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Injectivity Sets for Spherical Means on the Heisenberg Group

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ABSTRACT. We prove that the boundary of a bounded domain is a set of injectivity for the twisted spherical means on $\mathbb{C}^n$ for a certain class of functions on $\mathbb{C}^n$. As a consequence we obtain results about injectivity of the spherical mean operator in the Heisenberg group and the complex Radon transform.

1. Introduction

Let $f \in C(\mathbb{R}^n)$, $n \geq 2$. The spherical means of f are $Mf(x, r) = \int_{S(x,r)} f d\sigma$ where $S(x, r)$ is the $(n-1)$-dimensional sphere of radius r centered at x, and σ is the normalized surface measure on $S(x, r)$. A set $S \subseteq \mathbb{R}^n$ is a set of injectivity for the spherical means in a subclass C of continuous functions on $\mathbb{R}^n$ if $f \in C$ with $Mf(x, r) = 0$, for all $r \geq 0$ and $x \in S$, imply that $f = 0$. Some of the references where the problem of finding injectivity sets for the spherical means has been investigated are [2, 3, 4, 7]. More references can be found in [3].

In this paper, we consider the analogous question for the twisted spherical means on $\mathbb{C}^n$, $n \geq 1$. Let μ_r be the normalized surface measure on the sphere $\{z \in \mathbb{C}^n : |z| = r\}$. For $f \in C(\mathbb{C}^n)$, the twisted spherical means of f are defined as follows:

$$f \times \mu_r(z) = \int_{|w|=r} f(z-w) e^{i\frac{1}{2}\operatorname{Im} z \cdot \overline{w}} d\mu_r(w), \ z \in \mathbb{C}^n, r > 0 , \tag{1.1}$$

where $z \cdot \overline{w} = \sum_{j=1}^n z_j \overline{w_j}$, for $z = (z_1, z_2, \ldots, z_n)$ and $w = (w_1, w_2, \ldots, w_n)$.

Definition 1. A set $S \subseteq \mathbb{C}^n$ is a set of injectivity for the twisted spherical means in a subclass C of continuous functions of $\mathbb{C}^n$ if $f \in C$ and $f \times \mu_r(z) = 0$, for all $r \geq 0$ and $z \in S$, imply that $f = 0$.

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The motivation for considering the twisted spherical means on $\mathbb{C}^n$ comes from the Heisenberg group. The Heisenberg group $H^n = \mathbb{C}^n \times \mathbb{R}$, has the following group structure:

$$(z,t)(w,s) = \left(z + w, t + s + \frac{1}{2}\operatorname{Im} z \cdot \overline{w}\right) .$$

Now the unitary group $U(n)$, acts on H^n in a natural way: $\sigma \cdot (z,t) = (\sigma z, t)$. A sphere in H^n can then be thought of as an orbit of a point in H^n under the $U(n)$-action. Let $\mu_{r,t}$ be the normalized surface measure on the sphere $\{(z,t) : |z| = r\}$, $r > 0$, $t \in \mathbb{R}$. For $f \in L^1(H^n)$, we define the spherical mean operator on H^n as

$$f \mapsto f * \mu_{r,t} ,$$

where now $*$ denotes the group convolution on H^n. Then

$$f * \mu_{r,t}(z,s) = 0, \forall r > 0, (z,s) \in S \times \mathbb{R}$$

imply that $(f * \mu_{r,t})^\lambda(z) = 0$ for all $\lambda \in \mathbb{R}$, $r > 0$ and for $z \in S$. Here h^λ is the Fourier transform of $h \in L^1(H^n)$ in the second variable, i.e.,

$$h^\lambda(z) = \int_{\mathbb{R}} h(z,s) e^{i\lambda s} ds . \tag{1.2}$$

But this would in turn imply that for each $\lambda \in \mathbb{R}$, the λ-twisted spherical means

$$f^\lambda \times_\lambda \mu_r(z) = \int_{|w|=r} f^\lambda(z - w) e^{i\frac{\lambda}{2}\operatorname{Im} z \cdot \overline{w}} d\mu_r(w) \tag{1.3}$$

are zero for $z \in S$ and for all $r > 0$, where μ_r is now the normalized surface measure on the sphere $\{z \in \mathbb{C}^n : |z| = r\}$. Note that 1-twisted spherical means are the same as twisted spherical means.

Therefore, injectivity of the spherical mean operator on H^n, at least for the sets of the type $S \times \mathbb{R}$, would follow from the injectivity for the λ-twisted spherical means on $\mathbb{C}^n$ for every $\lambda \in \mathbb{R}$. We will see that the analysis for λ-twisted spherical means, $\lambda \neq 0$, is similar to that for twisted spherical means. So we will normalize λ to be 1 and study this case.

In the case of the spherical means on $\mathbb{R}^n$, $n \geq 2$, it is shown in [2] that if D is a bounded region in $\mathbb{R}^n$, $n \geq 2$, and Γ is the boundary of D, then Γ is a set of injectivity for $L^p(\mathbb{R}^n)$, $1 \leq p \leq \frac{2n}{n-1}$. This bound $\frac{2n}{n-1}$ is sharp since a sphere is never a set of injectivity for $L^p(\mathbb{R}^n)$, $p > \frac{2n}{n-1}$. A spherical eigenfunction ϕ of the Laplacian on $\mathbb{R}^n$, which is in $L^p(\mathbb{R}^n)$, for any $p > \frac{2n}{n-1}$ and satisfies $M\phi(x,r) = \phi(r)\phi(x)$, provides a counterexample with Γ as a sphere.

In the case of twisted spherical means, however, φ_k, the kth Laguerre function of type $(n-1)$, which is defined as

$$\varphi_k(z) = L_k^{n-1}\left(\frac{1}{2}|z|^2\right) e^{-\frac{1}{4}|z|^2}, z \in \mathbb{C}, k = 0, 1, 2, \ldots$$

satisfies $\varphi_k \times \mu_r(z) = \varphi_k(r)\varphi_k(z)$ [8]. Here L_k^{n-1} is the Laguerre polynomial of degree k and type $n-1$. Recall that for $\alpha > -1$, Laguerre polynomials of type α and degree k are defined by [9, p. 7],

$$e^{-x} x^\alpha L_k^\alpha(x) = \frac{1}{k!}\frac{d^k}{dx^k}\left(e^{-x} x^{k+\alpha}\right), x > 0 .$$

In explicit form [9, p. 7], we have

$$L_k^\alpha(x) = \sum_{j=0}^{k} \frac{\Gamma(k+\alpha+1)}{\Gamma(k-j+1)\Gamma(j+\alpha+1)} (-1)^j \frac{x^j}{j!}, x > 0 . \tag{1.4}$$

The functions $\varphi_k, k = 0, 1, 2, \ldots$ belong to all $L^p(\mathbb{C}^n)$, $1 \le p \le \infty$. Therefore if S is the zero set of some φ_k, $k \ne 0$ (note that $\phi_0 = e^{-\frac{1}{4}|z|^2}$ has no zeros), then S is *not* a set of injectivity for twisted spherical means in any $L^p(\mathbb{C}^n)$, $1 \le p \le \infty$, and S is a union of spheres. Note that for $k \ne 0$, $\varphi_k(z)e^{\frac{1}{4}|z|^2} \notin L^p(\mathbb{C}^n)$ for any p, $1 \le p \le \infty$.

In this paper, we work with functions f that satisfy $f(z)e^{(\frac{1}{4}+\epsilon)|z|^2} \in L^p(\mathbb{C}^n)$, for some $\epsilon > 0$ and $1 \le p \le \infty$, which have more decay than φ_ks. We show that in this class of functions, the boundary of a bounded domain in $\mathbb{C}^n$ is a set of injectivity for the twisted spherical means. The proof is similar in spirit to the arguments in [2] for the case of $L^p(\mathbb{R}^n)$, $1 \le p \le \frac{2n}{n+1}$, but it is more subtle and involved because of the nature of the differential operator and corresponding eigenfunctions in this case.

2. Preliminaries

Given functions $f, g \in L^1(\mathbb{C}^n)$ and $\lambda \in \mathbb{R}$, $\lambda \ne 0$, define the λ-twisted convolution $f \times_\lambda g$ as

$$f \times_\lambda g(z) = \int_{\mathbb{C}^n} f(z-w)g(w)e^{i\frac{\lambda}{2}\mathrm{Im} z\cdot\overline{w}}dw, \; z \in \mathbb{C}^n \,. \tag{2.1}$$

When $\lambda = 1$, we call the 1-twisted convolution as the twisted convolution and use the notation $f \times g$ for $f \times_1 g$.

In the case when g is radial, i.e., it is $U(n)$-invariant, then

$$f \times g(z) = \int_0^\infty f \times \mu_r(z)\, g(r) r^{2n-1} dr \,.$$

Therefore, if $f \times \mu_r(z) = 0$, for all $r \ge 0$ for some $z \in \mathbb{C}^n$, then $f \times g(z) = 0$ for any radial g provided that $f \times g(z)$ exists.

We now define a second order elliptic operator L on $\mathbb{C}^n$ which will be used to study the twisted spherical means.

Let

$$L = -\Delta_z + \frac{1}{4}|z|^2 - iN \,,$$

where

$$N = \sum_{j=1}^n \left(x_j \frac{\partial}{\partial y_j} - y_j \frac{\partial}{\partial x_j} \right), \; z = x + iy \,.$$

We are interested in the radial eigenfunctions of L [1]. For this purpose let

$$L\psi = \beta\psi \,. \tag{2.2}$$

Since ψ is radial, $N\psi = 0$, consequently the above equation reduces to

$$-\Delta_z \psi + \frac{1}{4}|z|^2\psi = \beta\psi \,.$$

Using the polar coordinates $z = rw$, $|w| = 1$, we have

$$\psi''(r) + \frac{2n-1}{r}\psi'(r) - \frac{1}{4}r^2\psi(r) + \beta\psi(r) = 0 \,.$$

Put $\psi(r) = u\left(\frac{1}{2}r^2\right)$. Then the above equation gives

$$ru''(r) + nu'(r) - \frac{r - 2\beta}{4}u(r) = 0 \,.$$

Another substitution $v(r) = e^{\frac{r}{2}} u(r)$ reduces the above to the following confluent hypergeometric equation:

$$r v''(r) + (n - r) v'(r) - \frac{n - \beta}{2} v(r) = 0 . \tag{2.3}$$

Let $a = \frac{n-\beta}{2}$. Then a solution of the above equation is given by

$$v_a(r) = M(a, n, r)$$

where $M(a, n, r)$ denotes ${}_1F_1(a, n, r)$, a confluent hypergeometric function [5, p. 252], or [6, p. 255], and is defined as

$$M(a, c, z) = {}_1F_1(a, c, z) = \sum_{s=0}^{\infty} \frac{(a)_s}{(c)_s} \frac{z^s}{s!}, z \in \mathbb{C}, a \in \mathbb{C}, c \in \mathbb{C}\, (c \neq 0, -1, -2, \ldots)$$

where we use the notation

$$(a)_0 = 1$$

and

$$(a)_s = a(a+1)(a+2)\ldots(a+s-1), s = 1, 2, \ldots$$

Retracing the various substitutions made in the reduction of Equation (2.2) to Equation (2.3), an eigenfunction of L corresponding to eigenvalue β is therefore given by

$$\tilde{\psi}_a(z) = M\left(a, n, \frac{1}{2}|z|^2\right) e^{-\frac{1}{4}|z|^2} ,$$

where $a = \frac{n-\beta}{2}$. When $a = -k, k = 0, 1, 2, \ldots$, then $(a)_j = 0$, $j = k+1, k+2, \ldots$ and therefore in this case $M(a, n, r)$ reduces to a polynomial. Using the explicit form (1.4) of Laguerre polynomials, we have

$$\frac{\Gamma(n-a)}{\Gamma(1-a)\Gamma(n)} M(a, n, r) = L_k^{n-1}(r) . \tag{2.4}$$

For the analysis that follows, we will be interested in eigenvalues β such that $-\infty < a < 1$, (and therefore $a < n$), where as before β and a are related by $a = \frac{n-\beta}{2}$. We normalize the radial eigenfunctions of L for these eigenvalues and take for the eigenfunction corresponding to the eigenvalue β, the following function

$$\psi_a(z) = \frac{\Gamma(n-a)}{\Gamma(1-a)\Gamma(n)} M\left(a, n, \frac{1}{2}|z|^2\right) e^{-\frac{1}{4}|z|^2} . \tag{2.5}$$

This normalization, together with (2.4) imply that for $a = -k, k = 0, 1, 2, \ldots$, the eigenfunction ψ_a coincides with the kth Laguerre function ϕ_k. Also note that using the above expression, we can define ψ_a for $a \in \mathbb{C}$ with Re $a < 1$. We will need to know the asymptotic behavior of ψ_a. Now for $a \in \mathbb{C}$, a not a nonpositive integer, $M(a, n, r)$ has the following asymptotic behavior [5, p. 278]:

$$M(a, n, r) \sim \frac{(n-1)!}{\Gamma(a)} r^{a-n} e^r, r \mapsto +\infty . \tag{2.6}$$

Consequently, for $a \in \mathbb{C}$ with Re $a < 1$ and a not a nonpositive integer,

$$C_a = \frac{\Gamma(n-a)}{\Gamma(1-a)\Gamma(a)}$$

is finite and nonzero. Moreover,

$$\psi_a(z) \sim C_a \frac{1}{2^{a-n}} |z|^{2(a-n)} e^{\frac{1}{4}|z|^2}, |z| \mapsto \infty . \tag{2.7}$$

For the various properties listed above of a confluent hypergeometric function, see [5, Chapter 6], [6, p. 254–259]. This concludes the preliminaries required to state our main result.

3. The Main Result

Theorem 1.

Let Γ be the boundary of a bounded domain Ω in $\mathbb{C}^n$. Let f be such that $f(z)e^{(\frac{1}{4}+\epsilon)|z|^2} \in L^p(\mathbb{C}^n)$, for some $\epsilon > 0$ and $1 \leq p \leq \infty$. Suppose that $f \times \mu_r(z) = 0, z \in \Gamma, r > 0$. Then $f = 0$ a.e.

Remark 1. By scaling arguments, Theorem 1 is true also for λ-twisted convolution $\times_\lambda$ given by (2.1) for $\lambda \in \mathbb{R}$, $\lambda \neq 0$, if we replace the integrability condition on f by $f(z)e^{(\frac{1}{4}+\epsilon)|\lambda||z|^2} \in L^p(\mathbb{C}^n)$, for some $\epsilon > 0$ and $1 \leq p \leq \infty$.

We need the following lemma to prove the theorem.

Lemma 1.

Let $f \in C^\infty(\mathbb{C}^n)$ be such that $f(z)e^{(\frac{1}{4}+\epsilon)|z|^2} \in L^p(\mathbb{C}^n)$, for some $\epsilon > 0$ and $1 \leq p \leq \infty$. Then $f \times \psi_a(z)$ exists for each $-\infty < \operatorname{Re} a < 1$ and $z \in \mathbb{C}^n$, where ψ_a is defined by (2.5). Moreover, for a fixed $z \in \mathbb{C}^n$, $a \mapsto f \times \psi_a(z)$ defines an analytic function of a on $\operatorname{Re} a < 1$. Also for a fixed a with $0 < \operatorname{Re} a < 1$, and $z = x + iy \in \mathbb{C}^n$ being thought of as $(x, y) \in \mathbb{R}^{2n}$, the function $(x, y) \mapsto f \times \psi_a(x, y)$ can be extended to an entire function on $\mathbb{C}^n \times \mathbb{C}^n$.

Proof. For $a = -k, k = 0, 1, 2, \ldots$, ψ_a coincides with the kth Laguerre function ϕ_k. Therefore, in this case both ψ_a and f are rapidly decreasing functions. As the twisted convolution on $\mathbb{C}^n$ in absolute value is dominated by the usual convolution on $\mathbb{C}^n$ of the absolute values, $f \times \psi_a(z)$ exists in this case. Let us confine our attention to the case when a is not a nonpositive integer and $-\infty < \operatorname{Re} a < 1$. Using the asymptotic behavior of ψ_a near $+\infty$ given by (2.7), we have for sufficiently large $|w|$, say for $|w| > R$,

$$\begin{aligned}
&\int_{|w|>R} |\psi_a(w)|\, e^{-\left(\frac{1}{4}+\frac{\epsilon}{2}\right)|w|^2} dw \\
&\quad \leq \text{Const}\,|C_a| \int_{|w|>R} |w|^{2(\operatorname{Re} a-n)} e^{\frac{1}{4}|w|^2} e^{-(\frac{1}{4}+\frac{\epsilon}{2})|w|^2} dw \\
&\quad \leq \text{Const}\,|C_a| \int_{|w|>R} |w|^{2(\operatorname{Re} a-n)} e^{-\frac{\epsilon}{2}|w|^2} dw\,,
\end{aligned}$$

which is finite. Therefore, $\psi_a(w)e^{-(\frac{1}{4}+\frac{\epsilon}{2})|w|^2} \in L^1(\mathbb{C}^n, dw)$. Let us define a new function h by $h(z) = f(z)e^{(\frac{1}{4}+\epsilon)|z|^2}$. Then

$$\begin{aligned}
|f \times \psi_a(z)| &\leq \int_{\mathbb{C}^n} \left| f(z-w)\psi_a(w)e^{i\frac{1}{2}\operatorname{Im} z\cdot\overline{w}} \right| dw \\
&\leq \int_{\mathbb{C}^n} |f(z-w)|\,|\psi_a(w)|\, dw \\
&= \int_{\mathbb{C}^n} |h(z-w)| e^{-\left(\frac{1}{4}+\epsilon\right)|z-w|^2} |\psi_a(w)| dw \\
&= e^{-\left(\frac{1}{4}+\epsilon\right)|z|^2} e^{\frac{\epsilon}{2}r^2|z|^2} \\
&\quad \int_{\mathbb{C}^n} |h(z-w)|\,|\psi_a(w)|\, e^{-\left(\frac{1}{4}+\frac{\epsilon}{2}\right)|w|^2} e^{-\frac{\epsilon}{2}|w-rz|^2} dw\,,
\end{aligned}$$

where r is such that $2\left(\frac{1}{4}+\epsilon\right) = r\epsilon$. This last integral is finite as the function $h(w) = f(w)e^{(\frac{1}{4}+\epsilon)|w|^2} \in L^p(\mathbb{C}^n, dw)$ by hypothesis and $\psi_a(w)e^{-(\frac{1}{4}+\frac{\epsilon}{2})|w|^2} \in L^1(\mathbb{C}^n, dw)$.

Now as $M(a, n, z)$ is an entire function of a for a fixed $z \in \mathbb{C}$,

$$\psi_a(z) = \frac{\Gamma(n-a)}{\Gamma(1-a)\Gamma(n)} M\left(a, n, \frac{1}{2}|z|^2\right) e^{-\frac{1}{4}|z|^2}$$

is an analytic function of a for Re $a < 1$. Appealing to Morera's theorem, we conclude that for fixed $z, a \mapsto f \times \psi_a(z)$ is an analytic function of a for Re $a < 1$.

To prove the second part of the lemma, let $0 < \text{Re}\, a < 1$. By definition (2.1), for $z = (x, y), x, y \in \mathbb{R}^n$,

$$f \times \psi_a(z) = f \times \psi_a(x, y) = \int_{\mathbb{C}^n} f(p, q)\psi_a(x-p, y-q) e^{i\frac{1}{2}(xq-yp)} dpdq \,.$$

The extension of $f \times \psi_a$ to $\mathbb{C}^n \times \mathbb{C}^n$ is then given as follows. Let $(z_1, z_2) \in \mathbb{C}^n \times \mathbb{C}^n, z_1 = x + is, z_2 = y + it$. Define

$$\begin{aligned} f \times \psi_a(z_1, z_2) &= f \times \psi_a(x+is, y+it) \\ &= \int f(p, q)\psi_a\,(z_1 - p, z_2 - q)\, e^{i\frac{1}{2}(z_1 q - z_2 p)} dpdq \,, \end{aligned}$$

where

$$\psi_a\,(z_1, z_2) = \frac{\Gamma(n-a)}{\Gamma(1-a)\Gamma(n)} M\left(a, n, \frac{1}{2}\left(z_1^2 + z_2^2\right)\right) e^{-\frac{1}{4}(z_1^2+z_2^2)} \,.$$

Now for $0 < \text{Re}\, a < 1$, the hypergeometric function $M(a, n, z)$ has the following integral representation [5, p. 255], [6, p. 255]:

$$M(a, n, z) = \frac{\Gamma(n)}{\Gamma(a)\Gamma(n-a)} \int_0^1 t^{a-1}(1-t)^{n-a-1} e^{zt} dt, z \in \mathbb{C} \,.$$

Consequently, in this case

$$|M(a, n, z)| \leq \frac{\Gamma(\text{Re}\, a)\Gamma(n - \text{Re}\, a)}{|\Gamma(a)\Gamma(n-a)|} M(\text{Re}\, a, n, \text{Re}\, z) \,,$$

i.e.,

$$|M(a, n, z)| \leq A_{a,n} M(\text{Re}\, a, n, \text{Re}\, z) \,.$$

This implies

$$\begin{aligned} &\left|\psi_a\,(z_1 - p, z_2 - q)\, e^{i\frac{1}{2}(z_1 q - z_2 p)}\right| \\ &\leq A_{a,n} \frac{|\Gamma(n-a)|}{|\Gamma(1-a)|\Gamma(n)} M\left(\text{Re}\, a, n, \frac{1}{2}\text{Re}\, \left((z_1 - p)^2 + (z_2 - q)^2\right)\right) \\ &\quad \times e^{-\frac{1}{4}\text{Re}\, ((z_1-p)^2+(z_2-q)^2)} e^{-\frac{1}{2}(sq-tp)} \\ &= B_{a,n} M\left(\text{Re}\, a, n, \frac{1}{2}\text{Re}\, \left((z_1 - p)^2 + (z_2 - q)^2\right)\right) \\ &\quad \times e^{-\frac{1}{4}\text{Re}\, ((z_1-p)^2+(z_2-q)^2)} e^{-\frac{1}{2}(sq-tp)}. \end{aligned}$$

Coming back to the integral, we have

$$\begin{aligned} |f \times \psi_a\,(z_1, z_2)| &\leq \int |f(p, q)|\, |\psi_a\,(z_1 - p, z_2 - q)| \left|e^{i\frac{1}{2}(z_1 q - z_2 p)}\right| dpdq \\ &\leq B_{a,n} \int |f(p, q)| M\left(\text{Re}\, a, n, \frac{1}{2}\text{Re}\, \left((z_1 - p)^2 + (z_2 - q)^2\right)\right) \\ &\quad \times e^{-\frac{1}{4}\text{Re}\, ((z_1-p)^2+(z_2-q)^2)} e^{-\frac{1}{2}(sq-tp)} dpdq \,. \end{aligned}$$

The integral near 0 is finite. The integral over, say, $|(p, q)| > R$, by the asymptotic behavior of M near $+\infty$, as given in (2.6), is less than or equal to

$$\begin{aligned}
&\text{Const}\frac{\Gamma(n)}{|\Gamma(a)|}\int_{|(p,q)|>R}|f(p,q)|\Big((x-p)^2-s^2+(y-q)^2-t^2\Big)^{a-n} \\
&\qquad\times e^{\frac{1}{4}((x-p)^2-s^2+(y-q)^2-t^2)}e^{-\frac{i}{2}(sq-tp)}dpdq \\
&\leq \text{Const}\frac{\Gamma(n)}{|\Gamma(a)|}\int_{|(p,q)|>R}|h(p,q)|e^{-(\frac{1}{4}+\epsilon)(p^2+q^2)}e^{\frac{1}{4}((x-p)^2-s^2+(y-q)^2-t^2)} \\
&\qquad\times\Big((x-p)^2+(y-q)^2-s^2-t^2\Big)^{a-n}e^{-\frac{i}{2}(sq-tp)}dpdq
\end{aligned}$$

where as before $h(p, q) = f(p, q)e^{(\frac{1}{4}+\epsilon)(p^2+q^2)}$. The integral is again finite using arguments similar to the ones used earlier. Using Morera's theorem, we conclude that for a with $0 < \operatorname{Re} a < 1$, the natural extension of the function $(x, y) \mapsto f \times \psi_a(x, y)$, to $\mathbb{C}^n \times \mathbb{C}^n$ is an entire function. Again $f \times \psi_a(z)$ is well defined for $-\infty < \operatorname{Re} a < 1$ and as $M(a, n, z)$ is an entire function of a for a fixed $z \in \mathbb{C}$

$$\psi_a(z) = \frac{\Gamma(n-a)}{\Gamma(1-a)\Gamma(n)}M\left(a, n, \frac{1}{2}|z|^2\right)e^{-\frac{1}{4}|z|^2}$$

is an analytic function of a for $\operatorname{Re} a < 1$. Again appealing to Morera's theorem we conclude that for fixed z, $a \mapsto f \times \psi_a(z)$ is an analytic function of a for $\operatorname{Re} a < 1$. This completes the proof of the lemma. □

We now prove the theorem.

Proof. Now f is such that $f(z)e^{(\frac{1}{4}+\epsilon)|z|^2} \in L^p(\mathbb{C}^n)$ for some $\epsilon > 0$ and $1 \leq p \leq \infty$ and satisfies

$$f \times \mu_r(z) = 0\,,$$

for all $r > 0$ and $z \in \Gamma$. By convolving with smooth radial compactly supported approximate identity, we can assume that $f \in C^\infty(\mathbb{C}^n)$. Also since $f \in L^1(\mathbb{C}^n, dz)$, by the same argument we can even assume $f \in L^2(\mathbb{C}^n, dz)$. By the lemma, $f \times \psi_a(z)$ exists for all $z \in \mathbb{C}^n$ and a with $-\infty < a < 1$. Moreover, as ψ_a are radial, i.e., $U(n)$-invariant, so

$$f \times \psi_a(z) = \int_0^\infty f \times \mu_r(z)\psi_a(r)r^{2n-1}dr\,,$$

and therefore

$$f \times \psi_a(z) = 0, \forall z \in \Gamma, -\infty < a < 1\,. \tag{3.1}$$

Also as $L(h \times g) = h \times Lg$, whenever both sides are well defined, we have that $L(f \times \psi_a) = f \times L\psi_a = \beta(f \times \psi_a)$, and hence whenever $f \times \psi_a$ is nontrivial, it is also an eigenfunction for L.

Now, the operator L is a self-adjoint and, therefore, the spectrum of the Dirichlet problem in Ω is discrete. On the other hand, each of the functions $f \times \psi_a$ is an eigenfunction of L with eigenvalue $\beta = n - 2a$ and satisfies the Dirichlet condition (3.1). We conclude from here that $f \times \psi_a \equiv 0$, on Ω, for all but countably many a, $a < 1$.

However, for any fixed $z \in \mathbb{C}^n$, again by the lemma, $a \mapsto f \times \psi_a(z)$ is an analytic function on $\operatorname{Re} a < 1$ and, in particular, a continuous function of a in $a < 1$. Hence, $f \times \psi_a \equiv 0$, on Ω, for all a, $a < 1$.

Again by the lemma for each fixed a, $0 < a < 1$, $f \times \psi_a$ is a real analytic function on $\mathbb{R}^{2n}$. Hence for a, $0 < a < 1$, we have that $f \times \psi_a \equiv 0$ in $\mathbb{R}^{2n}$. On the other hand, for any fixed $z \in \mathbb{C}^n$, the lemma states that $a \mapsto f \times \psi_a(z)$ is an analytic function on $\operatorname{Re} a < 1$. Therefore, $f \times \psi_a(z) = 0$ for all $\operatorname{Re} a < 1$. Since $z \in \mathbb{C}^n$ is arbitrary, we conclude that $f \times \psi_a \equiv 0$ for all $-\infty < a < 1$.

In particular, when $a = -k, k = 0, -1, -2, \dots$, we have that $f \times \psi_{-k}(z) \equiv 0$. But ψ_{-k} is nothing but the kth Laguerre function ϕ_k, and since $f \in L^2(\mathbb{C}^n, dz)$, this forces $f \equiv 0$. This completes the proof of Theorem 1. $\square$

We conjecture that Theorem 1 is true also with $\epsilon = 0$ but this may require more refined arguments.

4. Geometric Interpretation

Theorem 1 can be interpreted as a statement about injectivity of the complex Radon transform restricted on a certain quadratic hypersurface in $\mathbb{C}^{n+1}$.

Let

$$\Omega_{n+1} = \left\{ \left(w', w_{n+1}\right) : w' \in \mathbb{C}^n, w_{n+1} \in \mathbb{C}, \operatorname{Im} w_{n+1} > \frac{1}{4} \left|w'\right|^2 \right\}$$

be the Siegel domain in $\mathbb{C}^{n+1}$. The Heisenberg group H^n acts on the $\bar{\Omega}_{n+1}$ (via biholomorphic maps) in the following way: For $(z, t) \in H^n$, $(w', w_{n+1}) \in \bar{\Omega}_{n+1}$,

$$(z, t) \cdot \left(w', w_{n+1}\right) = \left(w' + z, w_{n+1} + t + \frac{i}{4}|z|^2 + \frac{i}{2} w' \cdot \bar{z} \right) .$$

Now H^n can be identified with the boundary, $\partial\bar{\Omega}_{n+1}$, of the Siegel domain via $\Phi(z, t) = \left(z, t + \frac{i}{4}|z|^2\right)$. Moreover, if $(w', w_{n+1}) \in \partial\bar{\Omega}_{n+1}$, then

$$\Phi^{-1}\left((z, t) \cdot \left(w', w_{n+1}\right)\right) = (z, t) \cdot \Phi^{-1}\left(w', w_{n+1}\right) .$$

Therefore, H^n−action on $\partial\bar{\Omega}_{n+1}$ coincides with the group multiplication in H^n after necessary identifications. Under this H^n−action, $\bar{\Omega}_{n+1}$ is divided in orbits

$$\mathcal{O}_r = \left\{ \left(w', w_{n+1}\right) : \operatorname{Im} w_{n+1} = \frac{1}{4} r^2 + \frac{1}{4} \left|w'\right|^2 \right\}$$

of points $\left(0, \frac{i}{4} r^2\right)$, $r \geq 0$. Let $\mathcal{L}_r$ be the complex hyperplane in the tangent space to $\mathcal{O}_r$ at the point $(0, \frac{i}{4} r^2)$. Then

$$\mathcal{L}_r = \left\{ \left(w', \frac{i}{4} r^2\right) : w' \in \mathbb{C}^n \right\}$$

and therefore

$$\mathcal{L}_r \cap \partial\bar{\Omega}_{n+1} = \left\{ \left(w', \frac{i}{4} r^2\right) : \left|w'\right| = r \right\} ,$$

which corresponds in H^n to the sphere

$$\left\{ \left(w', 0\right) : \left|w'\right| = r \right\} .$$

For a function h on the Heisenberg group, being thought of as a distribution on $\mathbb{C}^{n+1}$ with support on $\partial\bar{\Omega}_{n+1}$, one can define the complex Radon transform $\mathcal{R}h$ of h as follows:

$$\mathcal{R}h(z, t, r) = \int_{\mathcal{L}_r \cap \partial\bar{\Omega}_{n+1}} L_{(z,t)} h ,$$

where $L_{(z,t)} h(w', w_{n+1})) = h((z, t) \cdot (w', w_{n+1}))$ and the integration is with respect to surface measure on $\mathcal{L}_r \cap \partial\bar{\Omega}_{n+1}$.

It is worthwhile to mention that for a fixed $(z,t) \in H^n$ and arbitrary $r > 0$, we obtain integrals over the family of all parallel complex hyperplanes (projections in the integral geometry terminology).

In this set up, our result for the twisted spherical means can be interpreted in the following way:

Theorem 2.
Let h be a function in $L^1(H^n)$ such that $e^{(\frac{1}{4}+\epsilon)|\lambda||z|^2} h^\lambda(z) \in L^p(\mathbb{C}^n)$ a.e. $\lambda \in \mathbb{R}$ for some $\epsilon > 0$ and $1 \le p \le \infty$. Here h^λ is given by (1.2). If $\mathcal{R}h(z,t,r) = 0$ for $(z,t) \in \Gamma \times \mathbb{R}$ and for all $r \ge 0$, where Γ is as in Theorem 1, then $h = 0$ a.e.

Proof. By assumption,

$$\mathcal{R}h(z,t,r) = 0, \forall (z,t) \in \Gamma \times \mathbb{R}, r \ge 0 .$$

Therefore, for $(z,t) \in \Gamma \times \mathbb{R}$ and for all $r \ge 0$,

$$\int_{\{(w',0):|w'|=r\}} h(z,t)\left(w',0\right) d\sigma_r\left(w'\right) = 0 ,$$

where σ_r is the surface measure on $\{(w',0) : |w'| = r\}$. This implies

$$\int_{|w'|=r} h\left(z+w', t+\frac{1}{2}\mathrm{Im} z \cdot \bar{w'}\right) d\sigma_r\left(w'\right) = 0 ,$$

which leads to

$$\int_{|w'|=r} \int_{\mathbb{R}} h\left(z+w', t+\frac{1}{2}\mathrm{Im} z \cdot \bar{w'}\right) e^{i\lambda t} dt d\sigma_r\left(w'\right) = 0 ,$$

for $\lambda \in \mathbb{R}$. The left-hand side can be rewritten as the λ-twisted spherical means of the function h^λ, defined by the formula (1.2). Thus, we have

$$h^\lambda \times_\lambda \mu_r(z) = 0 .$$

Here $z \in \Gamma$ and $r \ge 0$ is arbitrary. Also by hypothesis, h^λ satisfies the integrability condition in the remark after Theorem 1. Therefore, $h^\lambda = 0$ *a.e.* λ. This implies $h = 0$. □

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