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On the Sampling Theorem for Wavelet Subspaces

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ABSTRACT. *In [13], Walter extended the classical Shannon sampling theorem to some wavelet subspaces. For any closed subspace V_0 of $L^2(\mathbf{R})$, we present a necessary and sufficient condition under which there is a sampling expansion for every $f \in V_0$. Several examples are given.*

1. Introduction and Main Results

The classical Shannon sampling theorem says that for each $f \in PW_\pi := \{f \in L^2(\mathbf{R}) : \operatorname{supp} \hat{f} \subset [-\pi, \pi]\}$,

$$f(x) = \sum_{n=-\infty}^{+\infty} f(n) \frac{\sin \pi(x-n)}{\pi(x-n)}, \tag{1.1}$$

where the convergence is both in $L^2(\mathbf{R})$ and uniform on $\mathbf{R}$.

In [13], Walter extended the Shannon sampling theorem to wavelet subspaces and proved the following result:

Proposition 1.

Suppose that $\varphi(t)$ is a real continuous scaling function such that $\varphi(t) = O(|t|^{-1-\varepsilon})$ and

$$\hat{\varphi}^*(\omega) = \sum_n \varphi(n) e^{-in\omega} \neq 0, \quad \omega \in \mathbf{R}. \tag{1.2}$$

Let $V_0 = \left\{\sum_n c_n \varphi(t-n) : \{c_n\} \in \ell^2\right\}$. Then there is an $S \in V_0$ such that for any $f \in V_0$, $f(t) = \sum_n f(n) S(t-n)$, where the convergence is both in $L^2(\mathbf{R})$ and uniform on $\mathbf{R}$.

In [9], Janssen considered the shifted sampling and the corresponding aliasing error by means of Zak transform.

For convenience, we say that the sampling theorem holds on $V_0 \subset L^2(\mathbf{R})$ if there exists $\{g(\cdot - n) : n \in \mathbf{Z}\} \subset V_0$ such that for any $f \in V_0$, $f(x) = \sum_k f(k) g(x-k)$, where the convergence

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is both in $L^2(\mathbf{R})$ and pointwise on $\mathbf{R}$. In sampling theory, it is natural to add the condition $V_0 \subset C(\mathbf{R})$. For example, $PW_\pi \subset C(\mathbf{R})$ in Shannon sampling theorem and $V_0 \subset C(\mathbf{R})$ in Proposition 1. But Shannon wavelet $\varphi(t) = \frac{\sin \pi t}{\pi t}$ does not decay as fast as $|t|^{-1-\varepsilon}$, so Proposition 1 is not applicable to this wavelet subspace although (1.1) holds.

In this paper we characterize the closed subspace $V_0 \subset L^2(\mathbf{R})$ on which the sampling theorem holds.

Notations.

$\ell^2 = \{c_k : \sum_{k=-\infty}^{+\infty} |c_k|^2 < \infty.\}$.

$\sum_n$ stands for summation over all $n \in \mathbf{Z}$.

$C(\mathbf{R})$ is the space of continuous function.

$AC_{loc} = \{f \in C(\mathbf{R}) : f$ is locally absolutely continuous.$\}$.

$L^2[-\pi, \pi] = \{f : f$ is 2π-periodic and square integrable on $[-\pi, \pi]\}$.

$G_f(\omega) = \sum_k |\hat{f}(\omega + 2k\pi)|^2$, where $\hat{f}(\omega) = \int_{\mathbf{R}} f(x) e^{-ix\omega} dx$. It is easy to see that G_f is defined only a.e.

$E_f = \{\omega \in \mathbf{R} : G_f(\omega) > 0\}, \forall f \in L^2(\mathbf{R})$.

χ_E is the characteristic function of the set E.

V_0 is a closed subspace of $L^2(\mathbf{R})$. □

As we know, a family of functions $\{\varphi_j : j \in J\}$ in a Hilbert space $\mathcal{H}$ is called a frame if there exist $A > 0$, $B < \infty$ so that, for all $f \in \mathcal{H}$,

$$A\|f\|^2 \leq \sum_{j \in J} \left| < f, \varphi_j > \right|^2 \leq B\|f\|^2 .$$

The constants A, B are called frame bounds. If $A = B$, then we call the frame a tight frame. For the details on frames and dual frames, see [7, p. 56–60].

Our main results are as follows.

Theorem 1.

Let V_0 be a closed subspace of $L^2(\mathbf{R})$ and $\{\varphi(\cdot - n) : n \in \mathbf{Z}\}$ is a frame for V_0. Then the following two assertions are equivalent:

(i) *$\sum_k c_k \varphi(x - k)$ converges pointwise to a continuous function for any $\{c_k\} \in \ell^2$ and there is a frame $\{S(\cdot - n)\}$ for V_0 such that*

$$f(x) = \sum_k f(k) S(x - k), \quad \forall f \in V_0 , \tag{1.3}$$

where the convergence is both in $L^2(\mathbf{R})$ and uniform on $\mathbf{R}$.

(ii) *$\varphi \in C(\mathbf{R})$, $\sum_k |\varphi(x - k)|^2$ is bounded on $\mathbf{R}$ and*

$$A\chi_{E_\varphi}(\omega) \leq |\Phi(\omega)| \leq B\chi_{E_\varphi}(\omega), \text{ a.e.} \tag{1.4}$$

for some constants $A, B > 0$, where

$$\Phi(\omega) = \sum_k \varphi(k) e^{-ik\omega} . \tag{1.5}$$

Theorem 2.

Let $\{\varphi(\cdot - k)\}$ be a frame for V_0. Suppose that $\varphi \in AC_{loc}$ and $\varphi' \in L^2(\mathbf{R})$. Let $\Phi(\omega)$ be defined as in (1.5) and satisfy (1.4). Then the first item of Theorem 1 holds. Moreover, for any $f \in V_0$,

$$f'(t) = \sum_k f(k) S'(t - k), \quad \text{a.e.} \tag{1.6}$$

2. Proof of Theorems

Lemma 1.

Suppose $\varphi \in L^2(\mathbf{R})$. The following two assertions are equivalent.

(i) *For any $\{c_k\} \in \ell^2$, $\sum_k c_k\varphi(x-k)$ converges pointwise to a continuous function.*

(ii) $\varphi \in C(\mathbf{R})$ *and* $\sup\limits_x \sum_k |\varphi(x-k)|^2 < +\infty$.

Proof. (i)⇒(ii): It is easy to see that $\varphi \in C(\mathbf{R})$. For each $x \in \mathbf{R}$, since $\sum_k c_k\varphi(x-k)$ is convergent for each $\{c_k\} \in \ell^2$, it is easy to see that $\sum_k |\varphi(x-k)|^2 < +\infty$. For each $x \in [0,1]$, define

$$\Lambda_x c = \sum_k c_k\varphi(x-k), \qquad \forall c = \{c_k\} \in \ell^2 .$$

Then Λ_x is a bounded linear functional on ℓ^2 with the norm $\|\Lambda_x\| = (\sum_k |\varphi(x-k)|^2)^{1/2}$. For any $\{c_k\} \in \ell^2$, define $f(t) = \sum_k c_k\varphi(t-k)$. Since $f(t)$ is continuous on $\mathbf{R}$, we have

$$\sup_{x\in[0,1]} |\Lambda_x c| = \sup_{x\in[0,1]} |f(x)| < +\infty .$$

By the Banach–Steinhaus theorem [12], $\sup\limits_{x\in[0,1]} \|\Lambda_x\| < +\infty$, i.e., $\sum_k |\varphi(x-k)|^2$ is bounded on $\mathbf{R}$.

(ii)⇒(i): By the Cauchy inequality, $\sum_k c_k\varphi(x-k)$ is convergent uniformly on $\mathbf{R}$, so the limit function is continuous. □

For any $\{c_k\} \in \ell^2$, define its Fourier transform as $\sum_k c_k e^{-ik\omega}$. The following lemma is easy to prove.

Lemma 2.

Suppose that $\{x_k\}, \{y_k\} \in \ell^2$ and $X(\omega), Y(\omega)$ are their Fourier transforms, respectively. Then

$$\sum_n \left|\sum_k x_k y_{n-k}\right|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(\omega)Y(\omega)|^2 d\omega .$$

When one side of the above equation is finite, the Fourier transform of $x * y(n) := \sum_k x_k y_{n-k}$ is $X(\omega)Y(\omega)$.

Proposition 2.

([4, Theorem 3.56], [2, Theorem 2.16], and [10, Lemma 4.4.8]). *Suppose $\varphi \in L^2(\mathbf{R})$ and $\{\varphi(\cdot - n)\}$ spans the closed subspace V_0. Then $\{\varphi(\cdot - n)\}$ constitutes a frame of V_0 with bounds A, B if and only if $A\chi_{E_\varphi}(\omega) \le G_\varphi(\omega) \le B\chi_{E_\varphi}(\omega)$, a.e.*

Lemma 3.

Let $\{\varphi(\cdot - n)\}$ and $\{S(\cdot - n)\}$ be two frames for V_0. Suppose $\varphi \in C(\mathbf{R})$ and $\sum_k |\varphi(x+k)|^2 \le L < +\infty$, $\forall x$. Then there exists a constant $C > 0$ such that

$$\sum_k |S(x+k)|^2 \le C \sum_k |\varphi(x+k)|^2 .$$

Proof. By Lemma 1, $V_0 \subset C(\mathbf{R})$. Let $S(x) = \sum_k c_k\varphi(x-k)$ for some $\{c_k\} \in \ell^2$, where the convergence is both in $L^2(\mathbf{R})$ and pointwise on $\mathbf{R}$. Put $C(\omega) = \sum_k c_k e^{-ik\omega}$. Then $\hat{S}(\omega) = C(\omega)\hat{\varphi}(\omega)$ and $G_S(\omega) = |C(\omega)|^2 G_\varphi(\omega)$. By Proposition 2, $C(\omega)$ is bounded on E_φ, so $\tilde{C}(\omega) := C(\omega)\chi_{E_\varphi}(\omega)$ is bounded on $[-\pi,\pi]$. Let $\tilde{C}(\omega) = \sum_k \tilde{c}_k e^{-ik\omega}$ for some $\{\tilde{c}_k\} \in \ell^2$. Since $C(\omega)\hat{\varphi}(\omega) = \tilde{C}(\omega)\hat{\varphi}(\omega)$, we also have $S(x) = \sum_k \tilde{c}_k \varphi(x-k)$. Hence

$$\begin{aligned} \sum_n |S(x+n)|^2 &= \sum_n \left| \sum_k \tilde{c}_k \varphi(x+n-k) \right|^2 \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} |\tilde{C}(\omega)|^2 \left| \sum_n \varphi(x+n) e^{-in\omega} \right|^2 d\omega \\ &\le \left\| \tilde{C}(\omega) \right\|_\infty^2 \sum_n |\varphi(x+n)|^2 . \end{aligned}$$

This completes the proof. □

Lemma 4.

Suppose $\{\varphi(\cdot - n)\}$ is a frame for V_0. Let

$$\hat{\tilde{\varphi}}(\omega) = \begin{cases} \hat{\varphi}(\omega)/G_\varphi(\omega), & \omega \in E_\varphi \\ 0, & \omega \notin E_\varphi \end{cases} \tag{2.1}$$

Then $\{\tilde{\varphi}(\cdot - n)\}$ is a dual frame of $\{\varphi(\cdot - n)\}$.

Proof. Let T be the associated frame operator from V_0 to ℓ^2 defined by $(Tf)_n = \langle f, \varphi(\cdot - n) \rangle$. By [7, Proposition 3.2.3], we need only to check that the function $\tilde{\varphi}$ defined by (2.1) satisfies $T^*T\tilde{\varphi} = \varphi$. Since G_φ is 2π−periodic and has a positive lower bound on E_φ, by (2.1), $\tilde{\varphi} \in V_0$ and

$$\begin{aligned} \langle \tilde{\varphi}, \varphi(\cdot - n) \rangle &= \frac{1}{2\pi} \int_{\mathbf{R}} \hat{\tilde{\varphi}}(\omega)\, \bar{\hat{\varphi}}(\omega) e^{in\omega} d\omega = \frac{1}{2\pi} \int_{E_\varphi} \frac{1}{G_\varphi(\omega)} |\hat{\varphi}(\omega)|^2 e^{in\omega} d\omega \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \chi_{E_\varphi}(\omega) e^{in\omega} d\omega . \end{aligned}$$

It follows that $\sum_n \langle \tilde{\varphi}, \varphi(\cdot - n) \rangle \cdot e^{-in\omega} \hat{\varphi}(\omega) = \chi_{E_\varphi}(\omega)\hat{\varphi}(\omega) = \hat{\varphi}(\omega)$. Hence, $\sum_n \langle \tilde{\varphi}, \varphi(\cdot - n) \rangle \varphi(\cdot - n) = \varphi$. That is, $T^*T\tilde{\varphi} = \varphi$. □

Now we are ready to prove the main results.

Proof of Theorem 1. (i)⇒(ii). By Lemma 1, it suffices to show that (1.4) holds. Take $f = \varphi$, then we have

$$\varphi = \sum_k \varphi(k) S(\cdot - k) .$$

So $G_\varphi(\omega) = |\Phi(\omega)|^2 G_S(\omega)$. Hence, $E_\varphi \subset E_S$. Since both $\{S(\cdot - n)\}$ and $\{\varphi(\cdot - n)\}$ are frames for V_0, by Proposition 2, there exist two constants $A, B > 0$ such that $A \le |\Phi(\omega)| \le B$, a.e. on E_φ.

Next we show that $\Phi(\omega)$ is equal to 0 almost everywhere on $[-\pi, \pi] \backslash E_\varphi$. Put $C(\omega) = 1 - \chi_{E_\varphi}(\omega)$, then $C(\omega) \in L^2[-\pi, \pi]$. Let $C(\omega) = \sum_k c_k e^{-ik\omega}$ for some $\{c_k\} \in \ell^2$. Since $C(\omega)\hat{\varphi}(\omega) = 0$, $\sum_k c_k \varphi(x-k) = 0$ for any $x \in \mathbf{R}$. In particular, $\sum_k c_k \varphi(n-k) = 0$ for any $n \in \mathbf{Z}$. By Lemma 2,

$$\begin{aligned} 0 = \sum_n \left| \sum_k c_k \varphi(n-k) \right|^2 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} |C(\omega)|^2 \left| \sum_n \varphi(n) e^{-in\omega} \right|^2 d\omega \\ &= \frac{1}{2\pi} \int_{[-\pi,\pi]\backslash E_\varphi} \left| \sum_n \varphi(n) e^{-in\omega} \right|^2 d\omega . \end{aligned}$$

Hence, $\Phi(\omega) = 0$, a.e. on $[-\pi, \pi] \backslash E_\varphi$.

(ii) ⇒(i) Let

$$\hat{S}(\omega) = \begin{cases} \frac{1}{\Phi(\omega)}\hat{\varphi}(\omega), & \omega \in E_\varphi\,, \\ 0, & \omega \notin E_\varphi\,, \end{cases} \qquad \hat{\tilde{S}}(\omega) = \begin{cases} \frac{\bar{\Phi}(\omega)}{G_\varphi(\omega)}\hat{\varphi}(\omega), & \omega \in E_\varphi\,, \\ 0, & \omega \notin E_\varphi\,. \end{cases} \tag{2.2}$$

Since $G_S(\omega)$ is equal to $\frac{1}{|\Phi(\omega)|^2}G_\varphi(\omega)$ for $\omega \in E_\varphi$, and 0 for $\omega \notin E_\varphi$, by Proposition 2 and Lemma 4, $\{S(\cdot - n)\}$ is a frame for some $\tilde{V}_0 \subset L^2(\mathbf{R})$ and $\{\tilde{S}(\cdot - n)\}$ is the dual. By the definition of $S(x)$, it is easy to see that $S \in V_0$ and $\varphi \in \tilde{V}_0$. Hence, $V_0 = \tilde{V}_0$. For any $f \in V_0$, there exists $C(\omega) \in L^2[-\pi, \pi]$ such that $\hat{f}(\omega) = C(\omega)\hat{\varphi}(\omega)$. Suppose $C(\omega) = \sum_k c_k e^{-ik\omega}$, then

$$\begin{aligned} < f, \tilde{S}(\cdot - n) > &= \frac{1}{2\pi}\int_{E_\varphi} C(\omega)\frac{\Phi(\omega)}{G_\varphi(\omega)}\left|\hat{\varphi}(\omega)\right|^2 e^{in\omega}d\omega \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} C(\omega)\Phi(\omega)e^{in\omega}d\omega = \sum_k c_k\varphi(n-k) \\ &= f(n)\,. \end{aligned}$$

Hence,

$$f(x) = \sum_k < f, \tilde{S}(\cdot - k) > S(x-k) = \sum_k f(k)S(x-k)\,.$$

By Lemma 3 and the Cauchy inequality, the above equation is also convergent uniformly on **R**. □

Proof of Theorem 2. Let S, $\tilde{S}$ be defined as in (2.2).

To prove the first part of Theorem 2, by Theorem 1, we need only to show that $\sum_k |\varphi(x-k)|^2$ is bounded on **R**. Since $\varphi \in L^2(\mathbf{R})$, $\sum_k |\varphi(x-k)|^2 < +\infty$, a.e. Hence, $\sum_k c_k\varphi(x-k)$ is convergent a.e. on **R** for any $\{c_k\} \in \ell^2$. Suppose that $\sum_k c_k\varphi(x-k)$ is convergent for some x and $y \in (x, x+1)$, then

$$\begin{aligned} \int_x^y \left|\sum_{k=n}^m c_k\varphi'(t-k)\right| dt &\le \int_x^y \left(\sum_{k=n}^m |c_k|^2\right)^{1/2} \left(\sum_{k=n}^m |\varphi'(t-k)|^2\right)^{1/2} dt \\ &\le \left(\sum_{k=n}^m |c_k|^2\right)^{1/2} \sqrt{y-x}\left(\int_x^y \sum_{k=n}^m |\varphi'(t-k)|^2 dt\right)^{1/2}\,. \end{aligned}$$

Since $\sum_k |\varphi'(t-k)|^2 \in L^1[x, y]$ due to $\varphi' \in L^2(\mathbf{R})$, it follows by the above inequality that $\sum_{k=-\infty}^{+\infty} c_k\varphi'(t-k)$ is convergent in $L^1[x, y]$. So

$$\begin{aligned} \int_x^y \sum_k c_k\varphi'(t-k)dt &= \sum_k \int_x^y c_k\varphi'(t-k)dt \\ &= \sum_k c_k\left[\varphi(y-k) - \varphi(x-k)\right]\,. \end{aligned}$$

Hence, $\sum_k c_k\varphi(y-k)$ is convergent and $f(x) = \sum_k c_k\varphi(x-k)$ is well defined everywhere. Since $\{c_k\}$ is arbitrary, it is easy to see that $\sum_k |\varphi(x-k)|^2$ is bounded on **R**.

Next, let us prove (1.6). By (1.4) and (2.2) we see that there exists a 2π−periodic function $\alpha(\omega) \in L^\infty$ such that $\hat{S}(\omega) = \alpha(\omega)\hat{\varphi}(\omega)$. So $i\omega\hat{S}(\omega) = \alpha(\omega)\hat{\varphi}'(\omega) \in L^2(\mathbf{R})$. This implies $S \in AC_{loc}$ and $S'(t) \in L^2(\mathbf{R})$ ([3, Theorem 5.2]). On the other hand, by Theorem 1, for any $f \in V_0$, $f(t) = \sum_k f(k)S(t-k)$. Since $\{f(k)\} = \{< f, \tilde{S}(\cdot - k) >\} \in \ell^2$, similar to the above we can show that $\sum_{k=-\infty}^{+\infty} f(k)S'(t-k)$ is convergent a.e. on **R** and

$$\int_x^y \sum_k f(k)S'(t-k)dt = \sum_k f(k)\left[S(y-k) - S(x-k)\right] = f(y) - f(x)\,.$$

By [12, Theorem 7.11], this implies (1.6). □

3. Applications

In this section we give some applications of the sampling theorem.

Example 1. Daubechies wavelets. It is easy to check that for Daubechies wavelets φ_N, ψ_N, the $\Phi(\omega)$ defined in Theorem 1 has no zero if $2 \le N \le 20$. So the sampling theorem holds on both V_0 and W_0. □

Example 2. Spline wavelets. Let

$$\hat{\varphi}_n(\omega) = \left(\frac{\sin\frac{\omega}{2}}{\frac{\omega}{2}}\right)^{n+1}, \qquad n \ge 1 .$$

For each $n \ge 1$, $\{\varphi_n(\cdot - k) : k \in \mathbf{Z}\}$ constitutes a Riesz basis for the subspace it spans and $\Phi_n(\omega) = \sum_k \varphi_n(k)e^{-ik\omega}$ has no zero on $\mathbf{R}$ ([6, p. 89–111]). □

Remark 1.

If we define $\hat{\varphi}_n(\omega) = \left[\frac{1-e^{-i\omega}}{i\omega}\right]^{n+1}$, then $\Phi_n(\pi) = 0$ for even n. By Theorem 1, there is no sampling theorem on V_0 (the case of n=2 was studied in [13]). Janssen presented an alternative approach to solve this problem. For even n, he chose $\{\frac{1}{2} + k\}$ to be the sampling points, which is equivalent to our choice. For details, see [9]. □

Example 3. Let $E \subset \mathbf{R}$ be a bounded measurable set. Define

$$\hat{\varphi}(\omega) = \chi_E(\omega) .$$

By Proposition 2, $\{\varphi(\cdot - k)\}$ constitutes a frame for some closed subspace $V_0 \subset L^2(\mathbf{R})$. Moreover, $\sup_x \sum_k |\varphi(x-k)|^2 < +\infty$ (see [5]) and

$$\begin{aligned}
\varphi(k) &= \frac{1}{2\pi}\int_E e^{ik\omega}d\omega = \sum_n \frac{1}{2\pi}\int_{E\bigcap[2n\pi-\pi,2n\pi+\pi]} e^{ik\omega}d\omega \\
&= \sum_n \frac{1}{2\pi}\int_{(E-2n\pi)\bigcap[-\pi,\pi]} e^{ik\omega}d\omega \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_n \chi_E(\omega+2n\pi)e^{ik\omega}d\omega .
\end{aligned}$$

So

$$\sum_k \varphi(k)e^{-ik\omega} = \sum_n \chi_E(\omega + 2n\pi) .$$

By Theorem 1, there is a sampling expansion for each $f \in V_0$. In particular, if $E = [-\pi, \pi]$, the above equation turns out to be

$$\sum_k \varphi(k)e^{-ik\omega} = 1 .$$

Consequently, the function $S(x)$ defined in (2.2) satisfies $S(x) = \varphi(x) = \frac{\sin \pi x}{\pi x}$ and $V_0 = PW_\pi :=$ $\{f \in L^2 : \operatorname{supp} \hat{f} \subset [-\pi, \pi]\}$. For any $f \in V_0$,

$$f(x) = \sum_k f(k)S(x-k) = \sum_k f(k)\frac{\sin \pi(x-k)}{\pi(x-k)} ,$$

which is just the Shannon sampling theorem. □

Example 4. Suppose that E is a measurable set and $\{E + 2k\pi : k \in \mathbf{Z}\}$ constitutes a partition of $\mathbf{R}$. Let

$$\varphi(x) = \frac{1}{2\pi}\int_E e^{i\omega x} d\omega .$$

Then $\sum_k |\hat{\varphi}(\omega + 2k\pi)|^2 = \sum_k \chi_E(\omega + 2k\pi) = 1$, a.e. Hence, $\{\varphi(\cdot - n)\}$ is an orthonormal basis for the space V_0 it spans. It is easy to see that $V_0 = \{f : \operatorname{supp} \hat{f} \subset E\}$.

For any x, define

$$C_x(\omega) = \sum_k e^{ix(\omega+2k\pi)}\chi_E(\omega + 2k\pi) .$$

Then $C_x(\omega) \in L^2[-\pi, \pi]$ and $C_x(\omega) = e^{ix\omega}$ for $\omega \in E$. Let $E_k = (E - 2k\pi)\bigcap[-\pi, \pi]$. Then we have $\bigcup\limits_k E_k = [-\pi, \pi]$ and

$$\begin{aligned}
\varphi(x+n) &= \frac{1}{2\pi}\int_E e^{i(x+n)\omega} d\omega = \frac{1}{2\pi}\int_E C_x(\omega)e^{in\omega} d\omega \\
&= \sum_k \frac{1}{2\pi}\int_{E\bigcap[2k\pi-\pi,2k\pi+\pi]} C_x(\omega)e^{in\omega} d\omega \\
&= \sum_k \frac{1}{2\pi}\int_{E_k} C_x(\omega)e^{in\omega} d\omega \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} C_x(\omega)e^{in\omega} d\omega .
\end{aligned}$$

Hence,

$$\sum_n |\varphi(x+n)|^2 = \frac{1}{2\pi}\|C_x(\omega)\|^2_{L^2[-\pi,\pi]} = 1 .$$

On the other hand, since

$$\varphi(n) = \frac{1}{2\pi}\int_E e^{in\omega} d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{in\omega} d\omega = \delta_{n,0} ,$$

by Theorem 1, the sampling theorem holds on V_0 with sampling function $S(x) = \varphi(x)$. □

Remark 2.

Both the scaling function of translation invariant multiresolution [11] and the minimal supported frequency wavelet [8] satisfy the conditions of Example 4. □

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