G is at least $n-\epsilon\sqrt{n}$.

Let $n-k$ be the chromatic number of G, where k is nonnegative (as G has less than $\binom{n+1}{2}$ edges), and $k \le \epsilon\sqrt{n}$. Let H be a vertex critical $n-k$ chromatic subgraph of G. Then the minimum degree in H is at least $n-k-1 \ge n-\epsilon\sqrt{n}-1$, and hence its number of vertices is at most $n+2\epsilon\sqrt{n}$. It follows that in any proper coloring of H by $n-k$ colors there are at least $n-4\epsilon\sqrt{n}$ color classes of size 1. Moreover, any two color classes are connected by an edge, implying that H, and hence G, contains a clique on $n-r$ vertices, with $0 \le r \le 4\epsilon\sqrt{n}$. Let U denote the set of vertices of this clique, and let W denote the set of all the remaining vertices of G. Note that

$$e(U) = \frac{n^2}{2} - (2r+1)\frac{n}{2} + \frac{r(r+1)}{2},$$

and hence

$$(5)\quad e(W)+e(U,W) = e - e(U) = (2r+1)\frac{n}{2} - \frac{r(r+1)}{2} = r(n-r) + \frac{n}{2} + \frac{r^2-r}{2}.$$

Since $r \le 4\epsilon\sqrt{n}$ and ϵ is small, the number of edges incident with the members of W, which is $e(W)+e(U,W) = r(n-r)+n/2+(r^2-r)/2$ differs from any integral multiple of $n-r$ by at least, say, $n/4$. (In particular, there are at least $n/4$ edges incident with the vertices in W.)

If $e(W) \ge n/32$ then, by Edwards' result (1), we can partition W into two classes W_1 and W_2 so that $e(W_1,W_2) \ge e(W)/2+\sqrt{e(W)/8}+O(1) \ge e(W)/2+\sqrt{n}/16+O(1)$. Similarly, one can partition the set of vertices of the complete graph U into two classes U_1, U_2 so that

$$e(U_1,U_2) \ge e(U)/2 + \sqrt{e(U)/8} + O(1) \ge e(U)/2 + \sqrt{e/8} - \epsilon\sqrt{n} + O(1).$$

It follows that

$$\begin{aligned} f(G) &\ge \mathrm{Max}\,\{e(U_1 \cup W_1, U_2 \cup W_2), e(U_1 \cup W_2, U_2 \cup W_1)\} \\ &\ge e(U)/2 + e(W)/2 + e(W,U)/2 + \sqrt{e/8} - \epsilon\sqrt{n} + \sqrt{n}/16 + O(1) \\ &\ge e/2 + \sqrt{e/8} + \Omega(e^{1/4}), \end{aligned}$$

as needed.

It remains to consider the case $e(W) \le n/32$. In this case $e(U,W) = r(n-r) + n/2+(r^2-r)/2-e(W)$ differs from an integral multiple of $n-r$ by at least $n/5$. Let $v_1, \dots v_{n-r}$ be the vertices in U, and let d_i denote the number of edges from W to v_i, where $d_1 \le d_2 \le \dots \le d_{n-r}$. Define

$$U_1 = \{v_1, \dots, v_{\lfloor (n-r)/2 \rfloor}\}, U_2 = \{v_{\lfloor (n-r)/2 \rfloor + 1}, \dots, v_{n-r}\}.$$

We next show that by the above assumption the number of edges from W to U_2 exceeds $e(U,W)/2$ by at least $n/10$. Let $\overline{d}$ denote the average value of the

numbers d_i and note that this number differs from an integer by at least $\frac{n}{5(n-r)}$. If $d_{\lfloor (n-r)/2 \rfloor} \geq \overline{d}$ then each vertex in U_2 has at least $\overline{d} + \frac{n}{5(n-r)}$ neighbors in W, and the desired estimate follows. Otherwise, each member of U_1 has at most $\overline{d} - \frac{n}{5(n-r)}$ neighbors in W, and hence $e(U_1, W) \leq e(U,W)/2 - n/10$ implying that $e(U_2, W) \geq e(U,W)/2 + n/10$, as needed.
Therefore, in this case

$$f(G) \geq e(U_1 \cup W, U_2) \geq e(U_1, U_2) + e(U,W)/2 + e(W)/2 - e(W)/2 + n/10$$
$$\geq e/2 + \sqrt{e/8} - \epsilon\sqrt{n} - n/64 + n/10 + O(1),$$

showing that in this case $f(G)$ is in fact much bigger than needed and completing the proof. ∎

We close this section with the easy proof of (2). Write e in the form

$$e = \binom{n_1}{2} + \binom{n_2}{2} + \cdots + \binom{n_k}{2}, \tag{6}$$

where the numbers n_i are chosen one by one, each n_i being the maximum possible integer for which

$$\binom{n_1}{2} + \binom{n_2}{2} + \cdots + \binom{n_i}{2} \leq e. \tag{7}$$

An easy computation shows that the union of the k complete graphs K_{n_i} has e edges and does not contain a bipartite graph with more than $e/2+\sqrt{e/8}+O(e^{1/4})$ edges.

3. Triangle-free graphs

In [16] Shearer proved that if $G=(V,E)$ is a triangle-free graph with n vertices and e edges, and if $d_1, d_2, \ldots, d_n$ are the degrees of its vertices, then

$$f(G) \geq \frac{e}{2} + \frac{1}{8\sqrt{2}} \sum_{i=1}^{n} \sqrt{d_i}. \tag{8}$$

As a consequence he showed that under the same conditions

$$f(G) \geq \frac{e}{2} + \frac{1}{8\sqrt{2}} e^{3/4}, \tag{9}$$

and that if G is d-regular, then

$$f(G) \geq \frac{e}{2} + \frac{1}{8\sqrt{2}} n\sqrt{d}. \tag{10}$$

Combining (8) with some additional ideas we next prove Theorem 1.2. We make no attempt to optimize the absolute constants in our estimates.

Proof of Theorem 1.2. Let $G=(V,E)$ be a triangle-free graph with $|V|=n$ vertices and $|E|=e$ edges. Define $d=\lfloor e^{2/5}\rfloor$. We consider two possible cases depending on the existence of dense subgraphs in G.

Case 1: G contains no subgraph with minimum degree at least d. In this case, as is well known, there exists a labeling $v_1,\dots,v_n$ of the vertices of G so that for every i, the number of neighbors v_j of v_i with $j<i$ is strictly smaller than d. (To see this, let u be a vertex of minimum degree in G, define $v_n=u$, delete it from G and repeat the process). Let $d^+(v_i)$ denote the number of neighbors v_j of v_i with $j<i$ and let $d(v_i)$ denote the total degree of v_i. Then

$$\sum_{i=1}^{n}\sqrt{d(v_i)}\ge\sum_{i=1}^{n}\sqrt{d^+(v_i)}>\frac{\sum_{i=1}^{n}d^+(v_i)}{\sqrt{d}}=\frac{e}{\sqrt{d}}=\Omega(e^{4/5}).$$

This, together with Shearer's result (8) implies (4), as needed.

Case 2: There exists a subset W of m vertices of G so that the induced subgraph H of G on W has minimum degree at least d. We first prove that in this case H (and hence G) contains an induced subgraph H' on a set W' of vertices of G, with at least $md/4$ edges, which is r-colorable for $r=\lceil\frac{2m}{d}\rceil$. To see this, let R be a random subset of at most r vertices of H obtained by picking, with repetitions, r vertices of H, each chosen randomly with uniform probability. Let u be a fixed vertex of H. The probability that u does not have a neighbor in R is

$$\left(1-\frac{d_H(u)}{m}\right)^r<\exp\{-(d/m)r\}<1/4,$$

where $d_H(u)$ denotes the degree of u in H. It follows that for every fixed edge uv of H, the probability that both u and v have neighbors in R is at least $1/2$. Let W' be the set of all vertices of W that have a neighbor in R, and let H' be the induced subgraph of G on W'. By linearity of expectation the expected number of edges of H' is at least $e(W)/2\ge md/4$. Hence there exists a set R of at most r vertices of H so that the corresponding graph H' has at least $md/4$ edges. Fix such an R and define a proper coloring of H' by $|R|\le r$ colors, by coloring each vertex of H' by the index of its smallest neighbor in R. Since G (and hence H) is triangle-free this is a proper coloring, proving that the subgraph H' with the required properties indeed exists.

By Lemma 2.1, there is a partition of W' into two disjoint subsets W_1 and W_2 so that

$$e(W_1,W_2)\ge\frac{e(W')}{2}+\frac{e(W')}{2r}\ge\frac{e(W')}{2}+\Omega\left(md\frac{d}{m}\right)$$

$$= \frac{e(W')}{2} + \Omega(d^2) = \frac{e(W')}{2} + \Omega(e^{4/5}).$$

We can now assign the remaining vertices of G, i.e., those in $V - W'$, one by one, either to W_1 or to W_2, where each vertex v_i in its turn is being assigned to W_1 if it has more neighbors in W_2 than in W_1, and otherwise it is being assigned to W_2. This ensures that at least half of the remaining edges of G lie in the bipartite subgraph obtained this way, and hence shows that in this case, too

$$f(G) \geq \frac{e - e(W')}{2} + \frac{e(W')}{2} + \Omega(e^{4/5}) = \frac{e}{2} + \Omega(e^{4/5}).$$

This completes the proof of (4).

It remains to show that (4) is tight, up to the constant c'. To this end, we need the following simple lemma which provides an upper bound for $f(G)$, for a regular graph G, in terms of the eigenvalues of its adjacency matrix.

Lemma 3.1. *Let $G=(V,E)$ be a d-regular graph with n vertices and $e=nd/2$ edges, and let $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ be the eigenvalues of (the adjacency matrix of) G. Then*

$$f(G) \leq (d - \lambda_n)n/4 = \frac{e}{2} - \lambda_n n/4.$$

(Note that since the trace of the adjacency matrix of G is zero, $\lambda_n \leq 0$.)

Proof. Let $A = (a_{u,v})_{u,v \in V}$ be the adjacency matrix of G in which $a_{u,v} = 1$ if $uv \in E$ and $a_{u,v} = 0$ otherwise. It is easy and well known that for every vector $\phi = (\phi(v) : v \in V)$,

$$(A\phi, \phi) \geq \lambda_n(\phi, \phi), \tag{11}$$

where (ϕ, g) denotes the inner product of ϕ and g. To see this simply write ϕ as a linear combination $\sum_{i=1}^{n} c_i h_i$ of the eigenvectors h_i of A which form an orthonormal basis and notice that

$$(A\phi, \phi) = \sum_{i=1}^{n} c_i^2 \lambda_i \geq \sum_{i=1}^{n} c_i^2 \lambda_n = \lambda_n(\phi, \phi).$$

Let $V = X \cup Y$ be an arbitrary partition of V into two disjoint subsets X and Y, where $|X| = xn$, $|Y| = (1-x)n$. Define a vector $\phi = (\phi(v) : v \in V)$ by putting $\phi(v) = 1 - x$ for each $v \in X$ and $\phi(v) = -x$ for each $v \notin X$. Then

$$(A\phi, \phi) = 2 \sum_{uv \in E} \phi(u)\phi(v)$$

$$= -\sum_{uv \in E} (\phi(u) - \phi(v))^2 + d \sum_{v \in V} \phi^2(v) = -e(X,Y) + d(\phi, \phi),$$

where $e(X,Y)$ is the number of edges between X and Y. Therefore, by (11)

$$-e(X,Y)+d(\phi,\phi)\geq\lambda_n(\phi,\phi),$$

and since

$$(\phi,\phi)=|X|(1-x)^2+(n-|X|)x^2=nx(1-x)$$

we conclude that

$$e(X,Y)\leq(d-\lambda_n)nx(1-x).$$

The desired result follows, as $x(1-x)\leq 1/4$. ∎

The following result is proved in [1] by an explicit construction. For every integer k which is not divisible by 3 there exists a triangle-free $d_n=2^{k-1}(2^{k-1}-1)$-regular graph G_n on $n=2^{3k}$ vertices, whose smallest eigenvalue is at least $-9\cdot 2^k-3\cdot 2^{k/2}-1/4$. By Lemma 3.1 this gives the following.

Proposition 3.2. *For every $n=2^{3k}$, where k is not divisible by 3, there is a triangle-free regular graph G_n with n vertices and $e=(\frac{1}{8}+o(1))n^{5/3}$ edges satisfying*

$$f(G_n)\leq\frac{e}{2}+(1+o(1))\frac{9}{4}n^{4/3}=\frac{e}{2}+(9\cdot 2^{2/5}+o(1))e^{4/5},$$

where the $o(1)$ terms tend to 0 as n tends to infinity.

By taking disjoint copies of appropriate graphs G_n as above (and by adding, if needed, a constant number of isolated edges) it is easy to deduce from Proposition 3.2 that there exists some absolute positive constant C' so that for every e there exists a triangle-free graph G with e edges satisfying

$$f(G)\leq\frac{e}{2}+C'e^{4/5}.$$

This shows that the exponent 4/5 in (4) cannot be improved and completes the proof of Theorem 1.2. ∎

Remark. In [12], [13] the authors construct, explicitly, for every prime $p\equiv 1 \pmod 4$, and for infinitely many values of n, a $d=p+1$ regular graph G_n on n vertices, whose smallest eigenvalue exceeds $-2\sqrt{d-1}$, and whose girth exceeds $\frac{2}{3}\log_{d-1}n$. Lemma 3.1 thus implies the following.

Proposition 3.3. *For every prime $p\equiv 1 \pmod 4$, and for infinitely many values of n, there is a d-regular graph G_n with n vertices, $e=nd/2$ edges, and girth at least $\frac{2}{3}\log_{d-1}n$ satisfying*

$$f(G_n)\leq\frac{e}{2}+\frac{1}{2}n\sqrt{d-1}.$$

This shows that Shearer's estimate in (10) is tight, up to the constant factor multiplying the error term, even if the girth of G is much larger than 4. A similar result can be proved for all values of d by a probabilistic argument. We omit the detailed computation but note that the algebraic approach and the construction in [1] seem to be essential for the proof of Proposition 3.2, which does not seem to follow from probabilistic arguments.

4. Concluding remarks

The technique described in Section 2 can be easily applied to improve the lower bound for $f(e)$ provided by [4], [5] for other values of e, and the choice of $e=n^2/2$ is just to simplify the presentation. It would be interesting to find the precise value of $f(e)$ for every integer e. By the procedure described in the end of Section 2, if e is given by (6) where the numbers n_i are chosen one by one, each being the maximal possible value satisfying (7), then

$$f(e) \leq \sum_{i=1}^{k} \lfloor n_i^2/4 \rfloor .$$

It is not difficult to see that this does *not* provide the precise value of $f(e)$ for every e, but it would be nice to decide for which values of e this is the correct value.

The problem of determining precisely the minimum possible value of $f(G)$ as G ranges over all triangle-free graphs with e edges seems more difficult,

The proof of Theorem 1.1 as well as that of Theorem 1.2 (including the proof of (8) given in [16]) are algorithmic in the sense that they provide simple randomized polynomial time algorithms that enable one to find in any given input graph G a bipartite subgraph of the size guaranteed in the theorems. In both cases the algorithms can be derandomized by a standard application of the method of conditional probabilities (see, e.g., [15] or [2]), thus yielding efficient deterministic algorithms for the corresponding problems. Note that the problem of finding the precise value of $f(G)$ for a given input graph G is the well studied $MAX-CUT$ problem, which is known to be NP-complete.

Finally, the more general problem of finding large k-colorable subgraphs in graphs has also been investigated extensively by various researchers (see, e.g., [11] and some of its references), and it is possible to apply our techniques here to this problem as well. Here, too, the problem of finding the best possible lower bounds precisely seems difficult.

Note added in proof: Independently of our work (and before us), J. Shearer proved a slightly weaker version of the lower bound in Theorem 1.2. He showed that for any $\epsilon > 0$, any triangle-free graph with e edges contains a bipartite subgraph with at least $\frac{e}{2} + \Omega(e^{4/5-\epsilon})$ edges, provided $m > m_0(\epsilon)$.

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References

[1] N. ALON: Explicit Ramsey graphs and orthonormal labelings, *The Electronic J. Combinatorics,* 1 (1994), 8pp.

[2] N. ALON, and J. H. SPENCER: *The Probabilistic Method,* John Wiley and Sons Inc., New York, 1992.

[3] L. D. ANDERSEN, D. D. GRANT, and N. LINIAL: Extremal k-colorable subgraphs, *Ars Combinatoria,* **16** (1983), 259–270.

[4] C. S. EDWARDS: Some extremal properties of bipartite subgraphs, *Canadian Journal of Mathematics,* **3** (1973), 475–485.

[5] C. S. EDWARDS: An improved lower bound for the number of edges in a largest bipartite subgraph, *Proc.* 2^{nd} *Czechoslovak Symposium on Graph Theory,* Prague, (1975), 167–181.

[6] P. ERDŐS: Problems and results in Graph Theory and Combinatorial Analysis, in: *Graph Theory and Related Topics,* J. A. Bondy and U. S. R. Murty (Eds.), Proc. Conf. Waterloo, 1977, Academic Press, New York, 1979, 153–163.

[7] P. ERDŐS: Some recent problems in Combinatorics and Graph Theory, *Proc.* 26^{th} *Southeastern International Conference on Graph Theory,* Combinatorics and Computing, Boca Raton, 1995, Congressus Numerantium, to appear.

[8] P. ERDŐS, A. GYÁRFÁS, and Y. KOHAYAKAWA: The size of the largest bipartite subgraphs, preprint, 1995.

[9] T. HOFMEISTER, and H. LEFMANN: On k-partite subgraphs, preprint, 1995.

[10] J. LEHEL, and ZS. TUZA: Triangle-free partial graphs and edge-covering theorems, *Discrete Math.,* **30** (1982), 59–63.

[11] S. C. LOCKE: Maximum k-colorable subgraphs, *J. Graph Theory,* **6** (1982), 123–132.

[12] A. LUBOTZKY, R. PHILLIPS, and P. SARNAK: Explicit expanders, and the Ramanujan conjectures, *Proc.* 18^{th} *ACM STOC*, (1986), 240–246. See also: A. LUBOTZKY, R. PHILLIPS, and P. SARNAK: Ramanujan graphs, *Combinatorica,* **8** (1988), 261–277.

[13] G. A. MARGULIS: Explicit group-theoretical constructions of combinatorial schemes and their application to the design of expanders and superconcentrators, *Problemy Peredachi Informatsii,* **24** (1988), 51–60 (in Russian). English translation in *Problems of Information Transmission,* **24** (1988), 39–46.

[14] S. POLJAK, and ZS. TUZA: Bipartite subgraphs of triangle-free graphs, *SIAM J. Discrete Math.,* **7** (1994), 307–313.

[15] P. RAGHAVAN: Probabilistic construction of deterministic algorithms: approximating packing integer programs, *Journal of Computer and System Sciences,* **37** (1988), 130–143.

[16] J. B. SHEARER: A note on bipartite subgraphs of triangle-free graphs, *Random Structures and Algorithms,* **3** (1992), 223–226.

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