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Spectral Radius Properties for Layer Potentials Associated with the Elastostatics and Hydrostatics Equations in Nonsmooth Domains

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ABSTRACT. By producing a L^2 convergent Neumann series, we prove the invertibility of the elastostatics and hydrostatics boundary layer potentials on arbitrary Lipschitz domains with small Lipschitz character and 3D polyhedra with large dihedral angles.

1. Introduction

Much progress has been made in the last two decades in the direction of employing the classical method of layer potentials in the treatment of elliptic boundary value problems in non-smooth domains in the Euclidean setting.

The essence of the method resides in reducing the whole problem to solving a system of integral equations over the boundary of the domain. In the case of domains with smooth boundaries, one is typically able to reduce matters to inverting operators of the form "identity + weakly singular" which are readily treatable via Fredholm theory. As non-smooth domains no longer yield boundary integral operators of compact type, new tools had to be developed and new approaches had to be designed to handle this case.

Following the breakthrough in [1], which settled the sensible issue of the boundedness of such layer operators in L^p spaces ($1 < p < \infty$) on arbitrary Lipschitz surfaces, there have been spectacular applications to many classical PDEs of mathematical physics in Lipschitz domains. For instance, the Laplace equation has been treated via layer potentials in [3, 25] (following the work in [2, 8, 13]), the Lamé system of elastostatics in [4], the Stokes system of hydrodynamics in [9]

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and Maxwell's system of electromagnetism in [22]. For a more detailed account of developments in this active area of research see, e.g., the excellent survey in [16]. There is also a rich literature for non-smooth domains with isolated singularities and we refer the reader to the expositions in [21] and [17], as well as to the references therein.

The aim of this paper is to investigate conditions under which the inverses of such boundary layer potentials can be expanded in a (L^2 norm convergent) Neumann series. Since, as alluded to before, our operators have the form $I + K$, this is equivalent to showing that $\rho(K)$, the spectral radius of K on L^2, is < 1. For arbitrary Lipschitz domains, this issue is occasionally referred to as "the spectral radius conjecture". Of course, if the domain is smooth, then matters come down to understanding the point spectrum of K, which is a much simpler task. For work on the spectral radius for the harmonic layer potential in non-smooth domains, see [10].

Our primary interest lies with the (appropriately defined) elastostatics and hydrostatics layer potentials in non-smooth domains. For these operators, call them K_{Lame}, K_{Stokes}, we produce partial results to the aforementioned question, to the effect that $\rho(K_{Lame}) < 1$, $\rho(K_{Stokes}) < 1$ in arbitrary Lipschitz domains with small Lipschitz constant (cf. Theorem 1) and three-dimensional polyhedra satisfying certain size restrictions for their dihedral angles. More specifically, in the case of the Lamé system in 3D polyhedra, we ask that

$$\left|\cos\left(\frac{\alpha}{2}\right)\right| < \left(\frac{3\mu+\lambda}{8\mu+6\lambda}\right), \tag{1.1}$$

where λ and μ are the usual Lamé coefficients. Whereas in the case of Stokes's system, we require

$$\left|\cos\left(\frac{\alpha}{2}\right)\right| < \frac{1}{6}. \tag{1.2}$$

See Theorem 4 for a complete statement. The tools employed to prove these results are those of harmonic analysis and Mellin transform.

Another perspective from which Theorem 4 can be understood comes from regarding the Lamé system as a perturbation of the vector Laplacian. This way, for any 3D polyhedron, there exists $\varepsilon > 0$ such that K_{Lame} associated to the system $\Delta\vec{u} + \varepsilon \nabla \operatorname{div} \vec{u} = 0$ has $\rho(K_{Lame}) < 1$. This work is an extension to the case of 3D systems of PDEs of results in [6] (cf. also [7, 20, 24] for the two-dimensional case) where it has been shown that $\rho(K_{Laplace}) < 1$ for arbitrary 3D polyhedra. In fact, we are able to recover this particular result in the present setting. This occurs for the choice $\lambda = -\mu$ for which the Lamé system reduces to the vector Laplacian. The point is that, in this case, (1.1) is automatically satisfied. However, the extent to which the restrictions (1.1) and (1.2) can be improved remains an open problem for the moment. Another interesting open problem is to extend these results to the higher dimensional setting.

The conditions (1.1) and (1.2) can be regarded as constraints on the size of the Lipschitz character of the domain so that the spectral radius conjecture holds for the operators under discussion. For practical purposes, it is important that they are explicit and easy to check. Indeed, we believe that our results may also prove useful for the numerical treatment of such systems of PDEs in nonsmooth domains, an issue to which we shall, hopefully, return soon.

The layout of this paper is as follows. In Section 2 we review some basic definitions concerning the elastostatics and hydrostatics layer potential operators corresponding, respectively, to the pseudostress and the stress conormal derivatives. In Section 3 we discuss spectral properties of these operators in the context of Lipschitz domains with small Lipschitz constant. Finally, Section 4 deals with 3D polyhedra, a setting in which we produce the results outlined above.

2. The Elastostatics and Hydrostatics Layer Potentials

We start by reviewing *elastostatics layer potentials.* Concretely, consider the system of linear elastostatics $L\vec{u} = 0$ in an open subset of $\mathbb{R}^n$, where

$$L\vec{u} := \mu \triangle \vec{u} + (\lambda + \mu) \nabla \operatorname{div} \vec{u} .$$

The displacement $\vec{u}$ has n components and μ and λ are the Lamé moduli which are assumed to satisfy $\mu > 0$ and $-\mu \leq \lambda$. The operator L can be represented in the following notation:

$$L = A(D) = \left(a_{ij}^{kl}\partial_i\partial_j\right)_{k,l}, \tag{2.1}$$

where

$$a_{ij}^{kl} = a_{ij}^{kl}(\theta) := \mu\delta_{ij}\delta_{kl} + (\mu + \lambda - \theta)\delta_{ik}\delta_{jl} + \theta\delta_{il}\delta_{jk} .$$

Above, $\theta \in \mathbb{R}$ is arbitrary, δ_{ij} is the Kronecker symbol, and $i, j, k, l \in \{1, \dots, n\}$. Hereafter we shall use Einstein's convention for summation, i.e., an index repeating in the same expression means that we are summing with respect to that index.

Let $\Omega \subset \mathbb{R}^n$ be a Lipschitz domain and let N be the outward unit normal vector to Ω which exists almost everywhere on $\partial\Omega$. Corresponding to $A := (a_{ij}^{kl})_{i,j,k,l}$, the conormal derivative for the operator L in (2.1) is given by

$$\left(\frac{\partial \vec{u}}{\partial N_A}\right)^j := N_i a_{ik}^{jl}(\theta)\partial_k u^l = \mu\frac{\partial u^j}{\partial N} + (\mu + \lambda - \theta)N_j \operatorname{div}\vec{u} + \theta N_i\partial_j u^i ,$$

where $j = 1, \dots, n$. The special choice $\theta := \frac{\mu(\mu+\lambda)}{3\mu+\lambda}$ gives rise to the so-called pseudostress conormal derivative which has the form

$$\frac{\partial \vec{u}}{\partial \nu} := \mu \nabla \vec{u} \cdot N + \frac{\mu(\mu + \lambda)}{3\mu + \lambda}(\nabla\vec{u})^t \cdot N + \frac{(2\mu + \lambda)(\mu + \lambda)}{3\mu + \lambda}(\operatorname{div}\vec{u})N ,$$

where the superscript t indicates transposition of matrices.

Let $G = (G_{ij})_{i,j}$ be the Kelvin matrix valued fundamental solution for the system of elastostatics (see, e.g., [18]),

$$G_{ij}(X) := \frac{1}{2\mu(2\mu + \lambda)\omega_n}\left[\frac{3\mu + \lambda}{n - 2}\frac{\delta_{ij}}{|X|^{n-2}} + (\mu + \lambda)\frac{X_i X_j}{|X|^n}\right], \quad X \in \mathbb{R}^n \setminus \{0\} ,$$

where $i, j = 1, 2, \dots, n$ and ω_n is the surface area of the unit sphere in $\mathbb{R}^n$. Also, denote by $K_{\mathcal{L},p}$ the double layer elastostatics operator corresponding to the pseudostress conormal derivative on the boundary of Ω. Specifically, if we denote by G^j the jth column in the fundamental matrix G, then

$$\left(K_{\mathcal{L},p}\left(\vec{f}\right)\right)^i(P) := 2\int_{\partial\Omega}\left(\frac{\partial G^j}{\partial \nu}(P - \cdot)\right)^i(Q) f^j(Q)\, d\sigma(Q), \quad P \in \partial\Omega , \tag{2.2}$$

where $\vec{f} : \partial\Omega \to \mathbb{R}^n$ and $i = 1, \dots, n$. Also, $d\sigma$ stands for the canonical surface measure on $\partial\Omega$.

A careful computation gives that the ith component of $2\frac{\partial G^j}{\partial \nu}(X)$, denoted by $k_{\mathcal{L},p}^{ij}(X)$, is

$$\begin{aligned} k_{\mathcal{L},p}^{ij}(X) \quad := \quad & \frac{-4\mu\delta_{ij}}{\omega_n(3\mu + \lambda)}\frac{\langle X, N(X)\rangle}{|X|^n} \\ & - \frac{2n(\mu + \lambda)}{\omega_n(3\mu + \lambda)}\frac{X_i X_j\langle X, N(X)\rangle}{|X|^{n+2}}, \quad X \in \mathbb{R}^n \setminus \{0\} . \end{aligned}$$

Next, we briefly discuss *hydrostatics layer potentials*. To this end, consider the linearized, homogeneous, time independent Navier–Stokes equations, i.e., the Stokes system

$$\begin{cases} \Delta\vec{u} = \nabla p\,, \\ \operatorname{div}\vec{u} = 0\,, \end{cases} \tag{2.3}$$

in an open set in $\mathbb{R}^n$, where $\vec{u}$ is the velocity field and p is the pressure function. If we define the matrix $A = A(\theta) := (a_{ij}^{kl}(\theta))_{i,j,k,l}$ by

$$a_{ij}^{kl} = a_{ij}^{kl}(\theta) := \delta_{ij}\delta_{kl} + \theta\delta_{il}\delta_{jk}\,,$$

for $\theta \in \mathbb{R}$, then $a_{ij}^{kl}\partial_i\partial_j u^l = \Delta u^k + \theta\partial_k(\operatorname{div}\vec{u})$. Hence, any solution $\vec{u}$, p of the Stokes system (2.3) satisfies

$$a_{ij}^{kl}\partial_i\partial_j u^l = \partial_k p\,.$$

As before, let $\Omega \subset \mathbb{R}^n$ be a Lipschitz domain and denote by N the outward unit normal vector a.e. on $\partial\Omega$. The conormal derivative that corresponds to the matrix $A := (a_{ij}^{kl})_{i,j,k,l}$ is

$$\left(\frac{\partial\vec{u}}{\partial N_A}\right)^j := N_i a_{ik}^{jl}(\theta)\partial_k u^l - N_j p, \tag{2.4}$$

where $j = 1, 2, \ldots, n$. The special choice $\theta := 1$ gives the so-called stress conormal derivative (see also, e.g., [4, 19]). This derivative has a physical interpretation and it is known as the *slip condition*.

Going further, denote by $G = (G_{ij})_{i,j}$ the Kelvin matrix valued fundamental solution for the system of hydrostatics (see, e.g., [19]),

$$G_{ij}(X) := \frac{1}{2\omega_n}\left(\frac{1}{n-2}\frac{\delta_{ij}}{|X|^{n-2}} + \frac{X_i X_j}{|X|^n}\right), \quad X \in \mathbb{R}^n \setminus \{0\}\,.$$

Let $K_{\mathcal{S},s}$ be the double layer hydrostatics operator corresponding to the stress conormal derivative on the boundary of Ω. Also, set G^j for the jth column in the fundamental matrix. Then

$$\left(K_{\mathcal{S},s}\left(\vec{f}\right)\right)^i(P) := 2\int_{\partial\Omega}\left(\frac{\partial G^j}{\partial\nu}(P-\cdot)\right)^i(Q)f^j(Q)\,d\sigma(Q), \quad P \in \partial\Omega\,, \tag{2.5}$$

where $i = 1, \ldots, n$ and $\frac{\partial G^j}{\partial\nu} := \frac{\partial G^j}{\partial N_{A(1)}}$. The ith component of $2\frac{\partial G^j}{\partial\nu}(X)$, denoted by $k_{\mathcal{S},s}^{ij}(X)$, is

$$k_{\mathcal{S},s}^{ij}(X) := \frac{-2n}{\omega_n}\frac{X_i X_j\langle X, N(X)\rangle}{|X|^{n+2}} \qquad X \in \mathbb{R}^n \setminus \{0\}\,.$$

The operators $K_{\mathcal{L},p} = (K_{\mathcal{L},p}^{ij})_{i,j}$ and $K_{\mathcal{S},s} = (K_{\mathcal{S},s}^{ij})_{i,j}$, with $i, j = 1, 2, \ldots, n$, acting from $\left(L^2(\partial\Omega)\right)^n$ to $\left(L^2(\partial\Omega)\right)^n$ are $n \times n$ matrices with entries $K_{\mathcal{L},p}^{ij} \in \mathcal{L}(L^2(\partial\Omega))$ and, respectively, $K_{\mathcal{S},s}^{ij} \in \mathcal{L}(L^2(\partial\Omega))$. For any $f \in \left(L^2(\partial\Omega)\right)^n$, $f = (f^1, f^2, \ldots, f^n)$,

$$\left(K_{\mathcal{L},p}f\right)^i(P) = K_{\mathcal{L},p}^{ij}f^j(P) := \int_{\partial\Omega} k_{\mathcal{L},p}^{ij}(P-Q)f^j(Q)\,d\sigma(Q), \qquad P \in \partial\Omega\,.$$

The operators $K_{\mathcal{S},s}^{ij}$, $i, j = 1, \ldots, n$, are defined as above replacing the subscripts $\mathcal{L}$, p by $\mathcal{S}$, s. We regard $\left(L^2(\partial\Omega)\right)^n$ as a Hilbert space endowed with the inner product given by $\langle f, g\rangle := \int_{\partial\Omega} f^j(Q)\overline{g^j(Q)}\,d\sigma(Q)$ for any $f, g \in \left(L^2(\partial\Omega)\right)^n$, $f = (f^1, f^2, \ldots, f^n)$, $g = (g^1, g^2, \ldots, g^n)$. The corresponding norm on $\left(L^2(\partial\Omega)\right)^n$ is given by

$$\|f\|_{(L^2(\partial\Omega))^n}^2 := \sum_{j=1}^n \|f^j\|_{L^2(\partial\Omega)}^2\,.$$

Occasionally, we may simply write $\|f\|$ for $\|f\|_{(L^2(\partial\Omega))^n}$. Let $T = (T_{ij})_{i,j}$, $i, j \in \{1, 2, \dots, n\}$, be a generic linear and continuous operator acting from $\left(L^2(\partial\Omega)\right)^n$ to $\left(L^2(\partial\Omega)\right)^n$, $(Tf)^i = T_{ij} f^j$ for any $f \in \left(L^2(\partial\Omega)\right)^n$, $f = (f^1, f^2, \dots, f^n)$. Then its operator norm is

$$|||T||| := \sup_{\|f\|\le 1} \|Tf\|_{(L^2(\partial\Omega))^n} .$$

We end this section with a simple remark which we shall find useful in the sequel.

Remark 1. Let $T \in \mathcal{L}\left(\left(L^2(\partial\Omega)\right)^n\right)$, $T = \left(T_{ij}\right)_{i,j}$, with $i, j \in \{1, 2, \dots, n\}$. Then

$$|||T||| \le n \max_{i,j} \left\| T_{ij} \right\| ,$$

where $\|T_{ij}\|$ is the norm of the operator T_{ij} in $\mathcal{L}(L^2(\partial\Omega))$.

3. The Case of Lipschitz Domains

Call a bounded domain $\Omega \subset \mathbb{R}^n$ Lipschitz with Lipschitz constant $\le M$ if for any $P \in \partial\Omega$ there exists $r, h > 0$ and a coordinate system $\{y_1, \dots, y_n\}$ in $\mathbb{R}^n$ (isometric to the canonical one) with origin at P and a Lipschitz function $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$ with $\|\nabla\varphi\|_{L^\infty} \le M$ so that, if $C(r, h)$ denotes the cylinder $\{(y_1, \dots, y_{n-1});\ |y_j| < r \text{ all } j\} \times (0, h) \subset \mathbb{R}^n$, then

$$\begin{aligned} \Omega \cap C(r,h) &= \left\{Y = (y_1, \dots, y_n)\ ;\ \left|y_j\right| < r \text{ all } j \text{ and } y_n > \varphi\left(y_1, \dots, y_{n-1}\right)\right\} , \\ \partial\Omega \cap C(r,h) &= \left\{Y = (y_1, \dots, y_n)\ ;\ \left|y_j\right| < r \text{ all } j \text{ and } y_n = \varphi\left(y_1, \dots, y_{n-1}\right)\right\} . \end{aligned}$$

The best constant M satisfying the conditions above is denoted by char(Ω), the Lipschitz character of Ω.

The goal of this section is to establish spectral radius estimates for the operators $K_{\mathcal{L},p}$ and $K_{\mathcal{S},s}$ on $\left(L^2(\partial\Omega)\right)^n$ for Lipschitz domains with small Lipschitz character.

To state the main result of this section, consider

$$L_0^2(\partial\Omega) := \left\{ f \in L^2(\partial\Omega)\ ;\ \int_{\partial\Omega} f\, d\sigma = 0 \right\} .$$

Also, let Ψ denote the space of vector valued functions ψ on $\mathbb{R}^n$ satisfying the $n(n+1)/2$ equations

$$\partial_i \psi^j + \partial_j \psi^i = 0, \qquad 1 \le i, j \le n ,$$

and define

$$L_\Psi^2(\partial\Omega) := \left\{ f \in \left(L^2(\partial\Omega)\right)^n\ ;\ \int_{\partial\Omega} f \cdot \psi\, d\sigma = 0, \text{ for all } \psi \in \Psi \right\} .$$

Note that $L_\Psi^2(\partial\Omega)$ is a subspace of codimension $n(n+1)/2$ of $\left(L^2(\partial\Omega)\right)^n$.

We have the following theorem.

Theorem 1.

There exists $C_0 = C_0(n, \mu, \lambda) > 0$ such that for any bounded Lipschitz domain Ω with char$(\Omega) < C_0$ the following holds. For any $\beta \in \mathbb{C}$, $|\beta| \ge 1$ and $\beta \ne 1$, the operator

$$\beta I - K_{\mathcal{L},p} : \left(L^2(\partial\Omega)\right)^n \to \left(L^2(\partial\Omega)\right)^n \tag{3.1}$$

is invertible, where I is the identity operator.

Moreover, the operator $I - K_{\mathcal{L},p} \in \mathcal{L}\left(\left(L^2(\partial\Omega)\right)^n\right)$ is Fredholm with index zero and $I - K^*_{\mathcal{L},p}$: $\left(L^2_0(\partial\Omega)\right)^n \longrightarrow \left(L^2_0(\partial\Omega)\right)^n$ is invertible.

Also, there exists a constant $C_1 = C_1(n) > 0$ such that for any bounded Lipschitz domain Ω with char $(\Omega) < C_1$ the following holds. For any $\beta \in \mathbb{C}$, $|\beta| \geq 1$ and $\beta \neq 1$ the operator

$$\beta I - K_{\mathcal{S},s} : \left(L^2(\partial\Omega)\right)^n \to \left(L^2(\partial\Omega)\right)^n \tag{3.2}$$

is invertible.

Furthermore, the operator $I - K_{\mathcal{S},s} \in \mathcal{L}\left(\left(L^2(\partial\Omega)\right)^n\right)$ is Fredholm with index zero and $I - K^*_{\mathcal{S},s} : L^2_\Psi(\partial\Omega) \longrightarrow L^2_\Psi(\partial\Omega)$ is invertible.

The Fredholmness (with index zero) of the operators $\beta I - K_{\mathcal{L},p}$ and $\beta I - K_{\mathcal{S},s}$ is obtained by estimating the operator norm of $K_{\mathcal{L},s}$ and $K_{\mathcal{S},s}$, whereas the surjectivity of these operators is dealt with separately, appealing to the equation itself.

Before presenting the proof of Theorem 1 let us discuss certain corollaries of it. First, we need some more notation. Let X be a Banach space and $\mathcal{L}(X)$ be the space of linear and continuous operators $T : X \longrightarrow X$. For $T \in \mathcal{L}(X)$, denote by $\sigma(T; X)$ the spectrum of the operator T and by $\sigma_e(T; X)$ its essential spectrum, i.e.,

$$\sigma(T; X) := \{\beta \in \mathbb{C}\,;\ \beta I - T \ \text{ is not invertible on } X\}\ ,$$

and

$$\sigma_e(T; X) := \{\beta \in \mathbb{C}\,;\ \beta I - T \ \text{ is not Fredholm on } X\}\ .$$

Also, let $\rho(T; X) := \sup\{|\beta|\,;\ \beta \in \sigma(T; X)\}$ be the spectral radius of the operator T (i.e., the radius of the smallest closed circular disc centered at the origin which contains $\sigma(T; X)$). Denote by $T^* \in \mathcal{L}(X^*)$ the adjoint operator of $T \in \mathcal{L}(X)$. Finally, for $D \subset \mathbb{C}$ let $[D] := \mathbb{C} \setminus D_\infty$, where D_∞ is the unbounded connected component of $\mathbb{C} \setminus D$.

The following is well known (cf., e.g, [14, p. 102]).

Proposition 1.

Let X be a Banach space and $T \in \mathcal{L}(X)$. Then the following hold.

(1) The spectrum and the essential spectrum of the operator T are compact sets in X. Moreover $\sigma_e(T; X) \subset \sigma(T; X)$.

(2) The set $\partial\sigma(T; X) \setminus \sigma_e(T; X)$ contains only isolated points. Moreover any such point is an eigenvalue of the operator T.

(3) If $Y \subset X$ is a closed subspace invariant under T, then $\sigma(T; Y) \subset [\sigma(T; X)]$.

(4) The spectra of the operator T and its adjoint T^* satisfy $\sigma(T^*; X^*) = \overline{\sigma(T; X)}$ and $\sigma_e(T^*; X^*) = \overline{\sigma_e(T; X)}$. Here the bar denotes the usual complex conjugation.

Several immediate consequences of Theorem 1 and Proposition 1 are as follows. If Ω is a bounded Lipschitz domain in $\mathbb{R}^n$ with Lipschitz character sufficiently small, then

(i) We have $\sigma_e\left(K_{\mathcal{L},p};\ \left(L^2(\partial\Omega)\right)^n\right) \subset D_r(0)$ for some $0 < r < 1$, where $D_r(0)$ stands for the closed disc of radius r centered at the origin.

Also, $\sigma_e\left(K_{\mathcal{S},s};\ \left(L^2(\partial\Omega)\right)^n\right) \subset D_r(0)$ for some $0 < r < 1$,

(ii) Since $\partial\sigma\left(K_{\mathcal{L},p};\ \left(L^2(\partial\Omega)\right)^n\right) \setminus \sigma_e\left(K_{\mathcal{L},p};\ \left(L^2(\partial\Omega)\right)^n\right)$ contains only isolated points it follows that

$$\sigma\left(K_{\mathcal{L},p};\ \left(L^2(\partial\Omega)\right)^n\right) \subseteq \{1\} \cup D_r(0)\ ,$$

for some $0 < r < 1$.

Also,

$$\sigma\left(K_{\mathcal{S},s};\left(L^2(\partial\Omega)\right)^n\right) \subseteq \{1\} \cup D_r(0)\,,$$

for some $0 < r < 1$.

(iii) We have $\sigma\left(K^*_{\mathcal{L},p};\left(L^2_0(\partial\Omega)\right)^n\right) \subset D_r(0)$ for some $0 < r < 1$ and, hence, the spectral radius of $K^*_{\mathcal{L},p}$ on $\left(L^2_0(\partial\Omega)\right)^n$ is strictly less than one. In particular

$$\left(I - K^*_{\mathcal{L},p}\right)^{-1} = \sum_{j=0}^{\infty}(K^*_{\mathcal{L},p})^j\,,$$

where the series converges absolutely in the operator norm on $\left(L^2_0(\partial\Omega)\right)^n$.

Furthermore, $\sigma\left(K^*_{\mathcal{S},s};L^2_\Psi(\partial\Omega)\right) \subset D_r(0)$ for some $0 < r < 1$ and, hence, the spectral radius of $K^*_{\mathcal{S},s}$ on $L^2_\Psi(\partial\Omega)$ is strictly less than one. In particular

$$\left(I - K^*_{\mathcal{S},s}\right)^{-1} = \sum_{j=0}^{\infty}(K^*_{\mathcal{S},s})^j\,,$$

where the series converges absolutely in the operator norm on $L^2_\Psi(\partial\Omega)$.

(iv) By passing to the dual we get that $\sigma\left(K_{\mathcal{L},p};\left(L^2(\partial\Omega)/\mathbb{R}\right)^n\right) \subset D_r(0)$ for some $0 < r < 1$. Hence, the spectral radius of $K_{\mathcal{L},p}$ on $\left(L^2(\partial\Omega)/\mathbb{R}\right)^n$ is strictly less than one. In particular

$$\left(I - K_{\mathcal{L},p}\right)^{-1} = \sum_{j=0}^{\infty}(K_{\mathcal{L},p})^j\,,$$

where the series converges absolutely in the operator norm on $\left(L^2(\partial\Omega)/\mathbb{R}\right)^n$. This is relevant for the Dirichlet problem for the Lamé system.

Also, $\sigma\left(K_{\mathcal{S},s};\left(L^2_\Psi(\partial\Omega)\right)^*\right) \subset D_r(0)$ for some $0 < r < 1$. Hence, the spectral radius of $K_{\mathcal{S},s}$ on $\left(L^2_\Psi(\partial\Omega)\right)^*$ is strictly less than one. In particular

$$\left(I - K_{\mathcal{S},s}\right)^{-1} = \sum_{j=0}^{\infty}(K_{\mathcal{S},s})^j\,,$$

where the series converges absolutely in the operator norm on $\left(L^2_\Psi(\partial\Omega)\right)^*$. This is relevant for the Dirichlet problem for the Stokes system.

Before we begin the actual proof of Theorem 1, we pause to record a version of Theorem 1.10 in [12] which is well suited for our purposes, i.e., when it comes to estimating the operator norms of certain layer potentials on Lipschitz boundaries. To state it, recall that if X,Y are two metric spaces and $0 < \alpha \leq 1$, then

$$Lip_\alpha(X,Y) := \left\{f : X \to Y;\ \text{there is } C > 0 \text{ such that } \sup_{x,y\in X}\frac{|f(x)-f(y)|}{|x-y|^\alpha} \leq C\right\}.$$

Theorem 2 [12].

Let $B \in Lip_1(\mathbb{R}^{n-1}, \mathbb{R})$ and $\Lambda \in Lip_\alpha(S^{n-2}, \mathbb{R})$ for some $0 < \alpha \leq 1$ and $F \in C^\infty(\mathbb{R}, \mathbb{R})$ with F and all its derivatives belonging to $L^1(\mathbb{R})$. Here S^{n-2} stands for the unit sphere in $\mathbb{R}^{n-1}$. Set

$$\begin{aligned} Tf(x) &= p.v. \int_{\mathbb{R}^{n-1}} (B(x) - B(y) - \nabla B(y) \cdot (x-y)) \cdot \\ & F\left(\frac{B(x) - B(y)}{|x-y|}\right) \frac{\Lambda(x-y)}{|x-y|^n} f(y) dy\,, \end{aligned}$$

for almost every $x \in \mathbb{R}^{n-1}$. Furthermore, assume that F and Λ are both odd or both even and that $F(t) \leq C(1+|t|)^{-1}$. Then there exists $C(n, F, \Lambda) > 0$ and $\nu > 0$ such that

$$\|Tf\|_{L^2(\mathbb{R}^{n-1})} \leq C(n, F, \Lambda) \|\nabla B\|_{L^\infty(\mathbb{R}^{n-1})} \left(1 + \|\nabla B\|_{L^\infty(\mathbb{R}^{n-1})}\right)^\nu \|f\|_{L^2(\mathbb{R}^{n-1})}\,,$$

where $C(n, F, \Lambda) > 0$ depends solely on n, F and Λ.

After these preparations, we are ready to present the

Proof of Theorem 1. We proceed in two steps.

Step I: Fredholmness with index zero outside the unit disk. Let us first note that matters regarding Fredholmness with index zero can be reduced to the graph case, (i.e., when Ω is the domain above the graph of a Lipschitz function $\Phi : \mathbb{R}^{n-1} \longrightarrow \mathbb{R}$) by the following localization argument.

Let $(\Upsilon_i)_{i=1,\ldots,I}$ be an open covering of $\partial\Omega$ such that for any $i = 1, \ldots, I$ there exists a coordinate system in a neighborhood $U_i \subseteq \mathbb{R}^n$ of Υ_i and a Lipschitz function $\Phi_i : \mathbb{R}^{n-1} \longrightarrow \mathbb{R}$ such that $\Upsilon_i = \text{graph}\,\Phi_i \cap U_i$ and $U_i \cap \Omega$ lies above the graph of Φ_i. Let K stand for either one of the operators $K_{\mathcal{L},p}$ or $K_{\mathcal{S},s}$ and denote by K_i, $i = 1, \ldots, I$ the corresponding layer potential on $\Gamma_i := \text{graph}\,\Phi_i$.

Lemma 1.

Assume that for each $i = 1, \ldots, I$ and any $\beta \in \mathbb{C}$, $|\beta| \geq 1$ the operator $\beta I - K_i$ is Fredholm with index zero on $\left(L^2(\Gamma_i)\right)^n$. Then, for all $\beta \in \mathbb{C}$, $|\beta| \geq 1$, the operator $\beta I - K$ is Fredholm with index zero on $\left(L^2(\partial\Omega)\right)^n$.

Proof. Consider $(\phi_i)_{1 \leq i \leq I}$, a family of Lipschitz functions on $\partial\Omega$ which forms a partition of unity subordinated to the open covering $(\Upsilon_i)_{1 \leq i \leq I}$ of $\partial\Omega$. Also, for each $i \in \{1, \ldots, I\}$, let η_i be a Lipschitz function on $\partial\Omega$, compactly supported and such that $\eta_i \equiv 1$ in a neighborhood of $\text{supp}\,\phi_i$.

For any $f \in \left(L^2(\partial\Omega)\right)^n$ we have $\phi_i f \in \left(L^2(\Gamma_i)\right)^n$. By hypothesis, for $\beta \in \mathbb{C}$, $|\beta| \geq 1$, the operator $\beta I - K_i$ is Fredholm with index zero on $\left(L^2(\Gamma_i)\right)^n$ for each $i = 1, \ldots, I$. Thus (cf., e.g., [5, p. 31]), there exist $C > 0$, a family of Banach spaces $(Y_i)_{1 \leq i \leq I}$, and a family of compact operators $T_i : \left(L^2(\Gamma_i)\right)^n \longrightarrow Y_i$, $i = 1, \ldots, I$, such that

$$\begin{aligned} \sum_{i=1}^{I} \|\phi_i f\|_{(L^2(\Gamma_i))^n} &\leq C \sum_{i=1}^{I} \|(\beta I - K_i)\,\phi_i f\|_{(L^2(\Gamma_i))^n} + \sum_{i=1}^{I} \|T_i\,(\phi_i f)\|_{Y_i} \\ &\leq C \sum_{i=1}^{I} \|\eta_i(\beta I - K)\phi_i f\|_{(L^2(\partial\Omega))^n} \\ &\quad + C \sum_{i=1}^{I} \|(1-\eta_i)\,(\beta I - K_i)\,\phi_i f\|_{(L^2(\Gamma_i))^n} \\ &\quad + \sum_{i=1}^{I} \|T_i \phi_i f\|_{Y_i}\,. \end{aligned} \tag{3.3}$$

First, it is obvious that for each $i = 1, \ldots, I$, the operator $T_i\phi_i : \left(L^2(\partial\Omega)\right)^n \longrightarrow Y_i$ is compact. Next, note that $1 - \eta_i$ and ϕ_i have mutually disjoint supports. Then, by standard arguments we may conclude that, for each $i = 1, \ldots, I$, the operator $(1-\eta_i)(\beta I - K_i)\phi_i : \left(L^2(\partial\Omega)\right)^n \longrightarrow \left(L^2(\Gamma_i)\right)^n$ is Hilbert-Schmidt and hence compact. Further, since ϕ_i is a Lipschitz function, the commutator operator

$$[K, \phi_i] := K\phi_i - \phi_i K : \left(L^2(\partial\Omega)\right)^n \longrightarrow \left(L^2(\partial\Omega)\right)^n$$

is weakly singular and therefore compact for all $i = 1, \ldots, I$. Now, (3.3) implies

$$\begin{aligned} \|f\|_{(L^2(\partial\Omega))^n} &\leq C \sum_{i=1}^{I} \|\phi_i(\beta I - K) f\|_{(L^2(\partial\Omega))^n} + \|\mathrm{Comp}(f)\| \\ &\leq C\|(\beta I - K) f\|_{(L^2(\partial\Omega))^n} + \|\mathrm{Comp}(f)\|\,, \end{aligned}$$

where "Comp" stands for a compact operator on $\left(L^2(\partial\Omega)\right)^n$. This shows that for all $\beta \in \mathbb{C}$, $|\beta| \geq 1$ the operator $\beta I - K$ is semi-Fredholm on $\left(L^2(\partial\Omega)\right)^n$. Now, since $\beta I - K$ is invertible on $\left(L^2(\partial\Omega)\right)^n$ for $|\beta|$ large, the conclusion of Lemma 1 follows. □

In the case when $\partial\Omega = \text{graph}\,\Phi$, where $\Phi : \mathbb{R}^{n-1} \to \mathbb{R}$ is a Lipschitz function, the elastostatics double layer potential in graph coordinates can be identified with

$$K^{ij}_{\mathcal{L},p} f(x) \equiv I^{ij}_1 f(x) + I^{ij}_2 f(x)\,,$$

where

$$\begin{aligned} I^{ij}_1 f(x) \; := \; & \frac{-4\mu}{\omega_n(3\mu+\lambda)} \\ & p.v. \int_{\mathbb{R}^{n-1}} \delta_{ij} \frac{\Phi(x) - \Phi(y) - \nabla\Phi(y)\cdot(x-y)}{\left(1 + \left(\frac{\Phi(x)-\Phi(y)}{|x-y|}\right)^2\right)^{n/2}} \frac{f(y)}{|x-y|^n} dy\,, \end{aligned}$$

and

$$I^{ij}_2 f(x) := p.v. \int_{\mathbb{R}^{n-1}} \frac{C_{ij}(x,y)((\Phi(x) - \Phi(y) - \nabla\Phi(y)\cdot(x-y))}{\left(1 + \left(\frac{\Phi(x)-\Phi(y)}{|x-y|}\right)^2\right)^{(n+2)/2}} \frac{f(y)}{|x-y|^{n+2}} dy\,,$$

with

$$C_{ij}(x,y) := \begin{cases} \frac{-2n(\mu+\lambda)}{\omega_n(3\mu+\lambda)}(x_i - y_i)(x_j - y_j) & \text{if } i, j < n\,, \\ \frac{-2n(\mu+\lambda)}{\omega_n(3\mu+\lambda)}(x_i - y_i)(\Phi(x) - \Phi(y)) & \text{if } i < n,\ j = n\,, \\ \frac{-2n(\mu+\lambda)}{\omega_n(3\mu+\lambda)}(\Phi(x) - \Phi(y))^2 & \text{if } i = n,\ j = n\,. \end{cases}$$

Next, observe that for any i, j, the operators I^{ij}_1 and I^{ij}_2 are precisely of the type described in Theorem 2. Let us show this for, e.g., the operator $I^{i,j}_2$ in the case $i, j < n$. All the other cases follow from similar considerations. Indeed, the kernel of I^{ij}_2 in the case under discussion has the form

$$\begin{aligned} k(x,y) \; := \; & \frac{-2n(\mu+\lambda)}{\omega_n(3\mu+\lambda)} \frac{1}{|x-y|^n} \cdot \frac{x_i - y_i}{|x-y|} \cdot \frac{x_j - y_j}{|x-y|} \\ & \cdot \frac{\Phi(x) - \Phi(y) - \nabla\Phi(y)\cdot(x-y)}{\left(1 + \left(\frac{\Phi(x)-\Phi(y)}{|x-y|}\right)^2\right)^{(n+2)/2}}\,. \end{aligned}$$

Thus, we may take $B := \Phi$, $\Lambda(\xi) := \xi_i\xi_j$, $\xi \in S^{n-2}$, and $F(t) := (1+t^2)^{-(n+2)/2}$. Note that both F and Λ are even and that the other hypothesis of Theorem 2 are satisfied. Therefore, there exists $C(n,\mu,\lambda) > 0$ depending only on n, μ, and λ and $\nu > 0$ such that

$$\left\|K_{\mathcal{L},p}\right\|_{\mathcal{L}((L^2(\partial\Omega))^n)} \leq C(n,\mu,\lambda)\|\nabla\Phi\|_{L^\infty(\mathbb{R}^{n-1})}\left(1+\|\nabla\Phi\|_{L^\infty(\mathbb{R}^{n-1})}\right)^\nu .$$

We conclude that there exists $C_0 = C_0(n,\mu,\lambda) > 0$ such that, for any bounded Lipschitz domain Ω with char $\Omega \leq C_0(n,\mu,\lambda)$, we get

$$\left\|K^{ij}_{\mathcal{L},p}\right\|_{\mathcal{L}(L^2(\partial\Omega))} < 1/n .$$

Now, Remark 1 easily implies that

$$\left\|K_{\mathcal{L},p}\right\|_{\mathcal{L}((L^2(\partial\Omega))^n)} < 1$$

so that for any $\beta \in \mathbb{C}$, $|\beta| \geq 1$, the operator $\beta I - K_{\mathcal{L},p}$ is invertible and therefore Fredholm with index zero on $\left(L^2(\partial\Omega)\right)^n$.

A similar argument works also for $\beta I - K_{\mathcal{S},s}$ and this completes the proof of Step I.

Step II: Surjectivity outside the unit disk. First note that as a consequence of the Fredholm property results obtained above, $\beta I - K_{\mathcal{L},p}$, $\beta I - K_{\mathcal{S},s} : \left(L^2(\partial\Omega)\right)^n \longrightarrow \left(L^2(\partial\Omega)\right)^n$ have closed ranges. Therefore, in order to conclude their surjectivity it is enough to show that they have dense ranges in $\left(L^2(\partial\Omega)\right)^n$. Now, the dense range property of the operators is equivalent to the injectivity of their adjoints. We approach this problem as follows.

More generally, let us consider $A(D)$ a $m \times m$ matrix of second order operators on $\mathbb{R}^n$ with real constant coefficients

$$A(D) = \left(a^{\alpha\beta}_{ij}\partial_i\partial_j\right)_{\alpha,\beta}, \quad a^{\alpha\beta}_{ij} \in \mathbb{R}$$
$$\text{for any } i,j = 1,\ldots,n \text{ and } \alpha,\beta = 1,\ldots,m .$$

Call $A = \left(a^{\alpha\beta}_{ij}\right)_{i,j,\alpha,\beta}$ symmetric if $a^{\alpha\beta}_{ij} = a^{\beta\alpha}_{ji}$. Also, we say that the matrix A satisfies the Legendre-Hadamard ellipticity condition provided there exists $c > 0$ such that

$$a^{\alpha\beta}_{ij}\xi_i\xi_j\eta^\alpha\eta^\beta \geq c|\xi|^2|\eta|^2 \text{ for any } \xi \in \mathbb{R}^n,\ \eta \in \mathbb{R}^m .$$

It is well known (cf., e.g., [23]) that if A is symmetric, satisfies the Legendre-Hadamard condition, and $n \geq 3$, then $A(D)$ has a fundamental solution $G(X) = (G^{\alpha\beta}(X))$ with the following properties:

(1) $G^{\alpha\beta}(X) \in C^\infty(\mathbb{R}^n \setminus \{0\})$, $G^{\alpha\beta}(X) = G^{\beta\alpha}(X)$, $G^{\alpha\beta}(X) = G^{\alpha\beta}(-X)$ and $G^{\alpha\beta}(tX) = t^{2-n}G^{\alpha\beta}(X)$ for any $t > 0$.

(2) The rows and the columns of $G(X)$ satisfy the system $A(D)u(X) = 0$ for $X \neq 0$. That is,

$$a^{\alpha\beta}_{ij}\partial_i\partial_j G^{\gamma\beta}(X) = a^{\alpha\beta}_{ij}\partial_i\partial_j G^{\beta\gamma}(X) = 0, \text{ for } X \neq 0 .$$

(3) If $u = (u_1, u_2, \ldots, u_m) \in \left(C^\infty_0(\mathbb{R}^n)\right)^m$, then

$$u(X) = -\int_{\mathbb{R}^n} G(X-Y)A(D)(u)(Y)\,dY ,$$

i.e.,

$$u^\alpha(X) = -\int_{\mathbb{R}^n} G^{\alpha\beta}(X-Y)a^{\beta\gamma}_{ij}\partial_i\partial_j u^\gamma(Y)\,dY .$$

(4) Let Ω be a Lipschitz domain in $\mathbb{R}^n$ and $N(P) = (N_1(P), N_2(P), \dots, N_n(P))$ be the outward unit normal vector (which exists at almost every $P \in \partial\Omega$). Set $d\sigma$ for the surface measure on $\partial\Omega$. Then

$$\int_{\partial\Omega} a_{ij}^{\alpha\beta} N_j(Q) \partial_i G^{\beta\gamma}(X - Q)\, d\sigma(Q) = \begin{cases} \delta_{\alpha\gamma} & \text{if } X \in \Omega\,, \\ \frac{1}{2}\delta_{\alpha\gamma} & \text{if } X \in \partial\Omega\,, \\ 0 & \text{if } X \notin \overline{\Omega}\,. \end{cases}$$

Let us recall from Section 2 that, for a given vector function u, we define its *conormal* derivative at almost every $P \in \partial\Omega$ by

$$\left(\frac{\partial u}{\partial N_A}\right)^{\alpha}(P) := N_i(P) a_{ij}^{\alpha\beta} \partial_j u^{\beta}(P)\,.$$

We consider the following layer potentials:

(i) *The single layer potential* operator $\mathcal{S}$ defined on $\left(L^2(\partial\Omega)\right)^m$

$$\mathcal{S}f(X) = 2\int_{\partial\Omega} G(X - Y) f(Y)\, d\sigma(Y), \quad X \in \Omega\,,$$

and its boundary version $S : \left(L^2(\partial\Omega)\right)^m \longrightarrow \left(L^2(\partial\Omega)\right)^m$

$$Sf(P) = 2\int_{\partial\Omega} G(P - Y) f(Y)\, d\sigma(Y), \quad P \in \partial\Omega\,.$$

(ii) *The double layer potential* operator $\mathcal{D}_A$ defined on $\left(L^2(\partial\Omega)\right)^m$,

$$\mathcal{D}_A f(X) = 2\int_{\partial\Omega} \left[\frac{\partial G}{\partial N_A}(X - \cdot)\right]^t (Q) f(Q)\, d\sigma(Q), \quad X \in \Omega$$

where $\frac{\partial}{\partial N_A}$ is applied to the columns of G; i.e.,

$$\frac{\partial G}{\partial N_A}(X - Q) = \left(N_i(Q) a_{ij}^{\alpha\beta} \partial_j G^{\beta\gamma}(X - Q)\right)_{\alpha,\gamma}\,,$$

and the superscript t stands for transposition of matrices. The boundary version of $\mathcal{D}_A$ is $K_A : \left(L^2(\partial\Omega)\right)^m \longrightarrow \left(L^2(\partial\Omega)\right)^m$,

$$K_A f(P) = 2\, p.v. \int_{\partial\Omega} \left[\frac{\partial G}{\partial N_A}(P - \cdot)\right]^t (Q) f(Q)\, d\sigma(Q), \quad P \in \partial\Omega\,.$$

The adjoint operator of K_A is $K_A^* : \left(L^2(\partial\Omega)\right)^m \longrightarrow \left(L^2(\partial\Omega)\right)^m$,

$$K_A^* f(P) = -2\, p.v. \int_{\partial\Omega} \frac{\partial G}{\partial N_A}(\cdot - Q)(P) f(Q)\, d\sigma(Q), \quad P \in \partial\Omega\,.$$

As a consequence of the results in [1], the operators K_A and K_A^* are well defined and bounded on $\left(L^2(\partial\Omega)\right)^m$.

Let $\{\Upsilon_+(P)\}_{P\in\partial\Omega}$ and $\{\Upsilon_-(P)\}_{P\in\partial\Omega}$ be two families of nontangential approach regions of conical type (as in, e.g., [25]), $\Upsilon_+(P) \subset \Omega = \Omega_+$ and $\Upsilon_-(P) \subset \mathbb{R}^n \setminus \Omega = \Omega_-$ for any $P \in \partial\Omega$. Denote by $u_\pm^*$ the nontangential maximal function of u defined on $\Omega_\pm$ by $u_\pm^*(P) := \sup_{X\in\Upsilon_\pm(P)} |u(X)|$, $P \in \partial\Omega$.

A basic result which follows from [1] and standard techniques is the following.

Proposition 2.

Assume $A = (a_{ij}^{\alpha\beta})_{i,j,\alpha,\beta}$ *is symmetric with real entries and satisfies the Legendre-Hadamard ellipticity condition. Then,*

(1) For any $f \in \big(L^2(\partial\Omega)\big)^m$, $(\mathcal{D}_A f)^* \in \big(L^2(\partial\Omega)\big)^m$. *Moreover, there exists* $C > 0$ *depending only on the Lipschitz character of* Ω *such that*

$$\big\|(\mathcal{D}_A f)^*\big\|_{(L^2(\partial\Omega))^m} \leq C\|f\|_{(L^2(\partial\Omega))^m}\,.$$

(2) For almost every $P \in \partial\Omega$ *we have*

$$\lim_{\substack{X\to P\\ X\in\Upsilon_\pm(P)}} \mathcal{D}_A f(X) = (\pm I + K_A)\, f(P)\,.$$

(3) For any $f \in \big(L^2(\partial\Omega)\big)^m$, $(\nabla\mathcal{S} f)^* \in \big(L^2(\partial\Omega)\big)^{mn}$. *Moreover, there exists* $C > 0$ *depending only on the Lipschitz character of* Ω *such that:*

$$\big\|(\nabla\mathcal{S} f)^*\big\|_{(L^2(\partial\Omega))^{mn}} \leq C\|f\|_{(L^2(\partial\Omega))^m}\,.$$

(4) For almost every $P \in \partial\Omega$ *we have*

$$\frac{\partial\mathcal{S}f}{\partial N_A}\bigg|_{\partial\Omega_\pm}(P) := \lim_{\substack{X\to P\\ X\in\Upsilon_\pm(P)}} N_i(P)a_{ij}^{\alpha\beta}\partial_j\mathcal{S}f(X) = \big(\pm I - K_A^*\big)\, f(P)\,.$$

Proposition 3.

Let $u, v \in \big(C^1(\Omega_\pm)\big)^m$ *be so that* $(\nabla u)^*, (\nabla v)^* \in \big(L^2(\partial\Omega)\big)^{mn}$. *Then we have*

$$\begin{aligned}\int_{\Omega_\pm} A(D)u(X)\cdot v(X)\,dX &= -\int_{\Omega_+} A\nabla u(X)\cdot\nabla v(X)\,dX\\ &\quad \pm\int_{\partial\Omega} v(Q)\cdot\frac{\partial u}{\partial N_A}(Q)\,d\sigma(Q)\,,\end{aligned}$$

where $A\nabla u\cdot\nabla v$ *stands for* $a_{ij}^{\alpha\beta}\partial_i u^\alpha\partial_j v^\beta$.

Proof. Integrating by parts we obtain

$$\begin{aligned}\int_{\Omega_+} A(D)u(X)\cdot v(X)\,dX &= \int_{\Omega_+} a_{ij}^{\alpha\beta}\partial_i\partial_j u^\alpha v^\beta(X)\,dX\\ &= -\int_{\Omega_+} a_{ij}^{\alpha\beta}\partial_i u^\alpha(X)\partial_j v^\beta(X)\,dX\\ &\quad +\int_{\partial\Omega} a_{ij}^{\alpha\beta}\partial_i u^\alpha(Q)N_j(Q)v^\beta(Q)\,d\sigma(Q)\\ &= -\int_{\Omega_+} A\nabla u(X)\cdot\nabla v(X)\,dX + \int_{\partial\Omega} v(Q)\cdot\frac{\partial u}{\partial N_A}(Q)\,d\sigma(Q)\,.\end{aligned}$$

The other equality follows in a similar manner. □

Let $f \in \big(L^2(\partial\Omega)\big)^m$ and $u = \bar{v} = \mathcal{S}f$. Since $A(D)\mathcal{S}f(X) \equiv 0$ for any $X \notin \partial\Omega$ we get

$$\int_{\partial\Omega} \mathcal{S}\bar{f}(Q)\frac{\partial\mathcal{S}f}{\partial N_A}(Q)\,d\sigma(Q) = \pm\int_{\Omega_\pm}(A\nabla\mathcal{S}f(X))\cdot\overline{(\nabla\mathcal{S}f(X))}\,dX\,.$$

As a consequence of Proposition 2 we have $\frac{\partial\mathcal{S}f}{\partial N_A}\Big|_{\partial\Omega_\pm} = (\pm I - K_A^*)f$. This implies the following.

Corollary 1.

For any $f \in \left(L^2(\partial\Omega)\right)^m$,

$$\int_{\partial\Omega} \mathcal{S}\bar{f}(Q)(\pm I - K_A^*)f(Q)\,d\sigma(Q) = \pm \int_{\Omega_\pm} (A \nabla \mathcal{S}f(X)) \cdot \overline{(\nabla \mathcal{S}f(X))}\,dX\,. \tag{3.4}$$

To continue, we need one more definition. Specifically, we shall call the matrix $A = (a_{ij}^{\alpha\beta})_{i,j,\alpha,\beta}$ *strictly positive definite* provided there exists $c > 0$ such that

$$a_{ij}^{\alpha\beta}\xi_i^\alpha\xi_j^\beta \ge c|\xi|^2 \text{ for any } \xi \in \mathbb{R}^{nm}\,,$$

and we call the matrix $A = (a_{ij}^{\alpha\beta})_{i,j,\alpha,\beta}$ *semi-positive definite* if

$$a_{ij}^{\alpha\beta}\xi_i^\alpha\xi_j^\beta \ge 0 \text{ for any } \xi \in \mathbb{R}^{nm}\,.$$

In this general framework, we are now ready to state the theorem which is needed in order to proceed with the proof of Step II.

Theorem 3.

Let A be symmetric with constant real entries and satisfy the Legendre-Hadamard ellipticity condition. Also, for $\beta \in \mathbb{C}$ *consider the operator*

$$\beta I - K_A^* : \left(L^2(\partial\Omega)\right)^m \longrightarrow \left(L^2(\partial\Omega)\right)^m\,. \tag{3.5}$$

Then, if $\beta \notin \mathbb{R}$, *the above operator is injective.*

If, in addition, A is strictly positive definite, then the operator in (3.5) is injective for any $\beta \notin (-1, 1]$.

Proof. The proof is an adaptation of an old argument of Kellogg [15]. Assume that there exists a function $f \in \operatorname{Ker}(\beta I - K_A^*)$, $f \not\equiv 0$ and define

$$I_\pm := \int_{\Omega_\pm} (A \nabla \mathcal{S}f(X)) \cdot \overline{(\nabla \mathcal{S}f(X))}\,dX\,.$$

The first observation is that I_+ and I_- are real numbers; this follows from the symmetry condition for A. Second, I_+ and I_- cannot vanish simultaneously, for in that case, by Plancherel and the Legendre-Hadamard condition,

$$0 = I_+ + I_- = \int_{\mathbb{R}^n} (A \nabla \mathcal{S}f(X)) \cdot \overline{(\nabla \mathcal{S}f(X))}\,dX \ge c \int_{\mathbb{R}^n} |\nabla \mathcal{S}f(X)|^2\,dX\,.$$

Let us point out here that $u : \mathbb{R}^n \to \mathbb{R}^n$ defined by $u := \mathcal{S}f$ in Ω^+ and $u := \mathcal{S}f$ in Ω^- is globally in the Sobolev space $H^{1,2}\left(\mathbb{R}^n\right)$ as the trace of $\mathcal{S}f$ in $H^{1/2,2}(\partial\Omega)$ is the same when restricted to $\partial\Omega$ from either side. In turn, the above inequality implies $\nabla\mathcal{S}f \equiv 0$ in $\Omega_\pm$ and, further, in view of the jump-relations in Proposition 2, $f = 0$, which is a contradiction.

Going further, note that

$$\int_{\partial\Omega} \mathcal{S}\bar{f}(Q) \cdot (\pm I - K_A^*)f(Q)\,d\sigma(Q) = (\pm 1 - \beta) \int_{\partial\Omega} \mathcal{S}\bar{f}(Q) \cdot f(Q)\,d\sigma(Q)$$

so that, by (3.4),

$$(1 - \beta) \int_{\partial\Omega} \mathcal{S}\bar{f}(Q) \cdot f(Q)\,d\sigma(Q) = I_+$$

$$(1 + \beta) \int_{\partial\Omega} \mathcal{S}\bar{f}(Q) \cdot f(Q)\,d\sigma(Q) = I_-\,. \tag{3.6}$$

Next, taking the quotient of the equalities (3.6) we get

$$\frac{1-\beta}{1+\beta}=\frac{I_+}{I_-}\Longrightarrow \beta=\frac{I_- - I_+}{I_- + I_+}\,.$$

This entails $\beta \in \mathbb{R}$, proving the first part of the theorem.

Assume next that A is strictly positive definite. Then I_+ and I_- are positive numbers which forces $\beta \in [-1, 1]$. Moreover, there is no loss of generality to assume in the present situation that f is real-valued. Also, in the case under discussion, $\beta = -1$ if and only if $I_- = 0$ which, so we claim, cannot happen. Indeed, since A is strictly positive definite we have that

$$(A \nabla \mathcal{S}f(X)) \cdot (\nabla \mathcal{S}f(X)) \geq c\,|\nabla \mathcal{S}f(X)|^2\,. \tag{3.7}$$

The fact that $I_- = 0$ and the inequality (3.7) imply that $\nabla \mathcal{S}f \equiv 0$ in Ω_-. Therefore, $\mathcal{S}f$ is constant on Ω_- and since the single layer decays at infinity, the constant must be zero. Taking the trace to the boundary in the Sobolev sense gives that $Sf \equiv 0$ on $\partial\Omega$ and by (3.6) $I_+ = 0$. But, from above, I_+ and I_- cannot vanish simultaneously. This concludes the proof of Theorem 3. □

Recall that the operator $K^*_{\mathcal{L},p}$ is the adjoint double layer elastostatics operator corresponding to the pseudostress conormal derivative. This choice of the conormal is based on a *strictly positive* definite matrix A. Thus, Theorem 3 applies (in this case $m = n$) and gives the desired results for the operator in (3.1).

Concerning the action of the same operator on $\left(L^2_0(\partial\Omega)\right)^n$, we first note that, since $K_{\mathcal{L},p}(e_j) = e_j$, where $\{e_j\}$ is the standard orthonormal basis in $\mathbb{R}^n$,

$$I - K^*_{\mathcal{L},p} : \left(L^2(\partial\Omega)\right)^n \longrightarrow \left(L^2_0(\partial\Omega)\right)^n \tag{3.8}$$

is well defined. Consider now the following diagram

$$\begin{array}{ccc} \left(L^2(\partial\Omega)\right)^n & \stackrel{I-K^*_{\mathcal{L},p}}{\longrightarrow} & \left(L^2(\partial\Omega)\right)^n \\ \uparrow \iota & & \pi \downarrow \\ \left(L^2_0(\partial\Omega)\right)^n & \stackrel{I-K^*_{\mathcal{L},p}}{\longrightarrow} & \left(L^2_0(\partial\Omega)\right)^n \end{array} \tag{3.9}$$

which, because of (3.8), is commutative; here ι, π are the inclusion and the orthogonal projection operators, respectively. Since $\left(L^2_0(\partial\Omega)\right)^n$ is a closed subspace of $\left(L^2(\partial\Omega)\right)^n$ with finite codimension, it follows that ι and π are Fredholm operators of opposite index. From the first part of the proof of Theorem 1 the operator $I - K^*_{\mathcal{L},p}$ is Fredholm with index zero on $\left(L^2(\partial\Omega)\right)^n$. This, together with the commutative diagram (3.9), implies that $I - K^*_{\mathcal{L},p}$ is also a Fredholm operator on $\left(L^2_0(\partial\Omega)\right)^n$ with index zero.

There remains to prove that its kernel is trivial on this latter space. To this end, let $f \in \mathrm{Ker}(I - K^*_{\mathcal{L},p}) \cap \left(L^2_0(\partial\Omega)\right)^n$. The first equality in (3.6) with $\beta = 1$ yields $I_+ = 0$ which, in view of (3.7), implies that $\mathcal{S}f$ is constant in Ω_+. Now, taking the trace to the boundary in the Sobolev sense gives that Sf is constant on $\partial\Omega$. In turn, since $f \in \left(L^2_0(\partial\Omega)\right)^n$, from (3.6) we have $I_- = 0$ and therefore $\mathcal{S}f$ is constant in Ω_-. Finally, the jump relations give $f \equiv 0$. This shows that $I - K^*_{\mathcal{L},p}$ is injective; therefore, invertible on $\left(L^2_0(\partial\Omega)\right)^n$.

This concludes the proof of the first part of Theorem 1 having to do with elastic potentials.

In the case of the Stokes system, the matrix A corresponding to the choice of the conormal stress derivative is symmetric with real entries but only *semi-positive definite.* Nonetheless, Theorem 3 applies and gives the desired conclusion for the operator in (3.2). The important remark is that, in this case, we have

$$(A \nabla \mathcal{S}f(X)) \cdot (\nabla \mathcal{S}f(X)) \geq C \sum_{i,j=1}^{n} \left(\partial_j\,(\mathcal{S}f(X))^i + \partial_i\,(\mathcal{S}f(X))^j\right)^2\,. \tag{3.10}$$

Therefore, if $I_- = 0$, then $\partial_i (\mathcal{S}f)^j + \partial_j (\mathcal{S}f)^i = 0$ in Ω_-. Since the nontrivial solutions of $\partial_i u^j + \partial_j u^i = 0$, for $i, j = 1, \dots, n$, are affine functions and $\mathcal{S}f$ decays at infinity, we get that $\mathcal{S}f \equiv 0$ in Ω_-. Then, the rest of the argument follows.

At this point, there remains to show that the operator $I - K^*_{\mathcal{S},s} : L^2_\Psi(\partial\Omega) \longrightarrow L^2_\Psi(\partial\Omega)$ is well defined and invertible. Another characterization of $L^2_\Psi(\partial\Omega)$ useful for this purpose is

$$L^2_\Psi(\partial\Omega) = \left\{ f \in \left(L^2(\partial\Omega)\right)^n ;\ K_{\mathcal{S},s} f = f \right\}^\perp . \tag{3.11}$$

Let $\Psi(\partial\Omega) := \{\psi|_{\partial\Omega},\ \psi \in \Psi\}$, where $|_{\partial\Omega}$ denotes the trace to $\partial\Omega$ in the Sobolev sense. Clearly (3.11) will follow from

$$\Psi(\partial\Omega) = \left\{ f \in \left(L^2(\partial\Omega)\right)^n ;\ K_{\mathcal{S},s} f = f \right\} . \tag{3.12}$$

We postpone the proof of the above equality for the moment and continue the argument concluding the last part of Theorem 1. To this end, let us first note that (3.11) and elementary functional analysis give

$$I - K^*_{\mathcal{S},s} : \left(L^2(\partial\Omega)\right)^n \longrightarrow L^2_\Psi(\partial\Omega) .$$

Based on this, proceeding as we did for (3.9), we can conclude that $I - K^*_{\mathcal{S},s}$ is Fredholm with index zero on $L^2_\Psi(\partial\Omega)$. Let us show that its kernel on this latter space is trivial. Consider $f \in \mathrm{Ker}(I - K^*_{\mathcal{S},s}) \cap L^2_\Psi(\partial\Omega)$. The first equality in (3.6) together with (3.10) imply that $Sf \in \Psi(\partial\Omega)$. In turn, since $f \in L^2_\Psi(\partial\Omega)$, the second part of (3.6) together with the decay at the infinity of the single layer imply that $\mathcal{S}f \equiv 0$ in Ω_-. Therefore, $Sf = 0$ on $\partial\Omega$. Next, since $Sf = 0$ if and only if $f = cN$ for some $c \in \mathbb{C}$ and $N \notin \mathrm{Ker}(I - K^*_{\mathcal{S},s})$ we get $f \equiv 0$. This shows that $I - K^*_{\mathcal{S},s}$ is injective, therefore invertible on $L^2_\Psi(\partial\Omega)$.

We are left now with proving (3.12). It is easy to check that if $\psi \in \Psi$ then $\Delta\psi = 0$, $\mathrm{div}\,\psi = 0$ in Ω, $\|(\nabla\psi)^*\|_{L^2(\partial\Omega)} < \infty$ and $\frac{\partial\psi}{\partial N_A} = 0$ on $\partial\Omega$, where $p \equiv 0$ [cf. (2.4)]. A simple application of the divergence theorem gives that $\Psi(\partial\Omega) \subseteq \left(L^2_1(\partial\Omega)\right)^n \cap L^2_N(\partial\Omega)$, where $L^2_N(\partial\Omega) = \{f \in \left(L^2(\partial\Omega)\right)^n ;\ \int_{\partial\Omega} f \cdot N\, d\sigma = 0\}$. On the other hand,

$$S : L^2_N(\partial\Omega) \to \left(L^2_1(\partial\Omega)\right)^n \cap L^2_N(\partial\Omega)$$

is invertible (see [9, Theorem 4.15]). This implies that $\psi = \mathcal{S}f$ for some $f \in L^2_N(\partial\Omega)$. Since $\frac{\partial\psi}{\partial N_A} = 0$ on $\partial\Omega$ we get $f \in \mathrm{Ker}(I - K^*_{\mathcal{S},s})$. Now, using that $K_{\mathcal{S},s}\mathcal{S} = \mathcal{S}K^*_{\mathcal{S},s}$ we obtain $\psi|_{\partial\Omega} \in \mathrm{Ker}(I - K_{\mathcal{S},s})$. This proves the inclusion $\Psi(\partial\Omega) \subseteq \mathrm{Ker}(I - K_{\mathcal{S},s})$ in (3.12). Next, since the operator $I - K_{\mathcal{S},s}$ is Fredholm with index zero on $\left(L^2(\partial\Omega)\right)^n$, we have that $\dim\left(\mathrm{Ker}(I - K_{\mathcal{S},s})\right) = \dim\left(\mathrm{Ker}(I - K^*_{\mathcal{S},s})\right)$. Therefore, (3.12) will follow from simple dimension considerations if we show that

$$S : \mathrm{Ker}(I - K^*_{\mathcal{S},s}) \longrightarrow \Psi(\partial\Omega) , \tag{3.13}$$

is well defined and invertible. It is easy to check that (3.4) together with (3.10) give that S is well defined. Next, since $Sf = 0$ if and only if $f = cN$ for some $c \in \mathbb{C}$ and $N + K^*_{\mathcal{S},s}(N) = 0$, it follows that S is one-to-one. On the other hand, as above, for any $\psi \in \Psi$ we have that $\psi = \mathcal{S}f$ with $f \in \mathrm{Ker}(I - K^*_{\mathcal{S},s})$, and this concludes the surjectivity of the operator S in (3.13). Thus, (3.12) follows.

This finishes the proof of Theorem 1. □

4. The Case of Three-Dimensional Polyhedra

The aim of this section is to give *explicit* bounds for $C_0(3, \mu, \lambda)$ and $C_1(3)$ in Theorem 1 for polyhedra in the three-dimensional Euclidean space. In this regard, our main result is the following.

Theorem 4.

Consider a bounded, simply connected Lipschitz polyhedron $\mathcal{P} \subset \mathbb{R}^3$ such that all its dihedral angles α satisfy the condition

$$\left|\cos\left(\frac{\alpha}{2}\right)\right| < \left(\frac{3\mu+\lambda}{8\mu+6\lambda}\right). \tag{4.1}$$

Then for any $\beta \in \mathbb{C}$, $|\beta| \geq 1$ and $\beta \neq 1$, the operator $\beta I - K_{\mathcal{L},p} \in \mathcal{L}\left(\left(L^2(\partial\mathcal{P})\right)^3\right)$ is invertible.

Moreover, the operator $I - K_{\mathcal{L},p} \in \mathcal{L}\left(\left(L^2(\partial\mathcal{P})\right)^3\right)$ is Fredholm with index zero and $I - K^*_{\mathcal{L},p}$: $\left(L^2_0(\partial\mathcal{P})\right)^3 \longrightarrow \left(L^2_0(\partial\mathcal{P})\right)^3$ is invertible.

Also, assume all dihedral angles α satisfy the condition

$$\left|\cos\left(\frac{\alpha}{2}\right)\right| < \frac{1}{6}. \tag{4.2}$$

Then, for any $\beta \in \mathbb{C}$, $|\beta| \geq 1$ and $\beta \neq 1$ the operator $\beta I - K_{\mathcal{S},s} \in \mathcal{L}\left(\left(L^2(\partial\mathcal{P})\right)^3\right)$ is invertible.

Moreover, the operator $I - K_{\mathcal{S},s} \in \mathcal{L}\left(\left(L^2(\partial\mathcal{P})\right)^3\right)$ is Fredholm with index zero and $I - K^*_{\mathcal{S},s}$: $L^2_\Psi(\partial\mathcal{P}) \longrightarrow L^2_\Psi(\partial\mathcal{P})$ is invertible.

Another way to interpret the first part of Theorem 4 is the following. For any Lipschitz polyhedron $\mathcal{P} \subset \mathbb{R}^3$ there exist Lamé moduli $\mu, \lambda \in \mathbb{R}$, $\mu > 0$ and $-\mu \leq \lambda$ such that the conclusions of Theorem 4 regarding the elastostatics layer potential hold. Note that in the limiting case $\mu + \lambda = 0$, which is the case of the vector Laplacian, the condition (4.1) reduces to $\left|\cos \frac{\alpha}{2}\right| < 1$ and therefore it is automatically satisfied for any Lipschitz polyhedron in $\mathbb{R}^3$. This way we recover the result of [6] for the harmonic layer potential operator.

We can also invoke Theorem 4 and Proposition 1 to conclude that the properties (i) through (iv) from Section 3 hold.

The Fredholmness (with index zero) of the operators $\beta I - K_{\mathcal{L},p}$ and $\beta I - K_{\mathcal{S},s}$ is obtained via Mellin transform techniques whereas their surjectivity has been concluded already in Theorem 1 and Theorem 2. We start with a couple of Mellin transform rudiments.

Let f be a function defined on $(0, \infty)$. The Mellin transform of f, denoted by $\mathcal{M}f$ is

$$\mathcal{M}(f)(z) := \int_{\mathbb{R}_+} t^z f(t) \frac{dt}{t}.$$

It is easy to see that the relationship between the Mellin and Fourier transforms (the latter denoted by $\mathcal{F}$) is $\mathcal{M}(f)(\cdot) = \mathcal{F}(f \circ \exp)(i\cdot)$. This implies the following well-known version of Plancherel's Theorem.

Proposition 4.

For any $\lambda \in \mathbb{R}$ and $f, g \in L^2(\mathbb{R}_+, t^{2\lambda-1}dt)$ we have

$$\frac{1}{2\pi i}\int_{\mathrm{Re}\, z=\lambda} \mathcal{M}(f)(z)\overline{\mathcal{M}(g)(z)}\, dz = \int_0^{+\infty} t^{2\lambda-1} f(t)\overline{g(t)}\, dt.$$

Therefore, the Mellin transform $\mathcal{M}$ is an isomorphism between $L^2(\mathbb{R}_+, t^{2\lambda-1}dt)$ and $L^2(Re\, z = \lambda)$.

With an eye toward proving Fredholmness for the operator $\beta I - K$ on $\left(L^2(\partial\mathcal{P})\right)^3$, where $K = K_{\mathcal{L},p}$ or $K = K_{\mathcal{S},s}$, we first reduce matters, via the localization argument given in Section 3, to working on the boundary of infinite polyhedral Lipschitz cones. In this setting we look at the operator K on the Mellin transform side with respect to the radial variable. In this way K is replaced by a family of operators indexed by $z \in \mathbb{C}$, $\operatorname{Re} z = 1$, all of which are defined on a spherical polygon in the unit sphere in $\mathbb{R}^3$. In this context, for each fixed z, the corresponding operator is further decomposed near each corner of the spherical polygon as a sum of an integral operator with operator norm strictly less than one and a compact operator.

Let V_i, $i = 1, \dots, I$, be the vertices of $\mathcal{P}$. Denote by Γ_i the boundary of the infinite tangent cone to $\partial\mathcal{P}$ with vertex V_i, $i = 1, \dots, I$. For each $i = 1, \dots, I$, consider the associated double layer potential $K_i \in \mathcal{L}\left(\left(L^2(\Gamma_i)\right)^3\right)$.

In the light of Lemma 1 it suffices to analyze the operator $\beta I - K_i$ on Γ_i, the boundary of a simply connected infinite polyhedral Lipschitz cone with vertex at the origin and all dihedral angles α satisfying the condition (4.1) or, respectively, (4.2). For convenience we will drop the subscript i.

Denote by F_j, $j = 1, \dots, M$, the faces (open plane sectors) and by l_j, $j = 1, \dots, M$, the edges of the cone Γ. Consider the spherical polygon γ obtained by intersecting Γ with the unit sphere $S^2 \subset \mathbb{R}^3$. The polygon γ consists of open arcs of great circles $\gamma_j := S^2 \cap F_j$ and corner points $e_j = l_j \cap S^2$, $j = 1, \dots, M$.

Let us introduce spherical coordinates such that $P = r\omega$, $Q = r'\omega'$ where $r := \operatorname{dist}(P, 0)$, $r' := \operatorname{dist}(Q, 0)$ and $\omega, \omega' \in S^2$. The elastostatics double layer potential operator on Γ takes the form:

$$
\begin{aligned}
\left(K_{\mathcal{L},p}(\vec{u})\right)^i(r\omega) &= \int_{\mathbb{R}_+\times\gamma} k^{ij}_{\mathcal{L},p}(r\omega - r'\omega')u^j(r'\omega')r'd\omega'\,dr' \\
&= \frac{1}{2\pi(3\mu+\lambda)}\int_{\mathbb{R}_+\times\gamma}\left(-2\mu\delta_{ij}\frac{n_{\omega'}\cdot\omega r r'}{|r\omega - r'\omega'|^3}\right. \\
&\quad \left. - 3(\mu+\lambda)\frac{n_{\omega'}\cdot\omega r r'(r\omega_i - r'\omega'_i)\,(r\omega_j - r'\omega'_j)}{|r\omega - r'\omega'|^5}\right)u^j(r'\omega')\,d\omega'\,dr' \\
&= \frac{1}{2\pi(3\mu+\lambda)}\int_{\mathbb{R}_+\times\gamma}\left(-2\mu\delta_{ij}\frac{n_{\omega'}\cdot\omega\frac{r}{r'}}{|\frac{r}{r'}\omega - \omega'|^3}\right. \\
&\quad \left. - 3(\mu+\lambda)\frac{n_{\omega'}\cdot\omega\frac{r}{r'}(\frac{r}{r'}\omega_i - \omega'_i)\,(\frac{r}{r'}\omega_j - \omega'_j)}{|\frac{r}{r'}\omega - \omega'|^5}\right)u^j(r'\omega')\,d\omega'\,\frac{dr'}{r'}\,,
\end{aligned}
$$

where $d\omega'$ is the arc element on γ. Now, let

$$
\begin{aligned}
k^{ij}_{\mathcal{L},p}\left(t, \omega, \omega'\right) &:= C\frac{\delta_{ij}n_{\omega'}\cdot\omega t}{|t\omega - \omega'|^3} + D\frac{n_{\omega'}\cdot\omega t(t\omega_i - \omega'_i)\,(t\omega_j - \omega'_j)}{|t\omega - \omega'|^5} \\
&= C\frac{\delta_{ij}n_{\omega'}\cdot\omega t}{(t^2 - 2t\omega\cdot\omega' + 1)^{3/2}} \\
&\quad + D\frac{n_{\omega'}\cdot\omega t(t\omega_i - \omega'_i)\,(t\omega_j - \omega'_j)}{(t^2 - 2t\omega\cdot\omega' + 1)^{5/2}}\,,
\end{aligned}
\tag{4.3}
$$

with

$$
C =: \frac{-2\mu}{2\pi(3\mu+\lambda)} \quad \text{and} \quad D =: -\frac{3\mu + 3\lambda}{2\pi(3\mu+\lambda)}\,.
$$

Similarly, the hydrostatics layer potential operator on Γ equals

$$
(K_{\mathcal{S},s}(\vec{u}))^i(r\omega) = \int_{\mathbb{R}_+\times\gamma} k^{ij}_{\mathcal{S},s}(r\omega - r'\omega')u^j(r'\omega')r'd\omega'\,dr'
$$

$$= -\frac{3}{2\pi}\int_{\mathbb{R}_+\times\gamma} \frac{n_{\omega'}\cdot\omega\frac{r}{r'}(\frac{r}{r'}\omega_i-\omega_i')\,(\frac{r}{r'}\omega_j-\omega_j')}{|\frac{r}{r'}\omega-\omega'|^5}$$

$$u^j(r'\omega')\,d\omega'\,\frac{dr'}{r'}\,.$$

We let

$$k^{ij}_{\mathcal{S},s}(t,\omega,\omega') := -\frac{3}{2\pi}\frac{n_{\omega'}\cdot\omega t(t\omega_i-\omega_i')\,(t\omega_j-\omega_j')}{(t^2-2t\omega\cdot\omega'+1)^{5/2}}\,.$$

Let us also denote by $L^2(\mathbb{R}_+, t\,dt)\otimes L^2(\gamma)$ the algebraic tensor product of the Hilbert spaces $L^2(\mathbb{R}_+, t\,dt)$ and $L^2(\gamma)$ with

$$\|f\otimes g\|_{L^2(\mathbb{R}_+,t\,dt)\otimes L^2(\gamma)} := \|f\|_{L^2(\mathbb{R}_+,t\,dt)}\|g\|_{L^2(\gamma)}\,. \tag{4.4}$$

Now, set $L^2(\mathbb{R}_+, t\,dt)\overline{\otimes}L^2(\gamma)$ for the topological completion of $L^2(\mathbb{R}_+, t\,dt)\otimes L^2(\gamma)$ in the norm given by (4.4). Similarly, $L^2(\mathrm{Re}\,z=1)\otimes L^2(\gamma)$ is the algebraic tensor product of $L^2(\mathrm{Re}\,z=1)$ with $L^2(\gamma)$ and $L^2(\mathrm{Re}\,z=1)\overline{\otimes}L^2(\gamma)$ is the topological completion of $L^2(\mathrm{Re}\,z=1)\otimes L^2(\gamma)$ in the norm

$$\|f\otimes g\|_{L^2(\mathrm{Re}\,z=1)\otimes L^2(\gamma)} := \|f\|_{L^2(\mathrm{Re}\,z=1)}\|g\|_{L^2(\gamma)}\,.$$

Now, as a consequence of Proposition 4 we have that $\mathcal{M}_r$ defined by

$$\mathcal{M}_r(f\otimes g) := \mathcal{M}(f)\otimes g$$

for any $f\in L^2(\mathbb{R}_+, t\,dt)$ and $g\in L^2(\gamma)$ *(i.e., the Mellin transform with respect to the radial variable)* is an isomorphism from $L^2(\mathbb{R}_+, t\,dt)\otimes L^2(\gamma)$ into $L^2(\mathrm{Re}\,z=1)\otimes L^2(\gamma)$. This uniquely extends to an isomorphism between the spaces $L^2(\mathbb{R}_+, t\,dt)\overline{\otimes}L^2(\gamma)$ and $L^2(\mathrm{Re}\,z=1)\overline{\otimes}L^2(\gamma)$.

For any complex number z such that $\mathrm{Re}\,z=1$, let $\mathcal{A}_{ij}(z): L^2(\gamma)\to L^2(\gamma)$ be the ijth entry in the matrix operator $\mathcal{A}: \left(L^2(\gamma)\right)^3 \longrightarrow \left(L^2(\gamma)\right)^3$ defined by

$$\mathcal{A}_{ij}(z)v(\omega) := \int_\gamma \left[\mathcal{M}\left(k^{ij}\left(\cdot,\omega,\omega'\right)\right)\right](z)v\left(\omega'\right)\,d\omega'\,,$$

for any $i,j=1,2,3$. Above $k^{ij} := k^{ij}_{\mathcal{L},p}$ or $k^{ij} := k^{ij}_{\mathcal{S},s}$, and corresponding to that we have the operators $\mathcal{A}_{\mathcal{L},p}$ and $\mathcal{A}_{\mathcal{S},s}$ depending if we refer to the elastostatics or hydrostatics layer potential, respectively.

Let

$$\mathrm{Id}\otimes\mathcal{A}(z): \left(L^2(\mathrm{Re}\,z=1)\otimes L^2(\gamma)\right)^3 \longrightarrow \left(L^2(\mathrm{Re}\,z=1)\otimes L^2(\gamma)\right)^3$$

defined by $\mathrm{Id}\otimes\mathcal{A}(z)(f\otimes g) := f\otimes\mathcal{A}(z)(g)$ for any $f\in L^2(\mathrm{Re}\,z=1)$ and $g\in L^2(\gamma)$. Denote by $A(z)$ the unique extension of $\mathrm{Id}\otimes\mathcal{A}(z)$ as an operator on the space $\left(L^2(\mathrm{Re}\,z=1)\overline{\otimes}L^2(\gamma)\right)^3$.

Proposition 5.

Consider an infinite Lipschitz cone in $\mathbb{R}^3$ *and denote by* Γ *its boundary. The following diagram is commutative:*

$$\begin{array}{ccccc}
\left(L^2(\Gamma)\right)^3 \simeq & \left(L^2(\mathbb{R}_+, t\,dt)\overline{\otimes}L^2(\gamma)\right)^3 & \stackrel{\mathcal{M}_r\mathbf{I}}{\longrightarrow} & \left(L^2(Re\,z=1)\overline{\otimes}L^2(\gamma)\right)^3 \\
& \downarrow K_{\mathcal{L},p} & & \downarrow A(z) \\
\left(L^2(\Gamma)\right)^3 \simeq & \left(L^2(\mathbb{R}_+, t\,dt)\overline{\otimes}L^2(\gamma)\right)^3 & \stackrel{\mathcal{M}_r\mathbf{I}}{\longrightarrow} & \left(L^2(Re\,z=1)\overline{\otimes}L^2(\gamma)\right)^3
\end{array}$$

where $\mathbf{I}$ *is the* 3×3 *matrix with entries* $\delta_{ij}I$ *and* $\gamma = \Gamma\cap S^2$.

As a consequence, the diagram

$$\begin{array}{ccc} \left(L^2(\Gamma)\right)^3 \simeq \left(L^2(\mathbb{R}_+, t\,dt)\bar{\otimes} L^2(\gamma)\right)^3 & \xrightarrow{\mathcal{M}_r \mathbf{I}} & \left(L^2(Re\, z = 1)\bar{\otimes} L^2(\gamma)\right)^3 \\ \downarrow \beta I - K_{\mathcal{L},p} & & \downarrow \beta I - A(z) \\ \left(L^2(\Gamma)\right)^3 \simeq \left(L^2(\mathbb{R}_+, t\,dt)\bar{\otimes} L^2(\gamma)\right)^3 & \xrightarrow{\mathcal{M}_r \mathbf{I}} & \left(L^2(Re\, z = 1)\bar{\otimes} L^2(\gamma)\right)^3 \end{array}$$

is also commutative.

Moreover, the above diagrams are commutative if we replace $K_{\mathcal{L},p}$ by $K_{\mathcal{S},s}$.

Proof. Let us drop the subscripts $\mathcal{L}$, p and $\mathcal{S}, s$. If $u^j(r\omega) = f^j(r)g^j(\omega)$, $j = 1, 2, 3$, with $f^j \in L^2(\mathbb{R}_+, t\,dt)$ and $g^j \in L^2(\gamma)$, the equalities (2.2) and (2.5) become

$$(K(\vec{u}))^i(r\omega) = \int_\gamma \int_{\mathbb{R}_+} k^{ij}(r/r', \omega, \omega') f^j(r') \frac{dr'}{r'} g^j(\omega')\, d\omega' . \tag{4.5}$$

The inner integral above is the Mellin convolution in the radial variable of the kernel $k^{ij}(\cdot, \omega, \omega')$ with $f^j(\cdot)$. Therefore, if one takes the Mellin transform with respect to the radial variable of both sides in (4.5) we get

$$\mathcal{M}_r((K(\vec{u}))^i(\cdot\omega))(z) = \mathcal{M}\left(f^j\right)(z) \int_\gamma \mathcal{M}\left(k^{ij}(\cdot, \omega, \omega')\right)(z) g^j(\omega')\, d\omega' . \tag{4.6}$$

The equality (4.6) reads

$$\mathcal{M}_r((K(\vec{u}))^i(\cdot\omega))(z) = \mathcal{M}\left(f^j\right)(z)\mathcal{A}_{ij}(z)g^j(\omega) .$$

In other words, $\mathcal{M}_r((K(\vec{u}))^i(\cdot\omega))(z) = (A(z)(\mathcal{M}_r\mathbf{I}(u(\cdot\omega))(z)))^i$. Now, a density argument completes the proof. □

The main technical result of this section is the following:

Proposition 6.

For any $z \in \mathbb{C}$, $Re\, z = 1$, we have

$$\mathcal{A}_{ij}(z) = \mathcal{A}^0_{ij}(z) + \mathcal{A}^1_{ij}(z), \quad i, j \in \{1, 2, 3\}$$

where $\mathcal{A}^1_{ij}(z)$ is a compact operator on $L^2(\gamma)$ and

$$\left\| \mathcal{A}^0_{ij}(z) \right\|_{\mathcal{L}(L^2(\gamma))} \le u_{ij}$$

with

$$u_{ij} := \frac{\left|\cos\left(\frac{\alpha}{2}\right)\right|}{3\mu + \lambda} \left[2\mu\delta_{ij} + \frac{3}{2}\left(1 + \delta_{ij}\right)(\mu + \lambda)\right] \quad \textit{in the elastostatic case ,}$$

and

$$u_{ij} := \frac{3}{2}\left(1 + \delta_{ij}\right) \left|\cos\left(\frac{\alpha}{2}\right)\right| \quad \textit{in the hydrostatic case .}$$

An immediate consequence of Proposition 6 is the following.

Corollary 2.

Let $z \in \mathbb{C}$ with $Re\, z = 1$. Consider α satisfying the condition (4.1). Then for any $\beta \in \mathbb{C}$, $|\beta| \ge 1$ the operator $\beta I - \mathcal{A}_{\mathcal{L},p}(z)$ is Fredholm with index zero on $(L^2(\gamma))^3$.

If α satisfies the condition (4.2), then for any $\beta \in \mathbb{C}$, $|\beta| \ge 1$, the operator $\beta I - \mathcal{A}_{\mathcal{S},s}(z)$ is Fredholm with index zero on $(L^2(\gamma))^3$.

Proof. Again we drop the subscripts $\mathcal{L}$, p and, respectively, $\mathcal{S}$, s and we will point out when it matters which of the settings we consider. Let $\mathcal{A}^m(z) = (\mathcal{A}^m_{ij}(z))_{ij}$ for $m = 0, 1$ and $i, j \in \{1, 2, 3\}$ be as in Proposition 6. Decompose

$$\beta I - \mathcal{A}(z) = \beta\left(I - \frac{\mathcal{A}^0(z)}{\beta}\right) - \mathcal{A}^1(z) .$$

By Proposition 6 we have

$$\left\|\mathcal{A}^0_{ij}(z)\right\|_{\mathcal{L}(L^2(\gamma))} \leq u_{ij} ,$$

for all $i, j \in \{1, 2, 3\}$. This easily implies that $\|\mathcal{A}^0(z)\|_{\mathcal{L}(L^2(\gamma))} \leq \|U\|_{\mathcal{L}(\mathbb{R}^3)}$, where $U := (u_{ij})_{i,j}$, $i, j = 1, 2, 3$. Note that

$$U = \begin{pmatrix} u_1 & u_2 & u_2 \\ u_2 & u_1 & u_2 \\ u_2 & u_2 & u_1 \end{pmatrix} ,$$

with

$$u_1 := \frac{5\mu + 3\lambda}{3\mu + \lambda}\left|\cos\left(\frac{\alpha}{2}\right)\right| \quad\text{and}\quad u_2 := \frac{3}{2}\frac{(\mu + \lambda)}{3\mu + \lambda}\left|\cos\left(\frac{\alpha}{2}\right)\right| ,$$

in the elastostatic case and

$$u_1 = 3\left|\cos\left(\frac{\alpha}{2}\right)\right| \quad\text{and}\quad u_2 := \frac{3}{2}\left|\cos\left(\frac{\alpha}{2}\right)\right| ,$$

in the hydrostatic case. Since the matrix U is real and symmetric, we have

$$\|U\|_{\mathcal{L}(\mathbb{R}^3)} = \max_{\xi}\{|\xi|;\ \xi \text{ is an eigenvalue of } U\} .$$

This and an explicit calculation give

$$\left\|\mathcal{A}^0(z)\right\|_{\mathcal{L}(L^2(\gamma))} \leq u_1 + 2u_2 < 1 ,$$

where the last inequality holds if α satisfies condition (4.1) or, respectively, condition (4.2).

This implies that the operator $\beta\left(I - \frac{\mathcal{A}^0(z)}{\beta}\right)$ is invertible. Going further, since $\mathcal{A}^1(z)$ is a compact operator on $\left(L^2(\gamma)\right)^3$ we have that $\mathcal{A}(z)$ is a compact perturbation of an invertible operator, and therefore it is a Fredholm operator on $(L^2(\gamma))^3$ with index zero. □

Now we are ready to present the proof of Proposition 6.

Proof of Proposition 6. Again, we shall drop the subscripts $\mathcal{L}$, p and $\mathcal{S}$, s when no confusion is likely to occur. Let us denote by $C^\infty_{\text{pw}}(\gamma)$ the space of all continuous functions on γ which are C^∞ on each closed arc $\overline{\gamma}_k$ of γ, $k = 1, \dots, M$. Whenever ω and ω' belong to the same arc γ_k of γ we have $n_{\omega'} \cdot \omega = 0$. Also

$$k^{ij}\left(t, \omega, \omega'\right) \sim \begin{cases} t^{\operatorname{Re} z} & \text{as } t \to 0 , \\ t^{\operatorname{Re} z - 3} & \text{as } t \to \infty , \end{cases}$$

where k^{ij} has been defined in (4.3). Consequently, for $z \in \mathbb{C}$, $\operatorname{Re} z = 1$ and $v \in C^\infty_{\text{pw}}(\gamma)$ the following operator

$$\mathcal{A}_{ij}(z)v(\omega) := \int_{\mathbb{R}_+}\int_{\gamma} t^{z-1}k^{ij}\left(t, \omega, \omega'\right)v\left(\omega'\right)\,d\omega'\,dt, \qquad \omega \in \gamma ,$$

is pointwise well defined.

Next, for each corner e_k of γ we choose $\phi_k \in \mathcal{C}^{\infty}_{\mathrm{pw}}(\gamma)$ such that $0 \leq \phi_k \leq 1$ and ϕ_k is supported in a small neighborhood of e_k which does not contain any other corner of γ. We set

$$\mathcal{A}^0_{ij}(z) := \sum_{1 \leq k \leq M} \phi_k \mathcal{A}_{ij}(z) \phi_k \quad \text{and} \quad \mathcal{A}^1_{ij}(z) := \mathcal{A}_{ij}(z) - \mathcal{A}^0_{ij}(z) \, .$$

The effect of the functions ϕ_k reflects in the properties of the kernel $\hat{k}_{ij}(t, \omega, \omega')$ of the operator $\mathcal{A}^1_{ij}$ (ω and ω' are far from each other). More specifically,

$$\mathcal{A}^1_{ij}(z)v(\omega) = \int_{\mathbb{R}_+ \times \gamma} t^{z-1} \hat{k}^{ij}\left(t, \omega, \omega'\right) v\left(\omega'\right) d\omega' \, dt$$

where the kernel function $\hat{k}_{ij} \in \mathcal{C}([0, \infty) \times \gamma \times \gamma)$ is $\mathcal{C}^{\infty}$ on each set $[0, \infty) \times \overline{\gamma}_k \times \overline{\gamma}_l$. That is

$$\mathcal{A}^1_{ij}(z)v(\omega) = \int_{\gamma} \kappa^{ij}\left(z; \omega, \omega'\right) v\left(\omega'\right) d\omega' \, ,$$

with $\kappa^{ij}(z; \omega, \omega') := \int_0^{\infty} t^{z-1} \hat{k}^{ij}(t, \omega, \omega') \, dt$. For any $z \in \mathbb{C}$ with $\operatorname{Re} z = 1$ we have

$$\left| t^{z-1} \hat{k}^{ij}\left(t, \omega, \omega'\right) \right| \leq C(\mu, \lambda) \frac{t}{(t+1)^3} \, ,$$

uniformly with respect to ω and ω', where $C(\mu, \lambda)$ is a positive constant depending only on the Lamé moduli μ and λ. This easily implies that $|\kappa^{ij}(z; \omega, \omega')| \leq C$ for some $C > 0$ independent of ω and ω' for all $z \in \mathbb{C}$, $\operatorname{Re} z = 1$. Since the polygon γ is compact, this in turn will imply that for $\operatorname{Re} z = 1$ the operator $\mathcal{A}^1_{ij}(z)$ is Hilbert–Schmidt and therefore compact.

Next, let us consider the operator $\mathcal{A}^0_{ij}(z)$ in a neighborhood of the corner $e_1 = \overline{\gamma_1} \cap \overline{\gamma_2}$. Let $\phi' = \chi_1 \phi_1$ and $\phi'' = \chi_2 \phi_1$, where χ_k is the characteristic function of $\overline{\gamma_k}$, $k = 1, 2$. Then the operator

$$\mathcal{B}_{ij}(z) := \phi'' \mathcal{A}_{ij}(z) \phi'$$

takes the form

$$\mathcal{B}_{ij}(z)v(s) = \int_0^{\infty} t^{z-1} \phi''(s) \int_0^{s_0} k^{ij}\left(t, \omega, \omega'\right) \phi'\left(s'\right) v\left(s'\right) ds' \, dt, \quad s \in (0, s_0) \, .$$

We can assume, without loss of generality, that e_1 is the north pole of S^2 and that γ_1 lies in the $x - z$ plane. If we denote by α the interior angle of the spherical polygon at e_1 and we consider $\omega \in \gamma_2$, $\omega' \in \gamma_1$ parameterized by the arc lengths from e_1, then

$$n_{\omega'} = (0, -1, 0), \quad \omega' = \left(\sin s', 0, \cos s'\right), \quad \omega = (\sin s \cos \alpha, \sin s \sin \alpha, \cos s) \, .$$

Let $b := n_{\omega'} \cdot \omega$ and $a := \omega \cdot \omega'$. We have

$$b = -\sin \alpha \sin s \quad \text{and} \quad a = \cos s \cos s' + \cos \alpha \sin s \sin s' \, .$$

In the new notation we have in the elastostatic case

$$\mathcal{B}_{ij}(z)v(s) =$$
$$\frac{\phi''(s)}{2\pi} \int_0^{\infty} \int_0^{s_0} t^z \left[\frac{\tilde{C} \delta_{ij} \, b}{(t^2 - 2at + 1)^{3/2}} + \tilde{D} \frac{b(t\omega_i - \omega'_i)(t\omega_j - \omega'_j)}{(t^2 - 2at + 1)^{5/2}} \right] \phi'(s') v(s') \, ds' \, dt \, ,$$

where $\tilde{C}$ and $\tilde{D}$ are given by

$$\tilde{C} := \frac{-2\mu}{3\mu + \lambda} \quad \text{and} \quad \tilde{D} := -\frac{3\mu + 3\lambda}{3\mu + \lambda} \, .$$

In the hydrostatic case

$$\mathcal{B}_{ij}(z)v(s) = -\frac{3\phi''(s)}{2\pi}\int_0^\infty \int_0^{s_0} t^z \frac{b(tw_i - \omega_i')(tw_j - \omega_j')}{(t^2 - 2at + 1)^{5/2}}\phi'(s')v(s')\,ds'\,dt \,.$$

Using that $tw_i - \omega_i'$ and $tw_j - \omega_j'$ are the ith and jth components of the vector $t\omega - \omega'$ with magnitude $(t^2 - 2at + 1)^{1/2}$, we get $\left|(tw_i - \omega_i')(tw_j - \omega_j')\right| \leq \frac{1}{2}(1 + \delta_{ij})(t^2 - 2at + 1)$. For $z = 1 + i\xi$, $\xi \in \mathbb{R}$, s_0 sufficiently small and $s \in (0, s_0)$, we obtain

$$\left|\mathcal{B}_{ij}(1 + i\xi)v(s)\right| \leq \frac{1}{2\pi}\int_0^{s_0} |b| \int_0^\infty m_{ij}\frac{t}{(t^2 - 2at + 1)^{3/2}}\,dt\,|v(s')|\,ds' \,, \tag{4.7}$$

where $m_{ij} := \frac{2\mu\delta_{ij} + \frac{3}{2}(1+\delta_{ij})(\mu+\lambda)}{3\mu+\lambda}$ in the elastostatic case and $m_{ij} := \frac{3}{2}(1 + \delta_{ij})$ in the hydrostatic case. Notice

$$\int_0^\infty \frac{t}{(t^2 - 2at + 1)^{3/2}}\,dt = 1 + a\int_0^\infty \frac{1}{(t^2 - 2at + 1)^{3/2}}\,dt = \frac{1}{1-a} \,. \tag{4.8}$$

For the last equality in (4.8) we have used that

$$\int_0^\infty \frac{1}{(t^2 - 2at + 1)^{3/2}}\,dt = \frac{1}{1-a}, \quad a < 1 \,,$$

(cf., e.g., [11, p. 296]). Using (4.8) in (4.7) gives

$$\left|\mathcal{B}_{ij}(1 + i\xi)v(s)\right| \leq \frac{1}{2\pi}m_{ij}\int_0^{s_0}\frac{|b|}{1-a}|v(s')|\,ds' \,.$$

That is

$$\begin{aligned} &\left|\mathcal{B}_{ij}(1 + i\xi)v(s)\right| \\ &\quad \leq \frac{1}{2\pi}m_{ij}\int_0^{s_0}\frac{\sin s|\sin\alpha|}{1 - \cos s\cos s' - \cos\alpha\sin s\sin s'}|v(s')|\,ds' \,. \end{aligned} \tag{4.9}$$

Let us make the change of variables $\theta = \tan(\frac{s}{2})$, $\theta' = \tan(\frac{s'}{2})$. Then we have $ds' = \frac{2d\theta'}{1+\theta'^2}$, $\sin s = \frac{2\theta}{\theta^2+1}$, $\cos s = \frac{1-\theta^2}{\theta^2+1}$, and similar identities for s' and θ'. Substituting this in (4.9) leads to

$$\begin{aligned} \left|\mathcal{B}_{ij}(1 + i\xi)v(\theta)\right| &\leq \frac{1}{\pi}m_{ij}\int_0^{\theta_0}\frac{\theta|\sin\alpha|}{\theta^2 + \theta'^2 - 2\theta\theta'\cos\alpha}|v(\theta')|\,d\theta' \\ &= \frac{1}{\pi}m_{ij}\int_0^{\theta_0}\frac{\frac{\theta}{\theta'}|\sin\alpha|}{\left(\frac{\theta}{\theta'}\right)^2 + 1 - 2\frac{\theta}{\theta'}\cos\alpha}|v(\theta')|\,\frac{d\theta'}{\theta'} \,, \end{aligned} \tag{4.10}$$

where $\theta_0 = \tan(\frac{s_0}{2})$ and $\theta \in (0, \theta_0)$. The right-hand side of (4.10) is a Mellin convolution operator with kernel

$$\tilde{k}_\alpha^{ij}(x) = \frac{1}{\pi}m_{ij}\frac{x|\sin\alpha|}{x^2 + 2x\cos(\pi - \alpha) + 1} \,.$$

By Proposition 4 the norm of this operator on the space $L^2(0, \theta_0)$ is equal to the absolute value of the Mellin transform of the kernel evaluated at $\operatorname{Re} z = 1/2$.

For any function f defined on the interval $(0, \infty)$ we have that $\mathcal{M}(xf(x))(z) = \mathcal{M}(f)(z+1)$. Therefore,

$$\begin{aligned}\mathcal{M}(\tilde{k}_{ij}^{\alpha})(z) &= \frac{m_{ij}}{\pi}|\sin\alpha|\mathcal{M}\left(\frac{1}{x^2+2x\cos(\pi-\alpha)+1}\right)(z+1)\\ &= m_{ij}\left[\frac{|\sin\alpha|}{\sin(\pi-\alpha)}(1)^{(\cdot-2)}\frac{\sin[(1-\cdot)(\pi-\alpha)]}{\sin(\pi\cdot)}\right](z+1)\\ &= m_{ij}\frac{|\sin\alpha|}{\sin\alpha}(1)^{(z-1)}\frac{\sin(z(\pi-\alpha))}{\sin(\pi z)},\end{aligned} \tag{4.11}$$

where the second equality in (4.11) holds for $0 < \operatorname{Re} z < 2$ and $-\pi < \alpha < \pi$ (see, e.g., [11, p. 297]). This implies

$$\left\|\mathcal{B}_{ij}(1+i\xi)\right\|_{\mathcal{L}(L^2(\gamma))} \leq m_{ij}\left|\cos\left(\frac{\alpha}{2}\right)\right| = u_{ij},$$

and completes the proof. □

With all the ingredients in place, we are ready to present the argument which gives the Fredholm property for the operators $\beta I - K_{\mathcal{L},p}$ and $\beta I - K_{\mathcal{S},s}$.

Theorem 5.

Let Γ be the boundary of an infinite polyhedral Lipschitz cone with all its dihedral angles α satisfying the condition (4.1). Then the operator $\beta I - K_{\mathcal{L},p}$ is Fredholm with index zero on $\left(L^2(\Gamma)\right)^3$ for any $\beta \in \mathbb{C}$, $|\beta| \geq 1$.

Also, if all dihedral angles α satisfy the condition (4.2), then the operator $\beta I - K_{\mathcal{S},s}$ is Fredholm with index zero on $\left(L^2(\Gamma)\right)^3$ for any $\beta \in \mathbb{C}$, $|\beta| \geq 1$.

Proof. Let $z \in \mathbb{C}$ be such that $\operatorname{Re} z = 1$. As a consequence of Corollary 2 we have that $\beta I - \mathcal{A}(z) \in \mathcal{L}\left(\left(L^2(\gamma)\right)^3\right)$ is Fredholm with index zero. This easily implies that $\operatorname{Id} \otimes (\beta I - \mathcal{A}(z))$ is Fredholm with index zero on $(L^2(\operatorname{Re} z = 1) \otimes L^2(\gamma))^3$.

A functional analysis argument (cf., e.g., [5]) guarantees the equivalence between the Fredholm property of the operator and the existence of a Banach space Y, a compact map $T : (L^2(\operatorname{Re} z = 1) \otimes L^2(\gamma))^3 \to Y$, and $C > 0$ such that

$$\|f \otimes g\| \leq C\|\operatorname{Id} \otimes (\beta I - \mathcal{A}(z))(f \otimes g)\| + \|T(f \otimes g)\|, \tag{4.12}$$

for any $f \otimes g \in (L^2(\operatorname{Re} z = 1) \otimes L^2(\gamma))^3$.

A limiting argument based on (4.12), in the sense of $(L^2(\operatorname{Re} z = 1)\overline{\otimes} L^2(\gamma))^3$, yields

$$\|h\| \leq C\|(\beta I - A(z))h\| + \|Th\|,$$

uniformly for $h \in (L^2(\operatorname{Re} z = 1)\overline{\otimes} L^2(\gamma))^3$. This implies that $\beta I - A(z)$ is Fredholm on $(L^2(\operatorname{Re} z = 1)\overline{\otimes} L^2(\gamma))^3$. By Proposition 4, since $\mathcal{M}_r I$ is an isomorphism, we get that $\beta I - K_{\mathcal{L},p}$ is Fredholm on $\left(L^2(\Gamma)\right)^3$ under the condition (4.1) and $\beta I - K_{\mathcal{S},s}$ is Fredholm on $\left(L^2(\Gamma)\right)^3$ under the condition (4.2). Since the index is constant on connected components in the space of Fredholm operators and $\beta I - K_{\mathcal{L},p}$ and $\beta I - K_{\mathcal{S},s}$ are invertible for $|\beta|$ large, the desired conclusion follows. □

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