LUNAR PERTURBATIONS OF ARTIFICIAL SATELLITES OF THE EARTH

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Abstract. Lunisolar perturbations of an artificial satellite for general terms of the disturbing function were derived by Kaula (1962). However, his formulas use equatorial elements for the Moon and do not give a definite algorithm for computational procedures. As Kozai (1966, 1973) noted, both inclination and node of the Moon's orbit with respect to the equator of the Earth are not simple functions of time, while the same elements with respect to the ecliptic are well approximated by a constant and a linear function of time, respectively. In the present work, we obtain the disturbing function for the Lunar perturbations using ecliptic elements for the Moon and equatorial elements for the satellite. Secular, long-period, and short-period perturbations are then computed, with the expressions kept in closed form in both inclination and eccentricity of the satellite. Alternative expressions for short-period perturbations of high satellites are also given, assuming small values of the eccentricity. The Moon's position is specified by the inclination, node, argument of perigee, true (or mean) longitude, and its radius vector from the center of the Earth. We can then apply the results to numerical integration by using coordinates of the Moon from ephemeris tapes or to analytical representation by using results from lunar theory, with the Moon's motion represented by a precessing and rotating elliptical orbit.

1. Elements for the Moon and Other Quantities

Let T be the time in centuries of 36525 ephemeris days from J.D. 2415020.0. The following values will be adopted:
Eccentricity of the Moon:

$$e_☾ = 0.054900489 .$$

Inclination of the Moon:

$$I_☾ = 5°8'43''.427$$
$$\sin(I_☾/2) = 0.044886967 .$$

Mass ratio, Moon to Earth:

$$m_☾/m_⊕ = 0.0123001 .$$

Mean equatorial parallax of the Moon:

$$p_☾ = 57'2''.70 ,$$

where

$$p_☾ = \arcsin(a_e/a_☾) .$$

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Mean equatorial radius of the Earth:

$$a_e = 6\,378\,160 \text{ m}.$$

$a_☾$ = perturbed semimajor axis of the Moon's orbit.
Mean motion in longitude of the Moon:

$$n_☾ = 13°10'34''.889\,902 \text{ day}^{-1}.$$

Mean anomaly of the Moon:

$$M_☾ = 36°55'16''.80 + 1\,724\,878\,768''.03\,T + 25''.61\,T^2 + 0''.0438\,T^3.$$

Argument of Moon's perigee from ecliptic node:

$$\omega_☾ = -\,25°40'13''.60 + 14\,648\,522''.51\,T - 37''.17\,T^2 - 0''.0450\,T^3.$$

Ecliptic longitude of Moon's ascending node:

$$\Omega_☾ = 259°10'59''.79 - 6\,962\,911''.23\,T + 7''.48\,T^2 + 0''.0080\,T^3.$$

Obliquity of the ecliptic:

$$\varepsilon = 23°27'08''.26 - 46''.845\,T - 0''.0059\,T^2 + 0''.001\,81\,T^3.$$

Explicit and precise variations of ε in terms of the elements of the Moon and the Sun are also available:

$$\varepsilon = \varepsilon_0 + \Omega + \mathrm{d}\omega$$

(e.g., *Connaissance des Temps*, 1971).

2. The Disturbing Function

The disturbing function can be written as

$$R = \frac{Gm_☾}{r_☾} \sum_{l \geqslant 2} \left(\frac{r}{r_☾}\right)^l P_l(\cos\psi'), \tag{1}$$

where ψ' is the geocentric elongation of the satellite from the Moon.
The following size considerations apply:

$$Gm_☾ = \frac{m_☾}{m_☾ + m_\oplus}\, n_☾^2 a_☾^3 \simeq 0.0123 n_☾^2 a_☾^3,$$

also written as

$$Gm_☾ = N_☾^2 a_☾^3, \qquad N_☾^2 \simeq 1.59 \times 10^{-5} \text{ rev}^2 \text{ day}^{-2}.$$

The satellite Keplerian negative energy is

$$F_0 = n^2 a^2/2,$$

so that the relative size of the perturbing force function is given by

$$\nu = R/F_0 = 2\, N_{☾}^2/n^2 .$$

For low satellites ($T \simeq 90$ min), $\nu \simeq 1.2 \times 10^{-7}$. For high satellites ($T \simeq 24$ h), $\nu \simeq 3.18 \times \times 10^{-5}$. It follows that, in the above range of periods, for moderate eccentricities, the dominant part of the disturbing function of a satellite is due to the Earth oblateness (J_2), and lunar perturbations are about second order with respect to this.

Let α, δ and α', δ' be the right ascension and declination of the satellite and of the Moon, respectively in an equatorial system. It follows that

$$\cos\psi' = \cos\delta \cos\delta' \cos(\alpha - \alpha') + \sin\delta \sin\delta' .$$

Therefore,

$$R = \sum_{l \geqslant 2} N_{☾}^2 a_{☾}^{2-l} R_l , \tag{2}$$

where

$$R_l = a^l \left(\frac{r}{a}\right)^l \left(\frac{a_{☾}}{r_{☾}}\right)^{l+1} P_l(\cos\psi') , \tag{3}$$

or, making use of Legendre's addition theorem,

$$R_l = a^l \left(\frac{r}{a}\right)^l \left(\frac{a_{☾}}{r_{☾}}\right)^{l+1} \sum_{m=0}^{l} \epsilon_m \frac{(l-m)!}{(l+m)!} P_l^m(\sin\delta')\, P_l^m(\sin\delta) \cos m(\alpha - \alpha') , \tag{4}$$

where

$$\epsilon_m = \begin{cases} 1, & m = 0 \\ 2, & m \neq 0 . \end{cases}$$

Let

$$A_l^m = \left(\frac{a_{☾}}{r_{☾}}\right)^{l+1} \frac{(l-m)!}{(l+m)!} \epsilon_m P_l^m(\sin\delta') \cos m\alpha' ,$$

$$B_l^m = \left(\frac{a_{☾}}{r_{☾}}\right)^{l+1} \frac{(l-m)!}{(l+m)!} \epsilon_m P_l^m(\sin\delta') \sin m\alpha' . \tag{5}$$

We can write

$$R_l = a^l \left(\frac{r}{a}\right)^l \sum_{m=0}^{l} (A_l^m \cos m\alpha + B_l^m \sin m\alpha)\, P_l^m(\sin\delta) . \tag{6}$$

Let

$v = \omega + f$ = argument of latitude of satellite,

Ω = longitude (equatorial) of node of satellite,

I = inclination of satellite to Earth's equator.

By considering the relations

$$\cos(\alpha - \Omega) \cos\delta = \cos v ,$$

$$\sin(\alpha - \Omega)\cos\delta = \sin v \cos I,$$
$$\sin\delta = \sin v \sin I, \tag{7}$$

it follows that

$$R_l = a^l \left(\frac{r}{a}\right)^l \sum_{m=0}^{l} \sum_{p=0}^{l} F_{lmp}(I) \left\{ \begin{bmatrix} A_l^m \\ -B_l^m \end{bmatrix}_{l-m\,\text{odd}}^{l-m\,\text{even}} \cos[(l-2p)v + m\Omega] + \right.$$
$$\left. + \begin{bmatrix} B_l^m \\ A_l^m \end{bmatrix}_{l-m\,\text{odd}}^{l-m\,\text{even}} \sin[(l-2p)v + m\Omega] \right\}, \tag{8}$$

where (Kaula, 1961)

$$F_{lmp}(I) = \sum_i \frac{(2l-2i)!\, 2^{2i-2l}}{i!\,(l-i)!\,(l-m-2i)!} \sin^{l-m-2i} I \times$$
$$\times \sum_j \binom{m}{j} \cos^j I \sum_k \binom{l-m-2i+j}{k} \binom{m-j}{p-i-k} (-1)^{k-q} \tag{9}$$

and

$q = [(l-m)/2]$, the integral part of $(l-m)/2$,
$i = 0, 1, 2, \ldots, \min(p, q)$,
$j = 0, 1, 2, \ldots, m$,
$k =$ all values for which the coefficient is not zero; that is, $p - i \geqslant k$.

3. Rotation of Spherical Harmonics for the Moon

The spherical harmonics $P_l^m(\sin\delta')\, e^{im\alpha'}$ are expressed in terms of $\delta_☾$ and $\alpha_☾$, the Moon's ecliptic latitude and longitude (Lee, 1971).

The relations are given by

$$\cos\delta'\, e^{i\alpha'} = \cos\delta_☾ \cos\alpha_☾ + i(\cos\delta_☾ \sin\alpha_☾ \cos\varepsilon - \sin\delta_☾ \sin\varepsilon),$$
$$\sin\delta' = \cos\delta_☾ \sin\alpha_☾ \sin\varepsilon + \sin\delta_☾ \cos\varepsilon. \tag{10}$$

From the well-known properties of spherical harmonics under rotation, we can write

$$P_l^m(\sin\delta')\, e^{im\alpha'} = \sum_{r=-l}^{l} \alpha_l^{m,r} P_l^r(\sin\delta_☾)\, e^{ir\alpha_☾},$$

where $\alpha_l^{m,r}$ is a function of ε only, and

$$P_l^{-m}(x) = (-1)^m \frac{(l-m)!}{(l+m)!} P_l^m(x).$$

Using the orthogonality conditions of spherical harmonics, we have

$$\alpha_l^{m,r} = \frac{\epsilon_r (2l+1)}{4\pi} \frac{(l-r)!}{(l+r)!} \int_{-\pi/2}^{+\pi/2} \cos\delta_☾\, d\delta_☾ \times$$
$$\times \int_0^{2\pi} P_l^m(\sin\delta')\, e^{im\alpha'} P_l^r(\sin\delta_☾)\, e^{ir\alpha_☾}\, d\alpha_☾.$$

Introducing Equations (10) and computing the above integral, we find

$$\alpha_l^{m,r} = \frac{(l-r)!}{(l-m)!}\, e^{i(m-r)\pi/2}\, U_l^{m,r}, \tag{11}$$

where, for $m+r \geqslant 0$,

$$U_l^{m,r} = (-1)^{l-m} \binom{l+m}{l-r} \left(\cos\frac{\varepsilon}{2}\right)^{m+r} \left(\sin\frac{\varepsilon}{2}\right)^{r-m} \times$$
$$\times F\left(-l+r,\, l+r+1,\, m+r+1;\, \cos^2\frac{\varepsilon}{2}\right) \tag{12a}$$

and, for $m+r \leqslant 0$,

$$U_l^{m,r} = (-1)^{l-r} \binom{l-m}{l+r} \left(\cos\frac{\varepsilon}{2}\right)^{-m-r} \left(\sin\frac{\varepsilon}{2}\right)^{m-r} \times$$
$$\times F\left(-l-r,\, l-r+1,\, -m-r+1;\, \cos^2\frac{\varepsilon}{2}\right). \tag{12b}$$

In the above relations, F is the usual hypergeometric series ${}_1F_2$, defined by

$$F(a, b, c; x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{x^n}{n!},$$

where

$$(a)_n = a(a+1)(a+2)\dots(a+n-1),$$
$$(a)_0 = 1.$$

In both cases, $U_l^{m,r}$ is a polynomial in $\sin\varepsilon/2$, $\cos\varepsilon/2$ since at least one of the parameters a, b is negative, and the above series terminate. The distinction between the cases $m+r \gtrless 0$ is necessary to avoid a singularity in F due to a negative value of c. Considering this fact, another possible form for $U_l^{m,r}$, valid in any case, is found to be

$$U_l^{m,r} = \frac{(-1)^{m-r}}{(l+r)!} \left(\cos\frac{\varepsilon}{2}\right)^{m+r} \left(\sin\frac{\varepsilon}{2}\right)^{r-m} \frac{d^{l+r}}{dz^{l+r}} \left[z^{l-m}(z-1)^{l+m}\right], \tag{13}$$

where $z = \cos^2(\varepsilon/2)$.

Now, let

$$2A_l^{m,r} = U_l^{m,r} + (-1)^r\, U_l^{m,-r},$$
$$2B_l^{m,r} = U_l^{m,r} - (-1)^r\, U_l^{m,-r}. \tag{14}$$

It follows that

$$A_l^m = \left(\frac{a_☾}{r_☾}\right)^{l+1} \frac{(-1)^m \epsilon_m}{(l+m)!} \sum_{r=0}^{l} (l-r)!\, \epsilon_r A_l^{m,r} P_l^r(\sin\delta_☾) \cos\left[r\left(\alpha_☾ + \frac{\pi}{2}\right)\right],$$

$$B_l^m = \left(\frac{a_☾}{r_☾}\right)^{l+1} \frac{(-1)^m \epsilon_m}{(l+m)!} \sum_{r=1}^{l} (l-r)!\, \epsilon_r B_l^{m,r} P_l^r(\sin\delta_☾) \sin\left[r\left(\alpha_☾ + \frac{\pi}{2}\right)\right]. \tag{15}$$

Making use of relations (7) for the Moon, we find that, for $l-m$ even:

$$A_l^m = \left(\frac{a_☾}{r_☾}\right)^{l+1} \frac{(-1)^m \epsilon_m}{(l+m)!} \sum_{r=0}^{l} (l-r)!\, \epsilon_r A_l^{m,r} \sum_{p=0}^{l} F_{lrp}(I_☾) \cos \theta'_{lpr},$$

$$B_l^m = \left(\frac{a_☾}{r_☾}\right)^{l+1} \frac{(-1)^m \epsilon_m}{(l+m)!} \sum_{r=0}^{l} (l-r)!\, \epsilon_r B_l^{m,r} \sum_{p=0}^{l} F_{lrp}(I_☾) \sin \theta'_{lpr}. \quad (16)$$

and, for $l-m$ odd:

$$A_l^m = \left(\frac{a_☾}{r_☾}\right)^{l+1} \frac{(-1)^m \epsilon_m}{(l+m)!} \sum_{r=0}^{l} (l-r)!\, \epsilon_r A_l^{m,r} \sum_{p=0}^{l} F_{lrp}(I_☾) \sin \theta'_{lpr},$$

$$B_l^m = \left(\frac{a_☾}{r_☾}\right)^{l+1} \frac{(-1)^{m+1} \epsilon_m}{(l+m)!} \sum_{r=0}^{l} (l-r)!\, \epsilon_r B_l^{m,r} \sum_{p=0}^{l} F_{lrp}(I_☾) \cos \theta'_{lpr}, \quad (17)$$

where

$$\theta'_{lpr} = (l-2p)\, v_☾ + r\left(\Omega_☾ + \frac{\pi}{2}\right) \quad (18)$$

and the functions $F_{lrp}(I_☾)$ are defined by Equation (9).

Finally, the function R_l can be written, with Equations (8), (14), (16), and (17) taken into account:

$$R_l = a'^l \left(\frac{r}{a}\right)^l \left(\frac{a_☾}{r_☾}\right)^{l+1} \sum_{m=0}^{l} \sum_{s=0}^{l} \sum_{p=0}^{l} \sum_{q=0}^{l} \frac{(-1)^m \epsilon_m \epsilon_s (l-s)!}{(l+m)!} F_{lmp}(I) \times$$
$$\times F_{lsq}(I_☾) \times [(-1)^{l+m-s} U_l^{m,-s} \cos(\theta_{lpm} + \theta'_{lqs}) + U_l^{m,s} \cos(\theta_{lpm} - \theta'_{lqs})], \quad (19)$$

where

$$\theta_{lpm} = (l-2p)\, v + m\Omega. \quad (20)$$

4. Secular and Long-Period Terms of the Disturbing Function

If we assume that no resonance occurs between the orbital motion of the satellite and that of the Moon – that is, $pn + p'n_☾$ is not small for small integers p, p' not simultaneously zero – then the elimination of short-period terms (depending on the mean anomaly of the satellite, M) from the disturbing function can be obtained by making use of the well-known integrals

$$\frac{1}{2\pi} \int_0^{2\pi} \left(\frac{r}{a}\right)^l \sin(l-2p) f \, \mathrm{d}M = 0,$$

$$\frac{1}{2\pi}\int_0^{2\pi}\left(\frac{r}{a}\right)^l \cos(l-2p) f \,\mathrm{d}M = (1+\beta^2)^{-l-1} X_{0,0}^{l,l-2p}(\beta) =$$

$$= (1+\beta^2)^{-l-1} H_{lp(2p-l)}(\beta). \tag{21}$$

In the above relations,

$$\beta = e(1+\sqrt{1-e^2})^{-1} \tag{22}$$

and the X's are Hansen's coefficients (e.g., see Plummer, 1960, p. 45) and the H's are Kaula's coefficients (Kaula, 1961). They are defined by, for $2p-l>0$,

$$H_{lp(2p-l)} = (-\beta)^{2p-l}\binom{2p+1}{2p-l} F(-l-1, 2p-2l-1, 2p-l; \beta^2) \tag{23}$$

and, for $2p-l\leqslant 0$,

$$H_{lp(2p-l)} = (-\beta)^{l-2p}\binom{2l-2p+1}{l-2p} F(-l-1, -2p-1, l-2p+1; \beta^2). \tag{24}$$

In both cases, they are polynomials in β. The distinction again is necessary in order to avoid singular representation.

The long-period and secular part of the function R_l is then found to be

$$\bar{R}_l = a^l\left(\frac{a_☾}{r_☾}\right)^{l+1} \sum_{m=0}^{l}\sum_{s=0}^{l}\sum_{p=0}^{l}\sum_{q=0}^{l} \frac{(-1)^m \epsilon_m \epsilon_s (l-s)!}{(l+m)!} F_{lmp}(I) F_{lsq}(I_☾) \times$$
$$\times (1+\beta^2)^{-l-1} H_{lp(2p-l)}(\beta)[(-1)^{l+m-s} U_l^{m,-s}\cos(\bar{\theta}_{lpm}+\theta'_{lqs}) + + U_l^{m,s}\cos(\bar{\theta}_{lpm}-\theta'_{lqs})], \tag{25}$$

where

$$\bar{\theta}_{lpm} = (l-2p)\,\omega + m\Omega. \tag{26}$$

5. Secular and Long-Period Linear Perturbations

The Lagrange planetary equations can be written as

$$\frac{\mathrm{d}a}{\mathrm{d}t} = \frac{2}{na}\frac{\partial R}{\partial M},$$

$$\frac{\mathrm{d}e}{\mathrm{d}t} = \frac{1-e^2}{na^2 e}\frac{\partial R}{\partial M} - \frac{\sqrt{1-e^2}}{na^2 e}\frac{\partial R}{\partial \omega},$$

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \frac{\cot I}{na^2\sqrt{1-e^2}}\frac{\partial R}{\partial \omega} - \frac{\operatorname{cosec} I}{na^2\sqrt{1-e^2}}\frac{\partial R}{\partial \Omega},$$

$$\frac{\mathrm{d}M}{\mathrm{d}t} = n - \frac{1-e^2}{na^2 e}\frac{\partial R}{\partial e} - \frac{2}{na}\frac{\partial R}{\partial a},$$

$$\frac{d\omega}{dt} = -\frac{\cot I}{na^2\sqrt{1-e^2}}\frac{\partial R}{\partial I} + \frac{\sqrt{1-e^2}}{na^2 e}\frac{\partial R}{\partial e},$$
$$\frac{d\Omega}{dt} = \frac{\operatorname{cosec} I}{na^2\sqrt{1-e^2}}\frac{\partial R}{\partial I}, \tag{27}$$

where $n = \mu^{1/2}a^{-3/2}$ if it appears outside trigonometric functions and

$$M = \sigma + \int n\,dt = \sigma - 3\int\left(\int \frac{1}{a^2}\frac{\partial R}{\partial M}\,dt\right)dt. \tag{28}$$

In the above relation, σ contains all perturbations defined in the fourth equation of (27). The last term of Equation (28) contains only short-period terms and will not be considered in this section.

Now, the function

$$\bar{R} = \sum_{l \geqslant 2} N_☾^2 a_☾^{2-l} \bar{R}_l \tag{29}$$

does not depend on M and is an explicit function of time only through $r_☾$, $v_☾$, and $\Omega_☾$, considering $I_☾$ and ε constants, which is a good approximation.

The integration of the pertinent equations can be performed numerically by using as input lunar ecliptic coordinates – or, for that matter, equatorial coordinates – stored in tapes. This will produce precise evaluation of the true lunar motion. However, such a method can be very expensive in time. A good approximation can be obtained by considering $I_☾$, $e_☾$, $a_☾$, and ε fixed values and $M_☾$, $\omega_☾$, and $\Omega_☾$ linear functions of time, as given in Section 1, neglecting accelerations of these elements. Also, an expansion in power series of $e_☾$ will converge rapidly owing to the small value of this eccentricity.

Along these lines, we can consider the expansions

$$\left(\frac{a_☾}{r_☾}\right)^{l+1}\begin{bmatrix}\sin\\ \cos\end{bmatrix}(\theta'_{ls}) = \sum_{k=-\infty}^{\infty} G_{lqk}(e_☾)\begin{bmatrix}\sin\\ \cos\end{bmatrix}(\theta^*_{lqsk}), \tag{30}$$

where the G's are Kaula's (1961) coefficients, which in turn can be written as Hansen's coefficients

$$G_{lqk} = X_{l-2q+k}^{-(l+1),\,l-2q} \tag{31}$$

and

$$\theta^*_{lqsk} = (l-2q)\,\omega_☾ + (l-2q+k)\,M_☾ + s\left(\Omega_☾ + \frac{\pi}{2}\right). \tag{32}$$

We also remark that $G_{lqk} = O(e_☾^{|k|})$. These functions are given by

$$G_{lqk}(e_☾) = (1+\beta_☾^2)^l \sum_{j=-\infty}^{\infty} J_j[(l-2q+k)\,e_☾]\,X_{l-2q+k,\,j}^{-l-1,\,l-2q}(\beta_☾), \tag{33}$$

where the $J_j(x)$ are Bessel functions with the usual definition

$$J_j(x) = \sum_{s=0}^{\infty} (-1)^s \frac{(x/2)^{j+2s}}{(j+s)!\, s!}. \tag{34}$$

Also, for $k-j-m \geqslant 0$,

$$X_{k,j}^{l,m} = (-\beta_{☾})^{k-j-m} \binom{l+1-m}{k-j-m} F(k-j-l-1,\, -m-l-1,\, k-j-m+1;\, \beta_{☾}^2) \tag{35}$$

and, for $k-j-m \leqslant 0$,

$$X_{k,j}^{l,m} = (-\beta_{☾})^{-k+j+m} \binom{l+1+m}{-k+j+m} F(-k+j-l-1,\, m-l-1,\, -k+j+m+1;\, \beta_{☾}^2). \tag{36}$$

Here again, the hypergeometric series terminate; that is, they are polynomials. However, the G's are infinite series that converge for all $e_{☾} < 1$, although for large $e_{☾}$ the convergence is slow.

It follows that

$$\begin{aligned}\bar{R}_l = \sum_{m=0}^{l} \sum_{s=0}^{l} \sum_{p=0}^{l} \sum_{q=0}^{l} \sum_{k=-\infty}^{\infty} a^l \frac{(-1)^m \epsilon_m \epsilon_s (l-s)!}{(l+m)!} F_{lmp}(I)\, F_{lsq}(I_{☾}) \times \\ \times (1+\beta^2)^{-l-1} H_{lp(2p-l)}(\beta)\, G_{lqk}(e_{☾}) \times \\ \times [(-1)^{l+m-s} U_l^{m,-s} \cos(\bar{\theta}_{lpm} + \theta^*_{lqsk}) + \\ + U_l^{m,s} \cos(\bar{\theta}_{lpm} - \theta^*_{lqsk})].\end{aligned} \tag{37}$$

In the foregoing expressions, the U's are functions of ε only and, therefore, supposed to be constant.

The following particular terms will be defined:

$$\begin{aligned}\bar{R}_{lmspqk} &= (-1)^m \frac{\epsilon_m \epsilon_s (l-s)!}{(l+m)!} F_{lsq}(I_{☾})\, G_{lqk}(e_{☾}) \times \\ &\quad \times a^l (1+\beta^2)^{-l-1} F_{lmp}(I)\, H_{lp(2p-l)}(\beta) \times \\ &\quad \times [(-1)^{l+m-s} U_l^{m,-s} C^+_{lmspqk} + U_l^{m,s} C^-_{lmspqk}] \\ &= \Phi_{lmspqk}(a, e, I)\, [(-1)^{l+m-s} U_l^{m,-s} C^+_{lmspqk} + U_l^{m,s} C^-_{lmspqk}],\end{aligned} \tag{38}$$

where the definition of Φ_{lmspqk} is obvious and

$$C^{\pm}_{lmspqk} = \cos(\bar{\theta}_{lpm} \pm \theta^*_{lqsk}). \tag{39}$$

The following quantities will also be useful:

$$S^{\pm}_{lmspqk} = \sin(\bar{\theta}_{lpm} \pm \theta^*_{lqsk}), \tag{40}$$

$$D^{\pm}_{lmspqk} = (l-2p)\,\dot{\omega} + m\dot{\Omega} \pm [(l-2q)\,\dot{\omega}_{☾} + (l-2q+k)\,\dot{M}_{☾} + s\dot{\Omega}_{☾}], \tag{41}$$

and

$$\int C^{\pm}_{lmspqk}\, \mathrm{d}t = S^{\pm}_{lmspqk} / D^{\pm}_{lmspqk}. \tag{42}$$

In the above relations, we are using the secular rates of the Moon's motion as given in Section 1 and, for the satellite, as given by the even zonal-harmonics coefficients. The dominant terms follow:

$$\dot{\omega} = -\tfrac{3}{4}J_2\left(\frac{a_e}{a}\right)^2 \frac{1 - 5\cos^2 I}{(1-e^2)^2},$$

$$\dot{\Omega} = -\tfrac{3}{2}J_2\left(\frac{a_e}{a}\right)^e \frac{\cos I}{(1-e^2)^2},$$

$$\dot{M} = n\left[1 + \tfrac{3}{4}J_2\left(\frac{a_e}{a}\right)^2 \frac{-1 + 3\cos^2 I}{(1-e^2)^{3/2}}\right]. \tag{43}$$

The relation between a and n is the perturbed Kepler law

$$n^2 a^3 = Gm_{\oplus}\left[1 + \tfrac{3}{2}J_2\left(\frac{a_e}{a}\right)^2 \frac{1 - 3\cos^2 I}{(1-e^2)^{3/2}}\right]. \tag{44}$$

Now, let $\delta_1 e_{lmspqk}, \ldots, \delta_1\Omega_{lmspqk}$ be the linear long-period and secular perturbations that are obtained from Equations (27), integrating the right-hand members (a, e, I fixed) and substituting $N_{\mathbb{C}}^2 a_{\mathbb{C}}^{2-l}\bar{R}_{lmspqk}$ for R. The partial derivatives entering Lagrange's Equations (27) are given by

$$\frac{\partial \bar{R}_{lmspqk}}{\partial \omega} = \Phi_{lmspqk}(2p - l)\left[(-1)^{l+m-s} U_l^{m,-s} S^+_{lmspqk} + U_l^{m,s} S^-_{lmspqk}\right], \tag{45}$$

$$\frac{\partial \bar{R}_{lmspqk}}{\partial \Omega} = -\Phi_{lmspqk} m\left[(-1)^{l+m-s} U_l^{m,-s} S^+_{lmspqk} + U_l^{m,s} S^-_{lmspqk}\right], \tag{46}$$

$$\frac{\partial \bar{R}_{lmspqk}}{\partial a} = \frac{l}{a}\bar{R}_{lmspqk}, \tag{47}$$

$$\frac{\partial \bar{R}_{lmspqk}}{\partial I} = \bar{R}_{lmspqk}(F_{lmp} \to F^I_{lmp}), \tag{48}$$

and

$$\frac{\partial \bar{R}_{lmspqk}}{\partial e} = \bar{R}_{lmspqk}\left[H_{lp(2p-l)} \to H^e_{lp(2p-l)}\right], \tag{49}$$

where, if only terms with positive powers are considered,

$$F^I_{lmp}(I) = \sum_i \frac{(2l-2i)!\, 2^{2i-2l}}{i!\,(l-i)!\,(l-m-2i)!} \sin^{l-m-2i-1} I \times$$
$$\times \sum_j \binom{m}{j}\left[(l-m-2i)\cos^2 I - j\sin^2 I\right]\cos^{j-1} I \times$$
$$\times \sum_k \binom{l-m-2i+j}{k}\binom{m-j}{p-i-k}(-1)^{k-q}, \tag{50}$$

with the same summation conventions of Equation (9): for $2p-l>0$,

$$H^e_{lp(2p-l)} = \frac{\beta}{e\sqrt{1-e^2}} \left\{ \left[(2p-l)\frac{1}{\beta} - \frac{l+1}{1+\beta^2} \right] H_{lp(2p-l)} + \right.$$
$$+ 2(1+\beta^2)(-\beta)^{2p-l+1} \binom{2p+1}{2p-l} \frac{(l+1)(2p-2l-1)}{2p-l} \times$$
$$\left. \times F(-l, 2p-2l, 2p-l+1; \beta^2) \right\} \tag{51}$$

and, for $2p-l\leqslant 0$,

$$H^e_{lp(2p-l)} = \frac{\beta}{e\sqrt{1-e^2}} \left\{ \left[(l-2p)\frac{1}{\beta} - \frac{l+1}{1+\beta^2} \right] H_{lp(2p-l)} - \right.$$
$$- 2(1+\beta^2)(-\beta)^{l-2p+1} \binom{2l-2p+1}{l-2p} \frac{(l+1)(2p+1)}{l-2p+2} \times$$
$$\left. \times F(-l, -2p, l-2p+2; \beta^2) \right\}. \tag{52}$$

The following definitions are introduced:

$$\Phi_{lmspqk} = (-1)^m \frac{\epsilon_m \epsilon_s (l-s)!}{(l+m)!} F_{lsq}(I_☾) G_{lqk}(e_☾) a^l (1+\beta^2)^{-l-1} \times$$
$$\times F_{lmp}(I) H_{lp(2p-l)}(\beta), \tag{53}$$

$$\Phi^e_{lmspqk} = \Phi_{lmspqk}(H \to H^e), \tag{54}$$

$$\Phi^I_{lmspqk} = \Phi_{lmspqk}(F \to F^I), \tag{55}$$

$$C_{lmspqk} = (-1)^{l+m-s} U_l^{m,-s} C^+_{lmspqk} + U_l^{m,s} C^-_{lmspqk}, \tag{56}$$

$$C^\omega_{lmspqk} = (l-2p)\left[(-1)^{l+m-s} U_l^{m,-s} C^+_{lmspqk}/D^+_{lmspqk} + \right.$$
$$\left. + U_l^{m,s} C^-_{lmspqk}/D^-_{lmspqk}\right], \tag{57}$$

$$C^\Omega_{lmspqk} = m\left[(-1)^{l+m-s} U_l^{m,-s} C^+_{lmspqk}/D^+_{lmspqk} + U_l^{m,s} C^-_{lmspqk}/D^-_{lmspqk}\right], \tag{58}$$

and

$$S_{lmspqk} = (-1)^{l+m-s} U_l^{m,-s} S^+_{lmspqk}/D^+_{lmspqk} + U_l^{m,s} S^-_{lmspqk}/D^-_{lmspqk}. \tag{59}$$

Thus, we can write

$$\int \frac{\partial \bar{R}_{lmspqk}}{\partial \omega} \mathrm{d}t = \Phi_{lmspqk} C^\omega_{lmspqk},$$

$$\int \frac{\partial \bar{R}_{lmspqk}}{\partial \Omega} \mathrm{d}t = \Phi_{lmspqk} C^\Omega_{lmspqk},$$

$$\int \frac{\partial \bar{R}_{lmspqk}}{\partial a} \mathrm{d}t = \frac{l}{a} \Phi_{lmspqk} S_{lmspqk},$$

$$\int \frac{\partial \bar{R}_{lmspqk}}{\partial e} \mathrm{d}t = \Phi^e_{lmspqk} S_{lmspqk},$$

$$\int \frac{\partial \bar{R}_{lmspqk}}{\partial I} \mathrm{d}t = \Phi^I_{lmspqk} S_{lmspqk}.$$

The above integrals are not valid if the integers

$$l - 2p,\ l - 2q + k,\ l - 2q,\ m,\ s$$

are simultaneously zero; that is, we must exclude the cases

$$\begin{aligned} m &= 0, \\ s &= 0, \\ 2p &= l = \text{even} = 2\gamma, \\ 2q &= l = \text{even} = 2\gamma, \\ k &= 0. \end{aligned} \tag{60}$$

They correspond to secular perturbations and, in this case,

$$\frac{\partial \bar{R}_{2\gamma, 0, 0, \gamma, \gamma, 0}}{\partial \omega} = \frac{\partial \bar{R}_{2\gamma, 0, 0, \gamma, \gamma, 0}}{\partial \Omega} = 0.$$

In order to evaluate the other three integrals, we must consider (l=even=2γ):

$$F^I_{2\gamma, 0, \gamma}(I) = \sum_{i=0}^{\gamma-1} \frac{(4\gamma - 2i)!\, 2^{2i-4\gamma}(2\gamma - 2i)}{i!\,(2\gamma - i)!\,(2\gamma - 2i)!} \sin^{2\gamma-2i-1} I \cos I, \tag{61}$$

$$\begin{aligned} H_{2\gamma, \gamma, 0} &= F(-2\gamma - 1, -2\gamma - 1, 1; \beta^2) = \\ &= \sum_{n=0}^{2\gamma+1} \frac{(-2\gamma - 1)_n (-2\gamma - 1)_n}{n!} \beta^{2n}, \end{aligned} \tag{62}$$

and

$$H^e_{2\gamma, \gamma, 0} = \frac{2}{e\sqrt{1 - e^2}} \sum_{n=1}^{2\gamma+1} \frac{(-2\gamma - 1)_n (-2\gamma - 1)_n}{(n-1)!} \beta^{2n}. \tag{63}$$

Therefore,

$$\frac{\partial \bar{R}_{2\gamma, 0, 0, \gamma, \gamma, 0}}{\partial a} = \frac{2\gamma}{a} \bar{R}_{2\gamma, 0, 0, \gamma, \gamma, 0},$$

$$\frac{\partial \bar{R}_{2\gamma, 0, 0, \gamma, \gamma, 0}}{\partial I} = \bar{R}_{2\gamma, 0, 0, \gamma, \gamma, 0}(F_{2\gamma, 0, \gamma} \to F^I_{2\gamma, 0, \gamma}),$$

$$\frac{\partial \bar{R}_{2\gamma, 0, 0, \gamma, \gamma, 0}}{\partial e} = \bar{R}_{2\gamma, 0, 0, \gamma, \gamma, 0}(H_{2\gamma, \gamma, 0} \to H^e_{2\gamma, \gamma, 0}).$$

The long-period perturbations are given (excluding cases (60)) by

$$\delta_1 \bar{e}_{lmspqk} = N^2_☾ a^{2-l}_☾ \left(- \frac{\sqrt{1 - e^2}}{na^2 e} \Phi_{lmspqk} C^{\omega}_{lmspqk} \right),$$

$$\delta_1 \bar{I}_{lmspqk} = N^2_☾ a^{2-l}_☾ \frac{1}{na^2\sqrt{1 - e^2}} (\cot I C^{\omega}_{lmspqk} - \operatorname{cosec} I C^{\Omega}_{lmspqk}) \times \Phi_{lmspqk},$$

$$\delta_1 \bar{M}_{lmspqk} = N^2_☾ a^{2-l}_☾ \frac{1}{na^2} \left(- \frac{1 - e^2}{e} \Phi^e_{lmspqk} - 2l\Phi_{lmspqk} \right) S_{lmspqk},$$

$$\delta_1 \bar{\omega}_{lmspqk} = N_{\text{☾}}^2 a_{\text{☾}}^{2-l} \frac{1}{na^2} \left(- \frac{\cot I}{\sqrt{1-e^2}} \Phi^I_{lmspqk} + \frac{\sqrt{1+e^2}}{e} \Phi^e_{lmspqk} \right) S_{lmspqk},$$

$$\delta_1 \bar{\Omega}_{lmspqk} = N_{\text{☾}}^2 a_{\text{☾}}^{2-l} \frac{\operatorname{cosec} I}{na^2 \sqrt{1-e^2}} \Phi^I_{lmspqk} S_{lmspqk}. \tag{64}$$

The secular perturbations are given by

$$\delta_1 \bar{M}_{2\gamma,0,0,\gamma,\gamma,0} = - N_{\text{☾}}^2 a_{\text{☾}}^{2-2\gamma} \frac{1}{na^2} \times$$
$$\times \left[\frac{1-e^2}{e} \bar{R}_{2\gamma,0,0,\gamma,\gamma,0} (H \to H^e) + 4\gamma \bar{R}_{2\gamma,0,0,\gamma,\gamma,0} \right] t,$$

$$\delta_1 \bar{\omega}_{2\gamma,0,0,\gamma,\gamma,0} = N_{\text{☾}}^2 a_{\text{☾}}^{2-2\gamma} \frac{1}{na^2} \left[- \frac{\cot I}{\sqrt{1-e^2}} \bar{R}_{2\gamma,0,0,\gamma,\gamma,0} (F \to F^I) + \right.$$
$$\left. + \frac{\sqrt{1-e^2}}{e} \bar{R}_{2\gamma,0,0,\gamma,\gamma,0} (H \to H^e) \right] t,$$

$$\delta_1 \bar{\Omega}_{2\gamma,0,0,\gamma,\gamma,0} = N_{\text{☾}}^2 a_{\text{☾}}^{2-2\gamma} \frac{\operatorname{cosec} I}{na^2 \sqrt{1-e^2}} [\bar{R}_{2\gamma,0,0,\ ,\gamma,0} (F \to F^I)] \, t. \tag{65}$$

6. Secular and Long-Period Second-Order Perturbations

In a second-order evaluation, that is, if terms of the order $J_2 N_{\text{☾}}^2 a_{\text{☾}}^{2-l}$ are considered, it will be necessary to take into account the secular changes in M, ω, Ω owing to J_2. Such changes produce the largest higher order perturbations since they produce amplitudes that increase linearly with time. Let $\delta_2 \bar{M}$, $\delta_2 \bar{\omega}$, and $\delta_2 \bar{\Omega}$ be these perturbations. Taking into account only secular coefficients and noting that $\delta_1 \bar{a} = 0$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} (\delta_2 \bar{M}) = \frac{\partial \dot{M}}{\partial e} \delta_1 \bar{e} + \frac{\partial \dot{M}}{\partial I} \delta_1 \bar{I},$$

$$\frac{\mathrm{d}}{\mathrm{d}t} (\delta_2 \bar{\omega}) = \frac{\partial \dot{\omega}}{\partial e} \delta_1 \bar{e} + \frac{\partial \dot{\omega}}{\partial I} \delta_1 \bar{I},$$

$$\frac{\mathrm{d}}{\mathrm{d}t} (\delta_2 \bar{\Omega}) = \frac{\partial \dot{\Omega}}{\partial e} \delta_1 \bar{e} + \frac{\partial \dot{\Omega}}{\partial I} \delta_1 \bar{I}.$$

By considering Equations (43), we find that

$$\frac{\mathrm{d}}{\mathrm{d}t} (\delta_2 \bar{M}) = 3e \sqrt{1-e^2} \frac{1 - 3\cos^2 I}{1 - 5\cos^2 I} \dot{\omega} \delta_1 \bar{e} + 3 \sqrt{1-e^2} \sin I \dot{\Omega} \delta_1 \bar{I},$$

$$\frac{\mathrm{d}}{\mathrm{d}t} (\delta_2 \bar{\omega}) = \frac{4e}{1-e^2} \dot{\omega} \delta_1 \bar{e} + 5 \dot{\Omega} \sin I \delta_1 \bar{I},$$

$$\frac{\mathrm{d}}{\mathrm{d}t} (\delta_2 \bar{\Omega}) = \frac{4e}{1-e^2} \dot{\Omega} \delta_1 \bar{e} - \dot{\Omega} \tan I \delta_1 \bar{I}, \tag{66}$$

where, again, $\dot{\omega}$, $\dot{\Omega}$ are given by Equations (43). It follows that

$$\delta_2\bar{M} = 3e\sqrt{1-e^2}\,\frac{1-3\cos^2 I}{1-5\cos^2 I}\,\dot{\omega}\int\delta_1\bar{e}\,\mathrm{d}t + 3\sqrt{1-e^2}\dot{\Omega}\sin I\int\delta_1\bar{I}\,\mathrm{d}t,$$
$$\delta_2\bar{\omega} = \frac{4e}{1-e^2}\,\dot{\omega}\int\delta_1\bar{e}\,\mathrm{d}t + 5\dot{\Omega}\sin I\int\delta_1\bar{I}\,\mathrm{d}t,$$
$$\delta_2\bar{\Omega} = \frac{4e}{1-e^2}\,\dot{\Omega}\int\delta_1\bar{e}\,\mathrm{d}t - \dot{\Omega}\tan I\int\delta_1\bar{I}\,\mathrm{d}t. \qquad (67)$$

Secular accelerations do not exist, since $\bar{e}$, $\bar{I}$ have only long-period terms. Therefore, conditions (60) have to be excluded. If we consider the first two equations of (64), it follows that

$$\int C^{\omega}_{lmspqk}\,\mathrm{d}t = S^{\omega}_{lmspqk},$$
$$\int C^{\Omega}_{lmspqk}\,\mathrm{d}t = S^{\Omega}_{lmspqk},$$

where

$$S^{\omega}_{lmspqk} = [(-1)^{l+m-s}U_l^{m,-s}S^{+}_{lmspqk}/(D^{+}_{lmspqk})^2 + U_l^{m,s}S^{-}_{lmspqk}/(D^{-}_{lmspqk})]^2\,(l-2p),$$
$$S^{\Omega}_{lmspqk} = [(-1)^{l+m-s}U_l^{m,-s}S^{+}_{lmspqk}/(D^{+}_{lmspqk})^2 + U_l^{m,s}S^{-}_{lmspqk}/(D^{-}_{lmspqk})^2]\,m. \qquad (68)$$

It follows that

$$\delta_2\bar{M}_{lmspqk} = N^2_{☾}a_{☾}^{2-l}\,\frac{3\sqrt{1-e^2}}{na^2}\left(-\frac{1-3\cos^2 I}{1-5\cos^2 I}\sqrt{1-e^2}\,\dot{\omega} + \frac{\cos I}{\sqrt{1-e^2}}\,\dot{\Omega}\right)\times\Phi_{lmspqk}S^{\omega}_{lmspqk} - N^2_{☾}a_{☾}^{2-l}\,\frac{3}{na^2}\,\dot{\Omega}\Phi_{lmspqk}S^{\Omega}_{lmspqk},$$
$$\delta_2\bar{\omega}_{lmspqk} = N^2_{☾}a_{☾}^{2-l}\,\frac{1}{na^2\sqrt{1-e^2}}\,(-4\dot{\omega} + 5\dot{\Omega}\cos I)\,\Phi_{lmspqk}S^{\omega}_{lmspqk} + - N^2_{☾}a_{☾}^{2-l}\,\frac{5\dot{\Omega}}{na^2\sqrt{1-e^2}}\,\Phi_{lmspqk}S^{\Omega}_{lmspqk},$$
$$\delta_2\bar{\Omega}_{lmspqk} = -N^2_{☾}a_{☾}^{2-l}\,\frac{5\dot{\Omega}}{na^2\sqrt{1-e^2}}\,\Phi_{lmspqk}S^{\omega}_{lmspqk} + N^2_{☾}a_{☾}^{2-l}\,\frac{\dot{\Omega}\sec I}{na^2\sqrt{1-e^2}}\,\Phi_{lmspqk}S^{\Omega}_{lmspqk}. \qquad (69)$$

The total long-period and secular perturbations, including leading coupling terms with J_2, are finally obtained by

$$\delta\bar{e} = \delta_1\bar{e} + \delta_2\bar{e},\ldots,\ \delta\bar{\Omega} = \delta_1\bar{\Omega} + \delta_2\bar{\Omega}.$$

Obviously, the above relations are not valid for cases of critical inclination or satellites whose periods are commensurable with the rotation period (24 h) of the Earth.

7. Computational Procedure for Long-Period and Secular Perturbations

In short, to obtain long-period and secular perturbations due to a term $\bar{R}_{lmspqk}$ (Equation (38)), we proceed as follows:

7.1. Long-period perturbations

Compute, given means elements a, e, I, $e_☾$, $I_☾$, ε, and J_2:

(1) $N^2 a_☾^{2-l}$
(2) β (22)
(3) $F_{lsq}(I_☾)$ (9)
(4) $F_{lmp}(I)$ (9)
(5) $G_{lqk}(e_☾)$ (33), (34), (35), or (36)
(6) $H_{lp(2p-l)}(\beta)$ (23) or (24)
(7) Φ_{lmspqk} (53)
(8) $U_l^{m,-s}$, $U_l^{m,s}$ (12a) or (12b) (ε from Section 1)
(9) ω, Ω (43)
(10) $\bar{\theta}_{lpm}$ (26)
(11) $\omega_☾$, $M_☾$, $\Omega_☾$ (Section 1)
(12) θ^*_{lqsk} (32)
(13) $C^{\pm}_{lmspqk}$ (39)
(14) $S^{\pm}_{lmspqk}$ (40)
(15) $\dot{\omega}_☾$, $\dot{\Omega}_☾$, $\dot{M}_☾$ (Section 1)
(16) $\dot{\omega}$, $\dot{\Omega}$, $\dot{M}$ (43)
(17) $D^{\pm}_{lmspqk}$ (41)
(18) F^I_{lmp} (51)
(19) $H^e_{lp(2p-l)}$ (51) or (52)
(20) Φ^e_{lmspqk} (54)
(21) Φ^I_{lmspqk} (55)
(22) C_{lmspqk} (56)
(23) S_{lmspqk} (59)
(24) C^{ω}_{lmspqk} (57)
(25) C^{Ω}_{lmspqk} (58)
(26) S^{ω}_{lmspqk} (68)
(27) S^{Ω}_{lmspqk} (68)
(28) δ_1 (element)$_{lmspqk}$ (64)
(29) δ_2 (element)$_{lmspqk}$ (69)
(30) δ (element) $= \delta_1$ (element) $+ \delta_2$ (element)
Complete long-period perturbations.

7.2. Secular (l even $= 2\gamma$)

Given a, e, I, $e_☾$, $I_☾$, ε, β (mean values):

(1) $F_{2\gamma,0,\gamma}(I_☾)$ (9)

(2) $F_{2\gamma,0,\gamma}(I)$ (9)
(3) $G_{2\gamma,0,0}(e_☾)$ (33), (34), (35), or (36)
(4) $H_{2\gamma,\gamma,0}(\beta)$ (62)
(5) $U_{2\gamma}^{0,0}$(12b)
(6) $\Phi_{2\gamma,0,0,\gamma,\gamma,0}$ (53)
(7) $C^{\pm}_{2\gamma,0\ 0,\gamma,\gamma,0}=1$
(8) $\bar{R}_{2\gamma,0,0,\gamma,\gamma,0}$ (38)
(9) $F^{I}_{2\gamma,0,\gamma}(I)$ (61)
(10) $H^{e}_{2\gamma,\gamma,0}(\beta)$ (63)
(11) $\bar{R}_{2\gamma,0,0,\gamma,\gamma,0}(F\to F^{I})$ (38)
(12) $\bar{R}_{2\gamma,0,0,\gamma,\gamma,0}(H\to H^{e})$ (38)
(13) δ_1 (element, secular) (65)
Complete secular perturbations.

8. Short-Period Perturbations. Low Satellites

During a few revolutions of the satellite, where short-period variations are of interest, the position of the Moon and ω, Ω change little for low satellites. Then, as Kozai (1966) suggested, we can consider the elements of the Moon, ω and Ω, fixed when performing the integrations. In this case, the appropriate expression for R_l is given in Equation (8), since here it is immaterial what frame of reference is being used for the coordinates of the Moon. The particular term R_l will be written as

$$R_l = a^l \left(\frac{r}{a}\right)^l \sum_{m=0}^{l} \sum_{p=0}^{l} F_{lmp}(I) \times$$
$$\times \{C_l^m \cos[(l-2p)\,v + m\Omega] + S_l^m \sin[(l-2p)\,v + m\Omega]\}, \tag{70}$$

where

$$C_l^m = \begin{cases} A_l^m, & l-m \text{ even} \\ -B_l^m, & l-m \text{ odd}, \end{cases}$$
$$S_l^m = \begin{cases} B_l^m, & l-m \text{ even} \\ A_l^m, & l-m \text{ odd}. \end{cases} \tag{71}$$

The coefficients C's and S's depend only on the Moon (5). The values of $r_☾$, v', and Ω', given mean elements $a_☾$, $e_☾$, $I_☾$, ε, and the time T, can be computed by considering $\omega_☾$, $M_☾$, and $\Omega_☾$ (Section 1), then solving Kepler's equation

$$M_☾ = E_☾ - e_☾ \sin E_☾, \tag{72}$$

computing $f_☾$ from

$$\tan\frac{f_☾}{2} = \sqrt{\frac{1+e_☾}{1-e_☾}}\tan\frac{E_☾}{2}, \tag{73}$$

computing

$$r_☾ = a_☾(1 - e_☾ \cos E_☾)$$

$$v_{\zeta} = f_{\zeta} + \omega_{\zeta},$$

computing α_{ζ}, δ_{ζ} from Equations (7), and finally α', δ' from Equations (10).

Let

$$R_{lmp} = a^l \left(\frac{r}{a}\right)^l F_{lmp}(I) \left(C_l^m \cos\theta_{lpm} + S_l^m \sin\theta_{lpm}\right), \tag{74}$$

where

$$\theta_{lpm} = (l - 2p)\, v + m\Omega.$$

The following relations are easily established:

$$\frac{\partial R_{lmp}}{\partial a} = \frac{l}{a} R_{lmp}; \tag{75}$$

$$\frac{\partial R_{lmp}}{\partial e} = l\, \frac{e^2 - 1}{e} \left(\frac{a}{r}\right)^2 R_{lmp} + l\, \frac{1}{e} \left(\frac{a}{r}\right) R_{lmp} + (l - 2p)\, a^l \left(\frac{r}{a}\right)^{l-1} \times$$

$$\times F_{lmp}(I) \left(- C_l^m \sin\theta_{lpm} + S_l^m \cos\theta_{lpm}\right) \sin f \left(1 + \frac{r}{a} \frac{1}{1 - e^2}\right) \tag{76}$$

$$\frac{\partial R_{lmp}}{\partial I} = a^l \left(\frac{r}{a}\right)^l F_{lmp}^I(I)\, C_l^m \cos\theta_{lpm} + S_l^m \sin\theta_{lpm}), \tag{77}$$

where $F_{lmp}^I(I)$ is given by Equation (50);

$$\frac{\partial R_{lmp}}{\partial \omega} = (l - 2p)\, a^l \left(\frac{r}{a}\right)^l F_{lmp}(I) \left(- C_l^m \sin\theta_{lpm} + S_l^m \cos\theta_{lpm}\right); \tag{78}$$

$$\frac{\partial R_{lmp}}{\partial \Omega} = m a^l \left(\frac{r}{a}\right)^l F_{lmp}(I) \left(- C_l^m \sin\theta_{lpm} + S_l^m \cos\theta_{lpm}\right); \tag{79}$$

and

$$\frac{\partial R_{lmp}}{\partial M} = l\, \frac{e}{\sqrt{1 - e^2}} \frac{a}{r} R_{lmp} \sin f + (l - 2p)\, a^l \sqrt{1 - e^2} \left(\frac{r}{a}\right)^{l-2} \times$$

$$\times \left(- C_l^m \sin\theta_{lpm} + S_l^m \cos\theta_{lpm}\right) F_{lmp}(I). \tag{80}$$

For short-period perturbations, we make use of Lagrange's equations (27) where R is replaced by

$$R_{\text{per.}} = R - \frac{1}{2\pi} \int_0^{2\pi} R \, \mathrm{d}M, \tag{81}$$

and integration is carried on with respect to $\mathrm{d}t = \mathrm{d}M/n$, considering all other angles and actions to be constant.

The computation of the average of R with respect to M involves the following integrals:

$$I_1^{l,p} = \frac{1}{2\pi} \int_0^{2\pi} \left(\frac{r}{a}\right)^l \cos(l - 2p) f \, \mathrm{d}M = (1 + \beta^2)^{-l-1} X_{0,0}^{l,\, l-2p}(\beta), \tag{82}$$

$$I_2^{l,p} = \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{r}{a}\right)^{l-2}\cos(l-2p)\,f\,\mathrm{d}M =$$

$$= (1+\beta^2)^{-l+1}\,X_{0,0}^{l-2,\,l-2p}(\beta), \tag{83}$$

$$I_3^{l,p} = \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{r}{a}\right)^{l-1}\cos(l-2p)\,f\,\mathrm{d}M = (1+\beta^2)^{-l}\,X_{0,0}^{l-1,\,l-2p}(\beta), \tag{84}$$

$$I_4^{l,p} = \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{r}{a}\right)^{l-1}\sin(l-2p)\,f\,\sin f\,\mathrm{d}M =$$

$$= (1+\beta^2)^{-l}\left(\tfrac{1}{2}X_{0,0}^{l-1,\,l-2p-1} - \tfrac{1}{2}X_{0,0}^{l-1,\,l-2p+1}\right), \tag{85}$$

and

$$I_5^{l,p} = \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{r}{a}\right)^{l}\sin(l-2p)\,f\,\sin f\,\mathrm{d}M =$$

$$= (1+\beta^2)^{-l-1}\left(\tfrac{1}{2}X_{0,0}^{l,\,l-2p-1} - \tfrac{1}{2}X_{0,0}^{l,\,l-2p+1}\right), \tag{86}$$

where $X_{0,0}^{k,j}(\beta)$ is Hansen's coefficient defined by Equation (23) or (24).

It follows that, by defining

$$\bar{S}_{lpm} = -\,C_l^m \sin\bar{\theta}_{lpm} + S_l^m \cos\bar{\theta}_{lpm}, \tag{87}$$

and

$$\bar{C}_{lpm} = C_l^m \cos\bar{\theta}_{lpm} + S_l^m \sin\bar{\theta}_{lpm}, \tag{88}$$

where $\bar{\theta}_{lpm}$ is given by Equation (26), we have

$$I_{lmp}^{(1)} \equiv \frac{1}{2\pi}\int_0^{2\pi}\frac{\partial R_{lmp}}{\partial M}\,\mathrm{d}M = 0,$$

$$I_{lmp}^{(2)} \equiv \frac{1}{2\pi}\int_0^{2\pi}\frac{\partial R_{lmp}}{\partial \omega}\,\mathrm{d}M = (l-2p)\,a^l F_{lmp}(I)\,I_1^{l,p}\bar{S}_{lpm}, \tag{89}$$

$$I_{lmp}^{(3)} \equiv \frac{1}{2\pi}\int_0^{2\pi}\frac{\partial R_{lmp}}{\partial \Omega}\,\mathrm{d}M = m a^l F_{lmp}(I)\,I_1^{l,p}\bar{S}_{lpm}, \tag{90}$$

$$I_{lmp}^{(4)} \equiv \frac{1}{2\pi}\int_0^{2\pi}\frac{\partial R_{lmp}}{\partial e}\,\mathrm{d}M = l\,\frac{e^2-1}{e}\,a^l F_{lmp}(I)\,I_2^{l,p}\bar{C}_{lpm} +$$

$$+\,l\,\frac{1}{e}\,a^l F_{lmp}(I)\,I_3^{l,p}\bar{C}_{lmp} - (l-2p)\,a^l F_{lmp}(I)\,I_4^{l,p}\bar{C}_{lpm} +$$

$$-\,(l-2p)\,a^l\,\frac{1}{1-e^2}\,F_{lmp}(I)\,I_5^{l,p}\bar{C}_{lpm}, \tag{91}$$

$$I_{lmp}^{(5)} \equiv \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial R_{lmp}}{\partial a} \, \mathrm{d}M = 2la^{l-1} F_{lmp}(I) \, I_1^{l,p} \bar{C}_{lpm}, \tag{92}$$

and

$$I_{lmp}^{(6)} \equiv \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial R_{lmp}}{\partial I} \, \mathrm{d}M = a^l F_{lmp}^I(I) \, I_1^{l,p} \bar{C}_{lpm}. \tag{93}$$

Now it is necessary to evaluate, in closed form, integrals of the type

$$J_{pq}^R + iJ_{pq}^I \equiv J_{pq} = \int \left(\frac{r}{a}\right)^p e^{iqf} \, \mathrm{d}M \tag{94}$$

for $q = 0, 1, 2, \ldots, p+1$ and $p = 0, 1, 2, \ldots$. Introducing $\mathrm{d}M = (r/a)\mathrm{d}E$, we obtain

$$J_{pq} = \int \left(\frac{r}{a}\right)^{p+1} e^{iqf} \, \mathrm{d}E,$$

where

$$\left(\frac{r}{a}\right)^{p+1} = (1 - e\cos E)^{p+1},$$

$$e^{iqf} = \left(\frac{a}{r}\right)^q \left[(\cos E - e) + i\eta \sin E\right]^q,$$

$$\eta = \sqrt{1 - e^2}, \qquad i = \sqrt{-1}.$$

Therefore,

$$J_{pq} = \int \left(\frac{r}{a}\right)^{p+1-q} \left[(\cos E - e) + i\eta \sin E\right]^q \, \mathrm{d}E$$

with

$$p + 1 - q \geqslant 0.$$

It is easily found that

$$J_{pq} = \sum_{\substack{\gamma = -p-1 \\ (\gamma \neq 0)}}^{p+1} \frac{1}{\gamma} K_{pq\gamma}(e) \left(\sin \gamma E - i \cos \gamma E\right) + K_{pq0}(e) \, E, \tag{95}$$

where

$$K_{pq\gamma} = \sum_\alpha \sum_k \sum_s 2^{-\alpha} (-1)^{q-\alpha} \binom{p+1-q}{k} \binom{k}{\frac{\alpha-\gamma}{2} - s} \binom{q}{\alpha - k} \times$$

$$\times \binom{\alpha - k}{s} e^{2k+q-\alpha} (1+\eta)^{\alpha-k-s} (1-\eta)^s, \tag{96}$$

in which

$$\alpha = |\gamma|, |\gamma| + 2, |\gamma| + 4, \ldots, p \text{ or } p+1,$$

$$k = 0, 1, 2, \ldots, \frac{\alpha - \gamma}{2},$$
$$s = 0, 1, 2, \ldots, \alpha - k.$$

Thus,

$$J^R_{pq} \equiv \int \left(\frac{r}{a}\right)^p \cos qf \, \mathrm{d}M = \sum_{\substack{\gamma = -p-1 \\ (\gamma \neq 0)}}^{p+1} \frac{1}{\gamma} K_{pq\gamma} \sin \gamma E + K_{pq0} E \tag{97}$$

and

$$J^I_{pq} = \int \left(\frac{r}{a}\right)^p \sin qf \, \mathrm{d}M = - \sum_{\substack{\gamma = -p-1 \\ (\gamma \neq 0)}}^{p+1} \frac{1}{\gamma} K_{pq\gamma} \cos \gamma E. \tag{98}$$

The following integrals are then established:

$$A_1^{lmp} \equiv \int \frac{\partial R_{lmp}}{\partial M} \mathrm{d}M = \frac{l}{2} a^l \frac{e}{\sqrt{1 - e^2}} F_{lmp}(I) \left[(J^I_{l-1, l-2p-1} - \right.$$
$$- J^I_{l-1, l-2p+1}) \bar{C}_{lpm} + (J^R_{l-1, l-2p-1} - J^R_{l-1, l-2p+1}) \bar{S}_{lpm} \big] +$$
$$+ (l - 2p) a^l \sqrt{1 - e^2} F_{lmp}(I) (- J^I_{l-2, l-2p} \bar{C}_{lpm} + J^R_{l-2, l-2p} \bar{S}_{lpm}), \tag{99}$$

$$A_2^{lmp} \equiv \int \frac{\partial R_{lmp}}{\partial \omega} \mathrm{d}M = (l - 2p) a^l F_{lmp}(I) (- J^I_{l, l-2p} \bar{C}_{lpm} + J^R_{l, l-2p} \bar{S}_{lpm}), \tag{100}$$

$$A_3^{lmp} \equiv \int \frac{\partial R_{lmp}}{\partial \Omega} \mathrm{d}M = m a^l F_{lmp}(I) (- J^I_{l, l-2p} \bar{C}_{lpm} + J^R_{l, l-2p} \bar{S}_{lpm}), \tag{101}$$

$$A_4^{lmp} \equiv \int \frac{\partial R_{lmp}}{\partial e} \mathrm{d}M = l a^l \frac{e^2 - 1}{e} F_{lmp}(I) (J^R_{l-2, l-2p} \bar{C}_{lpm} +$$
$$+ J^I_{l-2, l-2p} \bar{S}_{lpm}) + l \frac{1}{e} a^l F_{lmp}(I) (J^R_{l-1, l-2p} \bar{C}_{lpm} +$$
$$+ J^I_{l-1, l-2p} \bar{S}_{lpm}) + \tfrac{1}{2} a^l F_{lmp}(I) \left[- (J^R_{l-1, l-2p-1} - J^R_{l-1, l-2p+1}) \times \right.$$
$$\times \bar{C}_{lpm} + (J^I_{l-1, l-2p-1} - J^I_{l-1, l-2p+1}) \bar{S}_{lpm} \big] +$$
$$+ \tfrac{1}{2} a^l \frac{1}{1 - e^2} (l - 2p) F_{lmp}(I) \left[- (J^R_{l, l-2p-1} - J^R_{l, l-2p+1}) \times \right.$$
$$\times \bar{C}_{lpm} + (J^I_{l, l-2p-1} - J^I_{l, l-2p+1}) \bar{S}_{lpm} \big], \tag{102}$$

$$A_5^{lmp} \equiv \int \frac{\partial R_{lmp}}{\partial a} \mathrm{d}M = l a^{l-1} F_{lmp}(I) (J^R_{l, l-2p} \bar{C}_{lpm} + J^I_{l, l-2p} \bar{S}_{lpm}), \tag{103}$$

and

$$A_6^{lmp} \equiv \int \frac{\partial R_{lmp}}{\partial I} \mathrm{d}M = a^l F^I_{lmp}(I) (J^R_{l, l-2p} \bar{C}_{lpm} + J^I_{l, l-2p} \bar{S}_{lpm}). \tag{104}$$

Now, if we make use of Lagrange's equations and consider Equation (81), the short-period perturbations due to a term R_{lmp} are given by

$$\Delta a_{lmp} = N_{☾}^2 a_{☾}^{2-l} \frac{2}{n^2 a} A_1^{lmn},$$

$$\Delta e_{lmp} = \frac{N_{☾}^2 a_{☾}^{2-l}}{n^2 a^2 e} \sqrt{1-e^2} \left[\sqrt{1-e^2}\, A_1^{lmp} - \left(A_2^{lmp} - I_{lmp}^{(2)} M\right)\right],$$

$$\Delta I_{lmp} = \frac{N_{☾}^2 a_{☾}^{2-l}}{n^2 a^2 \sqrt{1-e^2}} \operatorname{cosec} I \left[\left(A_2^{lmp} - I_{lmp}^{(2)} M\right) \cos I - \left(A_3^{lmp} - I_{lmp}^{(3)} M\right)\right],$$

$$\Delta M_{lmp} = -\frac{N_{☾}^2 a_{☾}^{2-l}}{n^2 a^2 e} \left[(1-e^2)\left(A_4^{lmp} - I_{lmp}^{(4)} M\right) + 2ae\left(A_5^{lmp} - I_{lmp}^{(5)} M\right)\right] + \Delta' M_{lmp},$$

$$\Delta \omega_{lmp} = \frac{N_{☾}^2 a_{☾}^{2-l}}{n^2 a^2 e \sqrt{1-e^2}} \left[-e \cot I \left(A_6^{lmp} - I_{lmp}^{(6)} M\right) + (1-e^2)\left(A_4^{lmp} - I_{lmp}^{(4)} M\right)\right],$$

$$\Delta \Omega_{lmp} = \frac{N_{☾}^2 a_{☾}^{2-l} \operatorname{cosec} I}{n^2 a^2 \sqrt{1-e^2}} \left(A_6^{lmp} - I_{lmp}^{(6)} M\right), \tag{105}$$

where

$$\Delta' M_{lmp} = -\frac{3 N_{☾}^2 a_{☾}^{2-l}}{n^2 a^2} \int A_1^{lmp} \, \mathrm{d}M \tag{106}$$

remains to be evaluated. This last involves the evaluation of the integral

$$L_{pq} = \int J_{pq} \, \mathrm{d}M = \int J_{pq} (1 - e \cos E) \, \mathrm{d}E = L_{pq}^R + i L_{pq}^I.$$

It is readily found that

$$L_{pq}^R = \int J_{pq}^R \, \mathrm{d}M = -\sum_{\substack{\gamma=-p-1 \\ (\gamma \neq 0)}}^{p+1} \frac{1}{\gamma^2} K_{pq\gamma} \cos \gamma E + K_{pq0} \frac{E^2}{2} - e K_{pq0} (E \sin E + \cos E) + \frac{e}{2} \sum_{\substack{\gamma=-p-1 \\ (\gamma \neq 0,\, \gamma \neq -1)}}^{p+1} \frac{1}{\gamma(\gamma+1)} K_{pq\gamma} \times \cos(\gamma+1) E + \frac{e}{2} \sum_{\substack{\gamma=-p-1 \\ (\gamma \neq 0,\, \gamma \neq 1)}}^{p+1} \frac{1}{\gamma(\gamma-1)} K_{pq\gamma} \cos(\gamma-1) E, \tag{107}$$

and

$$L_{pq}^I = \int J_{pq}^I \, \mathrm{d}M = -\sum_{\substack{\gamma=-p-1 \\ (\gamma \neq 0)}}^{p+1} \frac{1}{\gamma^2} K_{pq\gamma} \sin \gamma E + \frac{e}{2} \left(K_{pq1} - K_{p,q,-1}\right) E +$$

$$+\frac{e}{2}\sum_{\substack{\gamma=-p-1\\(\gamma\neq 0,\,\gamma\neq -1)}}^{p+1}\frac{1}{\gamma(\gamma+1)}K_{pq\gamma}\sin(\gamma+1)E+\frac{e}{2}\sum_{\substack{\gamma=-p-1\\(\gamma\neq 0,\,\gamma\neq 1)}}^{p+1}\frac{1}{\gamma(\gamma-1)}\times$$
$$\times K_{pq\gamma}\sin(\gamma-1)E. \qquad (108)$$

Therefore,

$$\begin{aligned}\Delta' M_{lmp} = & -\tfrac{3}{2}N_{☾}^2 a_{☾}^{2-l}\frac{la^{l-2}}{n^2}\frac{e}{\sqrt{1-e^2}}F_{lmp}(I)\,[(L^I_{l-1,\,l-2p-1}+\\ & -L^I_{l-1,\,l-2p+1})\,\bar{C}_{lpm}+(L^R_{l-1,\,l-2p-1}-L^R_{l-1,\,l-2p+1})\,\bar{S}_{lpm}]+\\ & -3N_{☾}a_{☾}^{2-l}(l-2p)\frac{a^{l-2}}{n^2}\sqrt{1-e^2}\,F_{lmp}(I)\,(L^R_{l-2,\,l-2p}\bar{S}_{lpm}+\\ & -L^I_{l-2,\,l-2p}\bar{C}_{lpm}). \end{aligned} \qquad (109)$$

9. Computational Procedure for Short-Period Perturbations of Low Satellites

Corresponding to a particular term R_{lmp} (Equation (74)), the short-period perturbations (☾, ω, Ω fixed) are computed as follows.

Given mean elements a, e, I, $e_{☾}$, $I_{☾}$, and ε, we compute

(1) $N_{☾}^2 a_{☾}^{2-l}$
(2) β (22)
(3) Given $M_{☾}$, compute $E_{☾}, f_{☾}$
(4) Given $\omega_{☾}$, $\Omega_{☾}$, and $a_{☾}$, compute $r_{☾}$, $v_{☾}$
(5) $\alpha_{☾}$, $\delta_{☾}$ (7)
(6) α', δ' (10)
(7) $P_l^m(\sin\delta')\,{}^{\cos}_{\sin}\,m\delta'$
(8) A_l^m, B_l^m (5)
(9) C_l^m, S_l^m (71)
(10) Given M, ω, and Ω, compute E
(11) $\bar{\theta}_{lpm}$ (26)
(12) $\bar{C}_{lpm}$, $\bar{S}_{lpm}$ (87), (88)
(13) $F_{lmp}(I)$ (9)
(14) $F^I_{lmp}(I)$ (50)
(15) $X_{0,\,0}^{k,\,j}(\beta)$ (23) or (24)

k	j
l	$l-2p$
$l-2$	$l-2p$
$l-1$	$l-2p$
$l-1$	$l-2p-1$
$l-1$	$l-2p+1$
l	$l-2p-1$
l	$l-2p+1$

(16) $K_{pq\gamma}(e)$ (96), $\gamma = -p-1, -p, \ldots, p, p+1$

p	q
$l-1$	$l-2p-1$
$l-1$	$l-2p+1$
$l-2$	$l-2p$
l	$l-2p$
$l-1$	$l-2p$
l	$l-2p-1$
l	$l-2p+1$

(17) $I_1^{l,p}, I_2^{l,p}, \ldots, I_5^{l,p}$ (82) through (86)
(18) $I_{lmp}^{(2)}, I_{lmp}^{(3)}, \ldots, I_{lmp}^{(6)}$ (89) through (93)
(19) J_{pq}^R, J_{pq}^I (97), (98) (same range for p, q as in (15))
(20) $A_1^{lmp}, A_2^{lmp}, \ldots, A_6^{lmp}$ (99) through (104)
(21) L_{pq}^R, L_{pq}^I (107), (108)

p	q
$l-1$	$l-2p-1$
$l-1$	$l-2p+1$
$l-2$	$l-2p$

(22) $\Delta' M_{lmp}$ (109
(23) $\Delta e_{lmp}, \ldots, \Delta\Omega_{lmp}$ (105)
Short-period perturbations completed.

10. Short-Period Perturbations for High Satellites with Small Eccentricity

When the satellite is high, for example, close to a 24-h period, the Moon can no longer be considered fixed during a few revolutions of the satellite. Here we consider also the variations of ω, Ω, in contrast to what we have done in Section 8. In this case, the integrals found in that section have to take these variations into account. This can be done only if the eccentricity of the satellite is small so that power series in e will converge rapidly. Thus, the disturbing function obtained in Equation (19) has to be developed in terms of M, $M_☾$.

The following expansions are well known:

$$\left(\frac{a_☾}{r_☾}\right)^{l+1} \frac{\sin}{\cos}\left[(l-2q)\, v_☾ + s\left(\Omega_☾ + \frac{\pi}{2}\right)\right] = \sum_{k=-\infty}^{\infty} G_{lqk}(e_☾) \frac{\sin}{\cos} \theta_{lsqk}^☾, \tag{110}$$

and

$$\left(\frac{r}{a}\right)^{l} \frac{\sin}{\cos}[(l-2p)\, v + m\Omega] = \sum_{j=-\infty}^{\infty} H_{lpj}(e) \frac{\sin}{\cos} \theta_{lmpj}, \tag{111}$$

where

$$\theta^{☾}_{lsqk} = (l - 2q)\,\omega_☾ + (l - 2q + k)\,M_☾ + s\left(\Omega_☾ + \frac{\pi}{2}\right), \tag{112}$$

$$\theta_{lmpj} = (l - 2p)\,\omega + (l - 2p + j)\,M + m\Omega, \tag{113}$$

and $H_{lpj}(e)$ are Kaula's (1962) coefficients. These can also be written in terms of Hansen's coefficients by

$$H_{lpj}(e) = X^{l,\,l-2p}_{l-2p+j}(\beta).$$

The coefficients $G_{lqk}(e_☾)$ have been defined in Equations (33), (34), and (35) or (36).

The classical expressions for Kaula's coefficients are (e.g., Plummer, 1960):

$$H_{lpj}(e) = (1 + \beta^2)^{-l-1} \sum_{i=-\infty}^{\infty} J_i[(l - 2p + j)\,e]\,X^{l,\,l-2p}_{l-2p+j,\,i} = O(e^{|j|}), \tag{114}$$

where $J_i(x)$ are the usual Bessel functions and $X^{l,m}_{k,i}(\beta)$ are given in terms of hypergeometric series (which always terminate), as follows: for $k\text{-}i\text{-}m \geqslant 0$,

$$X^{l,m}_{k,i} = (-\beta)^{k-i-m}\binom{l+1-m}{k-i-m} F(k - i - l - 1, m - l - 1, k - i - m + 1; \beta^2) \tag{115}$$

and, for $k\text{-}i\text{-}m \leqslant 0$,

$$X^{l,m}_{k,i} = (-\beta)^{-k+i+m}\binom{l+1+m}{-k+i+m} F(-k + i - l - 1, m - l - 1, -k + i + m + 1; \beta^2). \tag{116}$$

It follows that

$$R_l = \sum_{m=0}^{l}\sum_{s=0}^{l}\sum_{p=0}^{l}\sum_{q=0}^{l}\sum_{k=-\infty}^{\infty}\sum_{j=-\infty}^{\infty} R_{lmspqkj},$$

where

$$R_{lmspqkj} = \frac{(-1)^m\,\epsilon_m\epsilon_s\,(l-s)!}{(l+m)!}\,a^l F_{lmp}(I)\,F_{lsq}(I_☾)\,G_{lqk}(e_☾)\,\times \times H_{lpj}(e)\left[(-1)^{l+m-s}\,U_l^{m,-s}\cos(\theta_{lmpj} + \theta^{☾}_{lsqk}) + U_l^{m,s}\cos(\theta_{lmpj} - \theta^{☾}_{lsqk})\right]. \tag{117}$$

In what follows, it will be clear that we should have

$$l - 2p + j \neq 0.$$

We can make use of the fact that

$$\frac{\partial F(a, b, c; x)}{\partial x} = \frac{ab}{c} F(a + 1, b + 1, c + 1; x)$$

and

$$\frac{\partial J_i(x)}{\partial x} = \tfrac{1}{2}[J_{i-1}(x) - J_{i+1}(x)]$$

and arrive at the following relations:

$$Y_k^{l,m} \equiv \frac{\partial X_k^{l,m}}{\partial \beta} \equiv \sum_{i<k-m} \left\{ J_i(ke)\, Y_{k,i}^{l,m} + \frac{e\sqrt{1-e^2}}{\beta}\frac{k}{2} \times \right.$$
$$\left. \times \left[J_{i-1}(ke) - J_{i+1}(ke)\right] X_{k,i}^{l,m} \right\} + \sum_{i \geqslant k-m} \times$$
$$\times \left\{ J_i(ke)\, Z_{k,i}^{l,m} + \frac{e\sqrt{1-e^2}}{\beta}\frac{k}{2}\left[J_{i-1}(ke) - J_{i+1}(ke)\right] X_{k,i}^{l,m} \right\}, \tag{118}$$

where, for $i<k-m$,

$$Y_{k,i}^{l,m} \equiv \frac{\partial X_{k,i}^{l,m}}{\partial \beta} = -(k-i-m)\frac{1}{\beta} X_{k,i}^{l,m} - 2(-\beta)^{k-i-m+1} \binom{l+1-m}{k-i-m} \times$$
$$\times \frac{(k-i-l-1)(-m-l-1)}{k-i-m+1} \times$$
$$\times F(k-i-l, -m-l, k-i-m+2; \beta^2) \tag{119}$$

and, for $i \geqslant k-m$,

$$Z_{k,i}^{l,m} \equiv \frac{\partial X_{k,i}^{l,m}}{\partial \beta} = -(-k+i+m)\frac{1}{\beta} \times$$
$$\times X_{k,i}^{l,m} - 2(-\beta)^{-k+i+m+1} \binom{l+1-m}{-k+i+m} \times$$
$$\times \frac{(-k+i-l-1)(m-l-1)}{-k+i+m+1} \times$$
$$\times F(-k+i-l, m-l, -k+i+m+2; \beta^2). \tag{120}$$

Finally,

$$N_{lpj} \equiv \frac{\partial H_{lpj}}{\partial e} = \frac{\beta}{e\sqrt{1-e^2}}\, Y_{l-2p+j}^{l,l-2p}. \tag{121}$$

Let us consider the definitions

$$D_{lmspqkj}^{+} = (l-2p)\,\dot{\omega} + (l-2p+j)\,\dot{M} + m\dot{\Omega} + (l-2q)\,\dot{\omega}_☾ +$$
$$+ (l-2q+k)\dot{M}_☾ + s\dot{\Omega}_☾ \tag{122}$$

and

$$D_{lmspqkj}^{-} = (l-2p)\,\dot{\omega} + (l-2p+j)\,\dot{M} + m\dot{\Omega} +$$
$$-(l-2q)\,\dot{\omega}_☾ - (l-2q+k)\,\dot{M}_☾ - s\dot{\Omega}_☾. \tag{123}$$

We easily establish that

$$B_{lmspqkj}^{(1)} \equiv \int \frac{\partial R_{lmspqkj}}{\partial M}\,dt = R_{lmspqkj} \left\{ \begin{array}{l} U_l^{m,s} \to U_l^{m,s}(l-2p+j)/D_{lmspqkj}^{-} \\ U_l^{m,-s} \to U_l^{m,-s}(l-2p+j)/D_{lmspqkj}^{+} \end{array} \right\}, \tag{124}$$

$$B_{lmspqkj}^{(2)} \equiv \int \frac{\partial R_{lmspqkj}}{\partial \omega}\,dt = R_{lmspqkj} \left\{ \begin{array}{l} U_l^{m,s} \to U_l^{m,s}(l-2p)/D_{lmspqkj}^{-} \\ U_l^{m,-s} \to U_l^{m,-s}(l-2p)/D_{lmspqkj}^{+} \end{array} \right\}, \tag{125}$$

$$B^{(3)}_{lmspqkj} \equiv \int \frac{\partial R_{lmspqkj}}{\partial \Omega}\, dt = R_{lmspqkj}\left\{\begin{matrix} U^{m,s}_l \to U^{m,s}_l m/D^-_{lmspqkj} \\ U^{m,-s}_l \to U^{m,-s}_l m/D^+_{lmspqkj} \end{matrix}\right\}, \tag{126}$$

$$B^{(4)}_{lmspqkj} \equiv \int \frac{\partial R_{lmspqkj}}{\partial e}\, dt = \frac{a}{l}\, B^{(5)}_{lmspqkj}\{H_{lpj} \to N_{lpj}\}, \tag{127}$$

$$B^{(5)}_{lmspqkj} \equiv \int \frac{\partial R_{lmspqkj}}{\partial a}\, dt = \frac{l}{a}\, R_{lmspqkj}\left\{\begin{matrix} U^{m,-s}_l \to U^{m,-s}_l/D^+_{lmspqkj} \\ U^{m,s}_l \to U^{m,s}_l/D^-_{lmspqkj} \\ \cos \to \sin \end{matrix}\right\}, \tag{128}$$

$$B^{(6)}_{lmspqkj} \equiv \int \frac{\partial R_{lmspqkj}}{\partial I}\, dt = \frac{a}{l}\, B^{(5)}_{lmspqkj}\{F_{lmp} \to F^I_{lmp}\}, \tag{129}$$

and

$$\begin{aligned} \Delta' M_{lmspqkj} &= -\frac{3}{a^2} N^2_{☾} a^{2-l}_{☾} \int B^{(1)}_{lmspqkj}\, dt = \\ &= -\frac{3}{a^2} N^2_{☾} a^{2-l}_{☾} R_{lmspqkj}\left\{\begin{matrix} U^{m,s}_l \to U^{m,s}_l/(D^-_{lmspqkj})^2 \\ U^{m,-s}_l \to U^{m,-s}_l/(D^+_{lmspqkj})^2 \\ \cos \to \sin \end{matrix}\right\}, \end{aligned} \tag{130}$$

Finally, the short-period perturbations are given, for all terms for which

$$l - 2p + j \neq 0, \tag{131}$$

by

$$\begin{aligned} \Delta a_{lmspqkj} &= N^2_{☾} a^{2-l}_{☾} \frac{2}{na} B^{(1)}_{lmspqkj}, \\ \Delta e_{lmspqkj} &= N^2_{☾} a^{2-l}_{☾} \frac{\sqrt{1-e^2}}{na^2 e} [\sqrt{1-e^2}\, B^{(1)}_{lmspqkj} - B^{(2)}_{lmpsqkj}], \\ \Delta I_{lmspqkj} &= N^2_{☾} a^{2-l}_{☾} \frac{\operatorname{cosec} I}{na^2 \sqrt{1-e^2}} [B^{(2)}_{lmspqkj} \cos I - B^{(3)}_{lmspqkj}], \\ \Delta M_{lmspqkj} &= -N^2_{☾} a^{2-l}_{☾} \frac{1}{na^2 e} [(1-e^2)\, B^{(4)}_{lmspqkj} + 2eB^{(5)}_{lmspqkj}] + \Delta' M_{lmspqkj}, \\ \Delta \omega_{lmspqkj} &= N^2_{☾} a^{2-l}_{☾} \frac{1}{na^2 e \sqrt{1-e^2}} [-eB^{(6)}_{lmspqkj} \cot I + (1-e^2)\, B^{(4)}_{lmspqkj}], \\ \Delta \Omega_{lmspqkj} &= N^2_{☾} a^{2-l}_{☾} \frac{\operatorname{cosec} I}{na^2 \sqrt{1-e^2}} B^{(6)}_{lmspqkj}, \end{aligned} \tag{132}$$

which completes the calculations.

11. Computational Procedure for Short-Period Perturbations of High Satellites with Small Eccentricity

The sequence of calculations to obtain short-period perturbations due to a particular term, $R_{lmspqkj}$, of the disturbing function (see (117)) is now given.

Given mean elements $a, e, I, M_☾, \omega_☾, \Omega_☾, M, \omega, \Omega, \varepsilon, \dot{M}_☾, \dot{\omega}_☾, \dot{\Omega}_☾, \dot{M}, \dot{\omega}, \dot{\Omega}, \beta$, compute, for any term $(l, m, s, p, q, k, j), l-2p+j \neq 0$:

(1) $\theta^☾_{lsqk}, \theta_{lmpj}$ (112), (113)
(2) $J_i[(l-2p+j)e]$ (Bessel function (34)) to the approximation required
(3) $X^{l,l-2p}_{l-2p+j,i}(\beta)$ (115) or (116)
(4) $F_{lmp}(I)$ (9)
(5) $F_{lsq}(I_☾)$ (9)
(6) $G_{lqk}(e_☾)$ (33), (34), (35), and (36)
(7) $H_{lpj}(e)$ (114)
(8) $Y^{l,m}_{k,i}(\beta)$ (119)
(9) $Z^{l,m}_{k,i}(\beta)$ (120)
(10) $Y^{l,m}_k(\beta)$ (118)
(11) $N_{lpj}(\beta)$ (121)
(12) $D^{\pm}_{lmspqkj}$ (122), (123)
(13) $U^{m,s}_l\ U^{m,-s}_l$ (12)
(14) $B^{(i)}_{lmspqkj}, i=1, 2, \ldots, 6$ (125) through (129)
(15) $\Delta' M_{lmspqkj}$ (130)
(16) $\Delta a_{lmspqkj}, \ldots, \Delta\Omega_{lmspqj}$ (132)

Complete short-period perturbations.

12. Remarks on Solar Perturbations

The previous formulations apply as well to solar perturbations, which are about of the same order of magnitude. In fact, for the Sun,

$$R = \frac{Gm_\odot}{r_\odot} \sum_{l \geqslant 2} \left(\frac{r}{r_\odot}\right)^l P_l(\cos\psi'_\odot), \tag{133}$$

so that

$$Gm_\odot = \frac{m_\odot}{m_☾ + m_\oplus + m_\odot} n^2_\odot a^3_\odot \simeq n^2_\odot a^3_\odot$$

or

$$Gm_\odot = N^2_\odot a^3_\odot, \qquad N^2_\odot \simeq 0.75 \times 10^{-5}\ \text{rev}^2\ \text{day}^{-2},$$

which is of the same size as $N^2_☾$. For the Sun, we have (to the mean equinox of date):

$$\begin{aligned}
\omega_\odot &= 281°13'15''.0 + 6189''.03\,T + 1''.63\,T^2 + 0''.012\,T^3, \\
M_\odot &= 358°28'33''.0 + 129\,596\,579''.10\,T - 0''.54\,T^2 - 0''.012\,T^3, \\
e_\odot &= 0.016\,751\,04 \text{ (supposed constant)}, \\
a_\odot &= 1.000\,001\,29 \text{ (astronomical units)}, \\
n_\odot &= 3548''.192\,83\ \text{day}^{-1}.
\end{aligned}$$

We can consider $I_\odot$, $\Omega_\odot$ to be zero. The mean inclination with respect to the equator is ε. For that matter, it could be considered a function of time, but such precision is hardly necessary. The disturbing function is given by Equation (8), while (2) is written

$$R = \sum_{l \geqslant 2} N_{\odot}^2 a_{\odot}^{2-l} R_l . \tag{134}$$

The transformation (10) is not necessary, so that the coefficients A_l^m, B_l^m (Equation (5)) retain their original form by using $I' = \varepsilon$, the inclination of the orbit of the Sun with respect to the equator. It follows that

$$R_l = a^l \left(\frac{r}{a}\right)^l \left(\frac{a_{\odot}}{r_{\odot}}\right)^{l+1} \sum_{m=0}^{l} \sum_{p=0}^{l} \sum_{q=0}^{l} \epsilon_m \frac{(l-m)!}{(l+m)!} F_{lmp}(I) F_{lmq}(\varepsilon) \times \\ \times \cos\left[(l-2p) v - (l-2q) v_{\odot} + m\Omega\right]. \tag{135}$$

The secular and long-period part of this is

$$\bar{R}_l = a^l \sum_{m=0}^{l} \sum_{p=0}^{l} \sum_{q=0}^{l} \sum_{k=-\infty}^{\infty} \epsilon_m \frac{(l-m)!}{(l+m)!} F_{lmp}(I) F_{lmq}(\varepsilon) \times \\ \times (1+\beta^2)^{-l-1} H_{lp(2p-l)}(\beta) G_{lqk}(e_{\odot}) \times \\ \times \cos\left[(l-2p)\omega - (l-2q)\omega_{\odot} - (l-2q+k) M_{\odot} + m\Omega\right], \tag{136}$$

which for the Sun is used in place of Equation (37). More precisely:

$$\bar{R}_{lmpqk} = \epsilon_m \frac{(l-m)!}{(l+m)!} F_{lmp}(I) F_{lmq}(\varepsilon) (1+\beta^2)^{-l-1} H_{lp(2p-l)}(\beta) \times \\ \times G_{lqk}(e_{\odot}) C_{lpqmk}, \tag{137}$$

where

$$C_{lpqmk} = \cos\left[(l-2p)\omega - (l-2q)\omega_{\odot} - (l-2q+k) M_{\odot} + m\Omega\right]. \tag{138}$$

From this point on, all formulas developed for the Moon can be easily adapted, a task not worth undertaking here. The complete expressions are given by Kaula (1962), and the computational procedure is similar to the ones given in the previous sections.

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