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Irregular Sampling and the Inverse Spectral Problem

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ABSTRACT. We are interested in finding necessary and sufficient conditions for irregular sampling to hold. We shall show that the inverse spectral problem can be used to construct sampling type theorems from the knowledge of the sampling points only. This improves Kramer's theorem as it reveals all possible distributions of the sampling points together with a construction of the sampling functions.

1. Introduction

We are concerned with sampling type theorems in Paley–Wiener spaces and are particularly interested in unifying the different existing approaches, see [2, 4, 8, 13]. In this paper we would like to show that irregular sampling formulae can be viewed as an interpolation generated by an inverse transform, whenever a certain inverse spectral problem can be solved. In general, we are given the set $\{\mu_n, \ S_n(\mu)\}$ and the issue is to recover a unique function F from its values $\{F(\mu_n)\}$ by a sampling expansion formula

$$F(\mu) = \sum_{n\geq 0} F(\mu_n)\, S_n(\mu)\,.$$

Using the Sinc functions, the set $\left\{n, \frac{\sin\pi(\mu-n)}{\pi(\mu-n)}\right\}$ defines a well-known example of regular sampling for band limited functions. Kramer pointed out (see [4]) that Sturm–Liouville systems with discrete spectra generate sampling type theorems where the sampling points μ_n are precisely the eigenvalues of a self-adjoint operator defined by

$$-\left(p(x)y'(x,\mu)\right)' + q(x)y(x,\mu) = \mu w(x)y(x,\mu) \quad \text{where} \quad 0 < x < b \tag{1.1}$$

together with appropriate boundary conditions. The main drawback of this interesting result lies in the fact that the sampling points, that is the eigenvalues, are not explicitly known. Given (1.1) one could approximate few eigenvalues and their eigenfunctions, but that is not very practical. A similar problem described in [12] questions whether the set $\{-an+b, \ an+d\}_{n\geq 0}$, where a, b, and d are positive real numbers, can be a sampling sequence.

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It is clear that in practice one is given a sequence of values $\{F(\mu_n)\}_{n\geq 0}$, where the μ_n can be randomly distributed, and is asked to recover uniquely the original function. Indeed one cannot always choose the sampling points μ_n while an experiment is in progress and in general data processing involves the rejection or loss of some of the values before the recovery process can take place. In other words, μ_n are not necessarily uniformly spaced or integers. This leads us to address the following problem:

Statement of the problem
Given a sequence of real numbers $\{\mu_n\}_{n\geq 0}$, can we find a space containing a sequence $\{S_n(\mu)\}_{n\geq 0}$ such that the values $\{F(\mu_n)\}_{n\geq 0}$ define a unique function by the following sampling expansion formula,

$$F(\mu) = \sum_{n\geq 0} F(\mu_n)\, S_n(\mu)\ ?$$

Thus, as stated, we are dealing with an inverse problem where we need to construct $\{S_n(\mu)\}_{n\geq 0}$ from the given sequence $\{\mu_n\}_{n\geq 0}$. The main tool shall be the inverse spectral problem which has been pioneered by Gelfand and Levitan, and Krein. For the sake of simplicity, we shall restrict ourselves to the Gelfand–Levitan theory, see [3, 6, 7].

2. The Inverse Spectral Problem

Let $\{\mu_n\}_{n\geq 0}$ and $\{\alpha_n\}_{n\geq 0}$ be two given sequences such that μ_n^2 are distinct and $\alpha_n > 0$. We are then asked to find a real function $q \in L^1(0,\pi)$ and two real constants h and H such that the self-adjoint operator

$$\begin{cases} -y''(x,\mu) + q(x)y(x,\mu) = \mu^2 y(x,\mu) & 0 \leq x \leq \pi \\ y'(0,\mu) - hy(0,\mu) = 0, \qquad y'(\pi,\mu) + Hy(\pi,\mu) = 0 \end{cases} \tag{2.1}$$

has a discrete spectrum defined by $\{\mu_n^2\}$, and the corresponding eigenfunctions satisfy

$$\|y(x,\mu_n)\|^2 := \int_0^\pi |y(x,\mu_n)|^2\, dx = \alpha_n\ .$$

For each $x \in [0,\pi]$, the solution $y(x,.)$ satisfying the initial conditions

$$y(0,\mu) = 1 \ \text{ and } \ y'(0,\mu) = h\ ,$$

is an entire function in μ, and the eigenvalues of (2.1) are then the simple zeroes of

$$y'(\pi,\mu) + Hy(\pi,\mu) = 0\ .$$

The Gelfand–Levitan theory is based on the fact that solutions $y(x,\mu)$ can be expressed in terms of $\cos(\mu x)$, see [7],

$$y(x,\mu) = \cos(\mu x) + \int_0^x K(x,t)\cos(\mu t)dt \qquad 0 \leq x \leq \pi \tag{2.2}$$

and conversely

$$\cos(\mu x) = y(x,\mu) + \int_0^x H(x,t)y(t,\mu)dt \qquad 0 \leq x \leq \pi \tag{2.3}$$

where the smoothness of the kernels K and H depends on the smoothness of the potential q only. Let us agree to denote transforms

$$\mathcal{F}_c(f)(t) := \int_0^\pi f(t)\cos(\mu t)d\mu \qquad \mathcal{F}_y(f)(\mu) := \int_0^\pi f(x)y(x,\mu)dx$$

where $\mathcal{F}_c$ and $\mathcal{F}_y$ are the cosine transform and y-transform, respectively, and the Paley–Wiener space of even functions

$$PW_\pi^e := \left\{F(\mu) \text{ entire: } F(-\mu) = F(\mu), \quad |F(\mu)| < Me^{\pi|Im\mu|} \quad F \in L^2(R)\right\}$$

where

$$L^p(a,b) := \left\{f \text{ measurable and } \int_a^b |f(x)|^p dx < \infty\right\}.$$

Using the Gelfand–Levitan machinery, we first obtain the following.

Proposition 1.

Let $q \in L^1(0,\pi)$, then $\mathcal{F}_y(f) \in PW_\pi^e$ for $f \in L^2(0,\pi)$.

Proof. Since $q \in L^1(0,\pi)$, K has its first order derivatives locally integrable (see [6, p 22]) and so it follows from (2.3) that $y(x,.)$ is entire with type π and order 1. Obviously from (2.3) the two transforms are related by

$$\mathcal{F}_y(f)(\mu) = \mathcal{F}_c(f)(\mu) + \mathcal{F}_c\left(\int_x^\pi K(t,x)f(t)dt\right)(\mu).$$

Thus, if $f \in L^2_{dx}(0,\pi)$, then $\mathcal{F}_y(f)$ is entire of order one, type π, and even. Together Parseval equality

$$\int_0^\infty |\mathcal{F}_c(f)(\mu)|^2 d\mu = \frac{\pi}{2}\int_0^\pi |f(x)|^2 dx < \infty$$

and

$$\begin{aligned}\int_0^\infty \left[\mathcal{F}_c\left(\int_x^\pi K(t,x)f(t)dt\right)(\mu)\right]^2 d\mu &= \frac{\pi}{2}\int_0^\pi \left(\int_x^\pi K(t,x)f(t)dt\right)^2 dx \\ &\leq \frac{\pi}{2}\int_0^\pi \int_0^x |K(x,t)|^2 dtdx \, ||f||^2\end{aligned}$$

imply that

$$\int_{-\infty}^\infty |\mathcal{F}_y(f)(\mu)|^2 d\mu < \infty$$

and therefore we deduce that $\mathcal{F}_y(f) \in PW_\pi^e$. □

We now recall the main ingredients needed for the existence and construction of the function q. The Gelfand–Levitan–Gasymov theory is based on the solution of the following Fredholm integral equation (see [6]):

$$\Gamma(x,t) + K(x,t) + \int_0^x K(x,s)\Gamma(s,t)ds = 0 \qquad 0 \leq t \leq x \tag{2.4}$$

where

$$\Gamma(x,t) = \frac{1}{\alpha_0}\cos(\mu_0 x)\cos(\mu_0 t) - \frac{1}{\pi} + \sum_{n\geq 1}\left[\frac{\cos(\mu_n x)\cos(\mu_n t)}{\alpha_n} - \frac{2}{\pi}\cos(nx)\cos(nt)\right]$$

is given. The function $q(x)$, and boundary conditions needed to form (2.1), are obtained from K once (2.4) is solved,

$$\begin{aligned} h &= K(0,0) \\ q(x) &= 2\frac{d}{dx}K(x,x) \\ H &= \pi\gamma - K(\pi,\pi) \end{aligned}$$

where the constant $\gamma := \lim_{n\to\infty} n\,(\mu_n - n)$ depends only on the behavior of μ_n at infinity.

Proposition 2 (Gelfand–Levitan–Gasymov).
The sequences $\{\mu_n,\}$ and $\{\alpha_n\}$ are the spectral characteristic of (2.1) with $q^{(m)} \in L^1(0,\pi)$ if and only if

$$\mu_n^2 \text{ are distinct, } \alpha_n > 0, \quad \mu_n = n + \frac{\gamma}{n} + o\left(\frac{1}{n}\right) \quad \alpha_n = \frac{\pi}{2} + o\left(\frac{1}{n}\right) \quad \text{as } n \to \infty \tag{2.5}$$

and the function Γ has integrable derivatives up to the order $m+1$.

The strength of the above proposition lies in the fact that (2.5) is both a necessary and a sufficient condition for the existence of a potential q, with $m \geq 0$, to form the regular Sturm–Liouville problem (2.1). It is interesting to observe that the distribution of the sampling points is arbitrary except as $\mu \to \infty$, which is totally different from the Kadec condition, namely $|\mu_n - n| < \frac{1}{4}$, which ensures the completeness of sequence $\{\exp(i\mu_n x)\}$, see [11]. A finite number of points μ_n can be negative, as long as the eigenvalues μ_n^2 are all distinct.

3. Interpolation

Basically we shall be given two sequences $\{\mu_n\}_{n\geq 0}$ and $\{\alpha_n\}_{n\geq 0}$ satisfying (2.5), where μ_n represent the sampling points. Hence, we are guaranteed the existence of a certain regular Sturm–Liouville problem on $[0,\pi]$, defined by (2.1), with spectrum $\left\{\mu_n^2\right\}_{n\geq 0}$ and $||y(x,\mu_n)||^2 = \alpha_n$.

We now briefly recall the main points used in Kramer's theorem (see [4]). Given a sequence $\{a_n\}_{n\geq 0}$, such that

$$\sum_{n\geq 0} \frac{|a_n|^2}{\alpha_n} < \infty\,,$$

we can construct, using the inverse $y-$transform associated with the Sturm–Liouville problem (2.1), an element $f \in L^2(0,\pi)$

$$f(x) := \sum_{n\geq 0} \frac{a_n}{\alpha_n} y\,(x,\mu_n)\ .$$

Now take the $y-$transform of f to obtain an even entire function of μ,

$$F(\mu) := \int_0^\pi f(x) y(x,\mu) dx\ .$$

From Proposition 1, we have that $F \in PW_\pi^e$, and

$$F(\mu) := \int_0^\pi \sum_{n\geq 0} \frac{a_n}{\alpha_n} y\,(x,\mu_n)\, y(x,\mu) dx\ .$$

The functions $y(x,\mu)$ being locally bounded

$$|y(x,\mu)| \leq \left[1 + \sup_{0<x<\pi} \int_0^x |K(x,t)|\,dt\right] \sup_{\mu\in D} |\exp(x\,|\mu|)| \qquad \text{for} \quad \mu \in D$$

leads to a simple result.

Proposition 3.

Assume that $\{\mu_n, \alpha_n\}_{n\geq 0}$ *satisfy (2.5) and let* $f(x) := \sum_{n\geq 0} \frac{a_n}{\alpha_n} y(x, \mu_n)$ *with* $\sum_{n\geq 0} |a_n| < \infty$ *and* F *be its* y*-transform. Then* F *is an entire function defined by*

$$F(\mu) := \sum_{n\geq 0} F(\mu_n)\, \tilde{S}_n(\mu) \tag{3.1}$$

where $F(\mu_n) = a_n$ *and* $\tilde{S}_n(\mu) = \frac{1}{\alpha_n}\int_0^\pi y(x, \mu_n) y(x, \mu) dx$.

Proof. The absolute convergence of the series $\sum a_n$ implies the uniform convergence of the series

$$\sum_{n\geq 0} \frac{a_n}{\alpha_n} y(x, \mu_n)\, y(x, \mu)$$

and so

$$\begin{aligned} F(\mu) &= \int_0^\pi f(x)\, y(x, \mu) dx \\ &= \int_0^\pi \sum_{n\geq 0} \frac{a_n}{\alpha_n} y(x, \mu_n)\; y(x, \mu) dx \\ &= \sum_{n\geq 0} \frac{a_n}{\alpha_n} \int_0^\pi y(x, \mu_n)\, y(x, \mu) dx\,. \end{aligned}$$

The equality $F(\mu_k) = a_k$ is readily verified from the fact that eigenfunctions $\{y(x, \mu_n)\}$ are orthogonal

$$\frac{1}{\alpha_n}\int_0^\pi y(x, \mu_n)\, y\left(x, \mu_j\right) dx = \delta(n, j)$$

and so the result.

The absolute convergence of the series $\sum a_n$ is only a sufficient condition but not necessary. Using the fact that the mapping

$$\mu \to \int_0^\pi |y(x, \mu)|^2 dx$$

is locally bounded, one can prove the sampling expansion [5], which leads us to the following result. □

Proposition 4.

Let μ_n *and* α_n *satisfy (2.5), then for any* $F \in PW_\pi^e$*, we have*

$$F(\mu) = \sum_{n\geq 0} F(\mu_n)\, \tilde{S}_n(\mu)\,.$$

Proof. We only need to show that the space [4]

$$\mathbf{K} := \left\{F \text{ entire}: \quad F(\mu) = \int_0^\pi f(x) y(x, \mu) dx \quad \text{where } f \in L^2(0, \pi)\right\}$$

coincides with PW_π^e. We already have that $\mathbf{K} \subset PW_\pi^e$ by Proposition 1. Conversely, if $F \in PW_\pi^e$, then by the classical Paley–Wiener theorem we can write

$$F(\mu) = \int_0^\pi \psi(x) \cos(x\mu) dx \quad \text{where} \quad \psi \in L^2(0, \pi)\,.$$

Using the transformation operator (2.3) we obtain

$$\begin{aligned} F(\mu) &= \int_0^\pi \psi(x)\cos(x\mu)dx \\ &= \int_0^\pi \psi(x)\,[1+H]\,y(x,\mu)dx \\ &= \int_0^\pi [1+H]^*\,\psi(x)\;y(x,\mu)dx\,. \end{aligned}$$

Since $[1+H]^*\,\psi(x) \in L^2(0,\pi)$, it follows that $F \in \mathbf{K}$. The rest is a consequence of Proposition 3. □

Looking at $\{\mu_n\}_{n\geq 0}$ as a perturbation of $\{n\}_{n\geq 0}$, we can estimate the deviation of $\tilde{S}_n$ from the well-known Shannon sampling sequence S_n by

$$\begin{aligned} \tilde{S}_n(\mu) &= \frac{1}{\alpha_n}\int_0^\pi y(t,\mu)y(t,\mu_n)\,dt \\ &= \frac{1}{\alpha_n}\int_0^\pi (1+K)\cos(\mu t)\;(1+K)\cos(\mu_n t)\,dt \\ &= \frac{1}{\alpha_n}\int_0^\pi \left(1+K^*\right)(1+K)\cos(\mu t)\;\cos(\mu_n t)\,dt \\ &= \frac{1}{\alpha_n}\int_0^\pi \cos(\mu t)\cos(\mu_n t)\,dt + \frac{1}{\alpha_n}\int_0^\pi \left[\left(K^*+K+K^*K\right)\cos(\mu t)\right]\cos(\mu_n t)\,dt \\ &= \frac{\pi}{2\alpha_n}S_n(\mu) + \frac{1}{\alpha_n}\int_0^\pi \left[\left(K^*+K+K^*K\right)\cos(\mu t)\right]\cos(\mu_n t)\,dt \qquad (3.2) \end{aligned}$$

where $Kf(x) := \int_0^x K(x,t)f(t)dt, \quad K^*f(x) := \int_x^\pi K(t,x)f(t)dt$ and

$$S_n(\mu) = \frac{\sin[\pi(\mu+\mu_n)]}{[\pi(\mu+\mu_n)]} + \frac{\sin[\pi(\mu-\mu_n)]}{[\pi(\mu-\mu_n)]}\,.$$

Using the nonlinear equation (see [1] and [7]), we have

$$\Gamma = K^* + K + KK^* \qquad (3.3)$$

from which follows that (3.2) can be rewritten as

$$\tilde{S}_n(\mu) - \frac{\pi}{2\alpha_n}S_n(\mu) = \frac{1}{\alpha_n}\int_0^\pi \left[\left(\Gamma + K^*K - KK^*\right)\cos(\mu_n t)\right]\;\cos(\mu t)dt\,. \qquad (3.4)$$

If the kernel $\Gamma \in C^{(m)}$, then it also follows that $K, K^* \in C^{(m)}$ (see [6, lemma 1.3.1]). Thus, we have $\left[(\Gamma + K^*K - KK^*)\cos(\mu_n t)\right] \in C^{(m)}$ and the Lebesgue–Riemann theorem we obtain the following.

Proposition 5.

Assume that μ_n and α_n satisfy (2.5) and $\Gamma \in C^{(m)}$, then we have the following estimates:

[i] $\int_0^\infty \left|\left(\tilde{S}_n - \frac{\pi}{2\alpha_n}S_n\right)(\mu)\right|^2 d\mu \leq \left(\frac{\pi}{2\alpha_n}\right)^2\left[2\,||K|| + ||K||^2\right]^2,$

[ii] $\tilde{S}_n(\mu) - \frac{\pi}{2\alpha_n}S_n(\mu) = o\left(\frac{1}{\mu^m}\right) \qquad$ *as* $\mu \to \pm\infty$*, and*

[iii] $\left|\tilde{S}_n(\mu) - \frac{\pi}{2\alpha_n}S_n(\mu)\right| = o\left(\frac{1}{\mu_n^m}\right) \qquad$ *as* $n \to \infty$.

Proof. [i] follows from Parseval equality applied to (3.2)

$$\begin{aligned}\int_0^\infty \left|\left(\tilde{S}_n - \frac{\pi}{2\alpha_n} S_n\right)(\mu)\right|^2 d\mu &= \frac{\pi}{2\alpha_n^2}\int_0^\pi \left|\left(K^* + K + K^*K\right)\cos(\mu_n t)\right|^2 dt \\ &\leq \frac{\pi}{2\alpha_n^2}\left\|K^* + K + K^*K\right\|^2 . \int_0^\pi |\cos(\mu_n t)|^2 dt \\ &\leq \frac{1}{2}\left(\frac{\pi}{\alpha_n}\right)^2 \left\|K^* + K + K^*K\right\|^2 \\ &\leq \frac{1}{2}\left(\frac{\pi}{\alpha_n}\right)^2 \left(\left\|K^*\right\| + \|K\| + \left\|K^*K\right\|\right)^2 \\ &\leq \frac{1}{2}\left(\frac{\pi}{\alpha_n}\right)^2 \left(2\|K\| + \left\|K^*K\right\|\right)^2 .\end{aligned}$$

[ii] is proved by applying the Riemann-Lebesgue theorem to $(K^* + K + K^*K)\cos(\mu_n t)$, which is $C^{(m)}$ and similarly [iii] is proved by applying the Riemann–Lebesgue theorem to $(K^* + K + K^*K)^* \cos(\mu t)$, which is also $C^{(m)}$. □

The upper bound which involves $\|K\| := \sqrt{\int_0^\pi \int_0^\pi |K(x,t)|^2\, dxdt}$ can be estimated in terms of q only (see [10]), and we can also bring in the sampling points by using the function Γ as done in (3.4).
We now show that $\tilde{S}_n$ is independent of $\{\alpha_n\}$ if the points $\{\mu_n\}_{n\geq 0}$ are fixed. To this end, denote by q_1 and q_2 two potentials corresponding to two different choices of α_n and let

$$\delta_n(\mu) = S_n^{(1)}(\mu) - S_n^{(2)}(\mu)$$

where $S_n^{(i)}(\mu) = \frac{1}{\alpha_n^{(i)}} \int_0^\pi y_i(x,\mu) y_i(x,\mu_n) dx, \quad i = 1, 2$. Clearly $\delta_n(\mu) \in PW_\pi^e$ and

$$\delta_n(\mu_k) = 0 \qquad k = 0, 1, 2, \ldots$$

so the inverse $y^{(1)}$–transform of δ_n is the zero function for $t \in (0, \pi)$. Thus, we deduce that $\delta_n(\mu) = 0$ and so $S_n^{(1)}(\mu) = S_n^{(2)}(\mu)$, i.e., the sampling functions are independent of the choice of α_n.

Proposition 6.
The functions $\left\{\tilde{S}_n\right\}_{n\geq 0}$ *depend on the sequence* $\{\mu_n\}$ *only.*

We shall see that in the particular case where $\mu_n = n, \alpha_n = \frac{\pi}{2}$, for $n = 0, 1, \ldots$, then the inverse spectral method yields the classical Shannon sampling theorem. Indeed if $\Gamma = 0$, then we have from (2.4) $K = 0$, and it follows from (3.4) that

$$\mu_n = n \qquad \longrightarrow \qquad \tilde{S}_n(\mu) = \frac{\sin[\pi(\mu + n)]}{[\pi(\mu + n)]} + \frac{\sin[\pi(\mu - n)]}{[\pi(\mu - n)]} .$$

4. Frames

We now ask whether the newly obtained system $\left\{\tilde{S}_n\right\}_{n\geq 0}$ is a frame in PW_π^e. We recall that $\tilde{S}_n(\mu)$ is a frame if

$$A\,\|F\|^2 \leq \sum_{n\geq 0}\left|\left(F, \tilde{S}_n\right)\right|^2 \leq B\,\|F\|^2 \qquad \forall F \in PW_\pi^e \tag{4.1}$$

where $-\infty < A \le B < \infty$ and the inner product is defined by

$$\left(F, \tilde{S}_n\right) = \int_{-\infty}^{\infty} F(\mu)\overline{\tilde{S}_n(\mu)}d\mu .$$

It is readily seen that from (2.2) follows

$$\begin{aligned} \mathcal{F}_y(f)(\mu) &= \int_0^{\pi} f(t)y(t,\mu)dt \\ &= \int_0^{\pi} f(t)\,(1+K)\cos(t\mu)dt \\ &= \int_0^{\pi} \left(1+K^*\right) f(t)\cos(t\mu)dt \\ &= \mathcal{F}_c\left(\left(1+K^*\right) f(t)\right)(\mu) . \end{aligned}$$

where

$$Kf(x) = \int_0^x K(x,t)f(t)dt \qquad \text{and} \quad K^*f(x) = \int_x^{\pi} K(t,x)f(t)dt .$$

By Proposition 4 $F \in PW_{\pi}^{e}$ means that $F = \mathcal{F}_y(f)$ and thus Parseval equality leads

$$\begin{aligned} \left(F, \tilde{S}_n\right) &= \int_{-\infty}^{\infty} \mathcal{F}_y(f(t))(\mu)\frac{1}{\alpha_n}\mathcal{F}_y\left(y\left(t,\mu_n\right)\right)(\mu)d\mu \\ &= \frac{2}{\alpha_n}\int_0^{\infty} \mathcal{F}_c\left(\left(1+K^*\right) f(t)\right)(\mu)\,\mathcal{F}_c\left(\left(1+K^*\right) y\left(t,\mu_n\right)\right)(\mu)d\mu \\ &= \frac{\pi}{\alpha_n}\int_0^{\pi} \left(1+K^*\right) f(t)\left(1+K^*\right) y\left(t,\mu_n\right)dt \\ &= \frac{\pi}{\alpha_n}\int_0^{\pi} (1+K)\left(1+K^*\right) f(t) y\left(t,\mu_n\right)dt \\ &= \frac{\pi}{\alpha_n}\int_0^{\pi} [1+\Gamma]\, f(t)\; y\left(t,\mu_n\right)dt \\ &= \frac{\pi}{\alpha_n}\mathcal{F}_y\left([1+\Gamma]\, f\right)(\mu_n) . \end{aligned}$$

Thus, we have

$$\sum_{n\geq 0}\left|\left(F, \tilde{S}_n\right)\right|^2 = \sum_{n\geq 0}\left(\frac{\pi}{\alpha_n}\right)^2 \left|\mathcal{F}_y\left([1+\Gamma]\, f\right)(\mu_n)\right|^2 . \tag{4.2}$$

We recall that Parseval equality associated with the y-transform yields

$$\|(1+\Gamma)\, f\|^2 = \sum_{n\geq 0}\frac{1}{\alpha_n}\left|\mathcal{F}_y\left([1+\Gamma]\, f\right)(\mu_n)\right|^2 . \tag{4.3}$$

and so from (4.2) and (4.3) we have

$$\pi\left(\inf\frac{\pi}{\alpha_n}\right)\|(1+\Gamma)\, f\|^2 \le \sum_{n\geq 0}\left|(F, \tilde{S}_n)\right|^2 \le \pi\left(\sup\frac{\pi}{\alpha_n}\right)\|(1+\Gamma)\, f\|^2 . \tag{4.4}$$

On the other hand, if $F \in PW_{\pi}^{e}$, then we obtain from the nonlinear equation (3.3)

$$\int_{-\infty}^{\infty} |F(\mu)|^2 d\mu \;=\; 2\int_0^{\infty} |F(\mu)|^2 d\mu$$

$$
\begin{aligned}
&= 2\int_0^\infty \left|\mathcal{F}_c\left[\left(1+K^*\right)f(\mu)\right]\right|^2 d\mu \\
&= \pi\int_0^\pi \left|\left(1+K^*\right)f(t)\right|^2 dt &(4.5)\\
&= \pi\int_0^\pi (1+K)\left(1+K^*\right)f(t)\ f(t)dt \\
&= \pi\int_0^\pi [(1+\Gamma)f(t)]\ f(t)dt\,. &(4.6)
\end{aligned}
$$

The Cauchy–Schwartz inequality then yields

$$\int_{-\infty}^\infty |F(\mu)|^2\,d\mu \le \pi\left\|(1+\Gamma)^{-1}\right\|\ \|(1+\Gamma)f\|^2\,. \tag{4.7}$$

Recall that since K is a Volterra operator, the inverse of $1+K$ exists

$$1+H=(1+K)^{-1}$$

where H is also a Volterra operator. Thus it follows that

$$
\begin{aligned}
(1+\Gamma)^{-1} &= \left[(1+K)\left(1+K^*\right)\right]^{-1} \\
&= \left(1+K^*\right)^{-1}(1+K)^{-1} \\
&= \left[(1+K)^{-1}\right]^*(1+K)^{-1} \\
&= [1+H]^*[1+H]
\end{aligned}
$$

which implies that

$$
\begin{aligned}
\left\|(1+\Gamma)^{-1}\right\| &\le \left\|[1+H]^*[1+H]\right\| \\
&\le \|1+H\|^2
\end{aligned}
$$

and so (4.7) reduces to

$$\frac{1}{\pi\,\|1+H\|^2}\int_{-\infty}^\infty |F(\mu)|^2\,d\mu \le \|(1+\Gamma)f\|^2\,. \tag{4.8}$$

Similarly using (4.5) we obtain

$$
\begin{aligned}
\|(1+\Gamma)f\|^2 &\le \left\|(1+K)\left(1+K^*\right)f\right\|^2 \\
&\le \|(1+K)\|^2\left\|\left(1+K^*\right)f\right\|^2 \\
&\le \frac{1}{\pi}\|(1+K)\|^2\int_{-\infty}^\infty |F(\mu)|^2\,d\mu &(4.9)
\end{aligned}
$$

Using (4.8) and (4.9), we can recast (4.4) into

$$\pi\frac{1}{\sup\alpha_n}\frac{\int_{-\infty}^\infty |F(\mu)|^2\,d\mu}{\|(1+H)\|^2} \le \sum_{n\cdot\ge 0}\left|\left(F,\tilde{S}_n\right)\right|^2 \le \pi\frac{1}{\inf\alpha_n}\|(1+K)\|^2\int_{-\infty}^\infty |F(\mu)|^2\,d\mu\,.$$

If we further use the fact that F is even and assume that $\alpha_n=\frac{\pi}{2}$, then

$$\frac{4\int_0^\infty |F(\mu)|^2\,d\mu}{\|(1+H)\|^2} \le \sum_{n\cdot\ge 0}\left|\left(F,\tilde{S}_n\right)\right|^2 \le 4\,\|(1+K)\|^2\int_0^\infty |F(\mu)|^2\,d\mu\,.$$

It remains to show that $\left\{\tilde{S}_n\right\}_{n\geq 0}$ will form an exact frame. To this end let us remove an element $\tilde{S}_j$, say where $j \geq 0$, and show that the remaining elements will not form a frame. If we consider f defined by

$$f(x) := (1+\Gamma)^{-1} y\left(x, \mu_j\right)$$

and let $F(\mu) := \mathcal{F}_y(f)(\mu)$, then it follows

$$\begin{aligned} \left(F, \tilde{S}_n\right) &= \frac{\pi}{\alpha_n}\mathcal{F}_y((1+\Gamma) f)(\mu_n) \\ &= \frac{\pi}{\alpha_n}\mathcal{F}_y\left(y\left(x, \mu_j\right)\right)(\mu_n) \\ &= \frac{\pi}{\alpha_n}\delta(j, n) \, . \end{aligned}$$

Therefore,

$$\sum_{n\geq 0, n\neq j} \left|\left(F, \tilde{S}_n\right)\right|^2 = \sum_{n\geq 0, n\neq j} \delta(j, n) = 0$$

whereas

$$\|F\|^2 = \left\|(1+\Gamma)^{-1} y\left(x, \mu_j\right)\right\|^2 \neq 0 \, .$$

Hence, the set $\left\{\tilde{S}_n\right\}_{n\neq j}$ is not a frame and so we have proved the following.

Proposition 7.

Assume that the sequences $\{\mu_n\}$ and $\{\alpha_n\}$ satisfy condition (2.5), then $\tilde{S}_n$ will form an exact frame in PW_π^e.

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References

[1] Boumenir, A. (1991). Comparison theorem for self-adjoint operators, *Proc. Am. Math. Soc.,* **111**(1), 161–175.

[2] Butzer, P.L. and Stens, R.L. (1992). Sampling theory for not necessarily band-limited functions, *SIAM Rev.,* **34**, 40–53.

[3] Dym, H. and McKean, H.P. (1976). *Gaussian Processes, Function Theory and Inverse Spectral Problem,* Academic Press, New York.

[4] Everitt, W.N. and Nasri-roudsari, G. Interpolation and sampling theories, and linear ordinary boundary value problems, The University of Birmingham, Pre-prints 96/2.

[5] Everitt, W.N., Nasri-roudsari, G., and Rehberg. J. (To appear). A note on the Kramer sampling theorem.

[6] Gasymov, M.G. and Levitan, B.M. (1964). Determination of a differential equation from two of its spectra, *Russian. Math. Surveys,* **19**(2), 1–63.

[7] Gelfand, I.M. and Levitan, B.M. (1951). On the determination of a differential equation from its spectral function, *Am. Math. Transl.,* **1**(2), 239–253.

[8] Nashed M.Z. and Walter, G. (1995). Reproducing kernel Hilbert spaces from sampling expansions, Contemporary Mathematics, *Am. Math. Soc.,* **190**, 221–227.

[9] Naimark, M.A. (1968). *Linear Differential Operators,* Part II, Ungar.

[10] Marchenko, V.A. (1986). Strum-Liouville operators and applications, in *Operator Theory: Advances and Applications,* Birkhauser, Boston, MA, 22.

[11] Young, R.M. (1980). *An Introduction to Nonharmonic Fourier Series,* Academic Press, New York.

[12] Zayed, A. (1992). Existence of a boundary value problem with prescribed eigenvalues, *SIAM Rev.,* **34**(1), 118–119.

[13] Zayed, A. (1993). *Advances in Shannon's Sampling Theory,* CRC Press, Boca Raton, FL.

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