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# Traces of Oscillating Functions

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*ABSTRACT. We show that the trace of an indefinitely oscillating function on a subspace of $\mathbb{R}^d$ is not always indefinitely oscillating. In the periodic case, the number of oscillations of the trace depends on the regularity of the function. In the general case, we exhibit a definitive counter-example.*

## 1. Introduction

The notion of *chirp* was introduced by Meyer [8, 10] to describe a function behaving like

$$\varphi(x) = |x - x_0|^h \sin\left(|x - x_0|^{-\beta}\right) \tag{1.1}$$

around a point $x_0 \in \mathbb{R}$, with $h > 0$ and $\beta > 0$.

This concept proved to be useful in many domains: chirps naturally appear in the study of mathematical functions like the famous Riemann "nowhere differentiable" function $x \mapsto \sum n^{-2} \sin(n^2 x)$ (which has a dense set of chirp points, where it is in fact differentiable [7, 8]); they are also expected to happen in natural phenomenons like gravitational waves and fully developed turbulence [1, 2].

### 1.1 Preliminaries

The sine function in (1.1) is too specific to make a general definition; actually we are only interested in its oscillatory nature, which can be expressed by the fact that $x \mapsto \sin(x)$ has bounded primitives of any order. This remark allows us to generalize this notion to higher dimensions.

**Definition 1.** A function $f \in L^\infty(\mathbb{R}^d)$ has $N$ oscillations in the strict sense if and only if it can be written

$$f = \sum_{|\alpha|=N} \partial^\alpha f_\alpha$$

the $f_\alpha$ belonging to $L^\infty(\mathbb{R}^d)$.

**Definition 2.** A function is indefinitely oscillating in the strict sense if and only if it has $N$ oscillations in the strict sense for any $N \in \mathbb{N}$.

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**Definition 3.** A function $f$ is said to have $N$ oscillations in the broad sense ($N \leq \infty$) if and only if it can be written $f = g + h$, where $g$ has $N$ oscillations in the strict sense and $h \in \mathcal{S}(\mathbb{R}^d)$ (the Schwartz class).

These definitions, for an infinite number of oscillations, were introduced in [8, 10, 11, 12]. The need for a finite number of oscillations will soon appear.

The general form of a chirp can then be written as:

$$\varphi(x) = |x - x_0|^h f\left(\frac{x - x_0}{|x - x_0|^{1+\beta}}\right) \tag{1.2}$$

where $h > 0$ and $\beta > 0$ are the chirp (or oscillating singularity) exponents of $\varphi$ at the point $x_0$ and $f$ is an *oscillating function* in the sense of (one of) the previous definitions. More precisely, which definition to take in (1.2) depends on whether we want $\varphi$ to be called a chirp (then $f$ must be indefinitely oscillating in the broad sense) or an oscillating singularity (then it suffices that $f$ has a finite number oscillations in the broad sense).

### 1.2 Statement of the Problem

The question of detecting chirps in a 3D turbulent flow is complicated by the fact that the only very precise experimental data we can get are 1D speed or pressure samples. Can we conclude from detecting (or not) chirps in these samples that chirps exist (or not) in the 3D flow? This problem amounts to a look at the trace $f_\Delta$ of an oscillating function $f$ on a straight line $\Delta$. We are especially interested in the case where $f$ is indefinitely oscillating. More generally, we consider the problem where $\Delta \subset \mathbb{R}^d$ is a subspace of dimension $s < d$.

We suppose that $f$ is continuous in order to ensure that its trace $f_\Delta$ always exists and belongs to $L^\infty(\mathbb{R}^s)$. Without loss of generality, one can suppose that $0 \in \Delta$ so that $\Delta$ is determined by an orthonormal basis $\mu = (\mu^1, \dots, \mu^s) \in (\mathbb{R}^d)^s$. If $t \in \mathbb{R}^s$ is a vector in this basis, its coordinates in $\mathbb{R}^d$ are

$$\mu.t = \begin{bmatrix} \sum_l \mu_1^l t_l \\ \vdots \\ \sum_l \mu_d^l t_l \end{bmatrix}.$$

We deduce the expression of the trace: $f_\Delta(t) = f(\mu.t)$. The orthogonal projector on $\Delta$ maps $x \in \mathbb{R}^d$ to $x.\mu \in \mathbb{R}^s$.

The purpose of this paper is to determine whether the function $f_\Delta$ is still indefinitely oscillating, and in case it is not, how many oscillations it can have.

## 2. Characterization of Oscillating Functions

Oscillating functions were first studied by Xu in his Ph.D. thesis [11, 12], where the following basic lemma is shown:

***Lemma 1.***

*If $f \in L^\infty(\mathbb{R}^d)$ and if $\widehat{f}(\xi)$ vanishes for $|\xi| \leq r$, then $f$ is indefinitely oscillating in the strict sense because $\forall N$,*

$$f = \sum_{|\alpha|=N} \partial^\alpha f_\alpha$$

*with*

$$\|f_\alpha|_{L^\infty} \leq Cr^{-N} \|f\|_{L^\infty} .$$

From this lemma we deduce a lower bound for the number of oscillations of a function.

***Proposition 1.***

*Let $g \in L^\infty(\mathbb{R}^d)$ and $\phi \in \mathcal{S}(\mathbb{R}^d)$ whose integral is non-zero. If there exists a constant $C < \infty$ such that $\forall \epsilon \in (0,1)$*

$$\sup_{t\in\mathbb{R}^d}\left|\int_{\mathbb{R}^d}\phi(t-\epsilon y)g(y)dy\right| \le C\epsilon^{N-d}\log(\epsilon)^{-2},$$

*then $g$ has (at least) $N$ oscillations in the strict sense.*

**Proof.** Suppose first that $\widehat{\phi}(\xi) = 1$ for $\xi$ in a neighborhood of 0. Then $\phi$ can be used to construct a family of Littlewood–Paley decomposition operators: we note for $j \in \mathbb{Z}$

$$\begin{aligned} \phi_j(x) &= 2^{dj}\phi(2^j x) \\ S_j g &= \phi_j \star g \\ \Delta_j g &= S_{j+1}g - S_j g \end{aligned}$$

For $j < 0$ and $\epsilon = 2^j$, the hypothesis implies

$$\begin{aligned} \sup_{t\in\mathbb{R}^d}\left|\int_{\mathbb{R}^d}\phi\left(2^j\left(2^{-j}t-y\right)\right)g(y)dy\right| &\le C2^{j(N-d)}j^{-2} \\ 2^{-dj}\sup_{t\in\mathbb{R}^d}\left|\phi_j\star g\left(2^{-j}t\right)\right| &\le C2^{j(N-d)}j^{-2} \\ \left\|S_j g\right\|_{L^\infty} &\le C2^{jN}j^{-2} \end{aligned}$$

One has

$$g = g - S_0 g + \sum_{-\infty<j<0}\Delta_j g$$

and by Lemma 1, each term of the series is indefinitely oscillating in the strict sense because its Fourier transform is zero in a neighborhood of 0.

Moreover, when we write

$$\Delta_j g = \sum_{|\alpha|=N}\partial^\alpha g_{j,\alpha}$$

we know by the same lemma that

$$\begin{aligned} \left\|g_{j,\alpha}\right\|_{L^\infty} &\le C2^{-jN}\left\|\Delta_j g\right\|_{L^\infty} \\ &\le C2^{-jN}\left(\left\|S_j g\right\|_{L^\infty} + \left\|S_{j+1}g\right\|_{L^\infty}\right) \\ &\le Cj^{-2} \end{aligned}$$

whose series converges. To conclude,

$$g = \underbrace{g - S_0 g}_{\text{indefinitely oscillating}} + \sum_{|\alpha|=N}\partial^\alpha \underbrace{\sum_{-\infty<j<0} g_{j,\alpha}}_{\in L^\infty(\mathbb{R}^d)}$$

Now if one has just $\widehat{\phi}(0) \neq 0$, by continuity this is also true in a neighborhood $V_1$ of 0. One can then construct $\widehat{\theta}(\xi) \in \mathcal{S}(\mathbb{R}^d)$ equal to $\widehat{\phi}(\xi)^{-1}$ on a compact set containing a neighborhood $V_2$ of 0. In this way, $\widehat{\phi\star\theta}(\xi) = 1$ in $V_2$, and

$$\begin{aligned} \sup_{t\in\mathbb{R}^d}\left|\int_{\mathbb{R}^d}\phi\star\theta\,(t-\epsilon y)\,g(y)dy\right| &\le \|\theta\|_{L^1}\sup_{t\in\mathbb{R}^d}\left|\int_{\mathbb{R}^d}\phi\,(t-\epsilon y)\,g(y)dy\right| \\ &\le C\epsilon^{N-d}\log(\epsilon)^{-2} \end{aligned}$$

so we can replace $\phi$ by $\phi \star \theta$ in the previous calculation. □

If $\psi \in \mathcal{S}(\mathbb{R}^d)$ is an admissible wavelet (see [4]),

$$C_{ab}(f) = \langle f, \psi_{ab} \rangle = \int_{\mathbb{R}^d} \frac{1}{a^d} \psi\left(\frac{x-b}{a}\right) f(x)dx$$

are the wavelet coefficients of a function $f$. We recall that by definition $f \in C^{s,s'}(x_0)$ (the inhomogeneous 2-microlocal space) if and only if $\exists C < \infty$ such that

$$|C_{ab}(f)| \leq Ca^s \left(1 + \frac{|b - x_0|}{a}\right)^{-s'}$$

for all $b$ in a neighborhood of $x_0$ and $a$ in a neighborhood of $0^+$.

The 2-microlocal regularity of $\widehat{g}$ at 0 gives an upper bound, that will be used later, for the number of oscillations of a function.

***Proposition 2.***

*Let $g \in L^\infty(\mathbb{R}^d)$. If $g$ has $N$ oscillations in the broad sense, then*

$$\widehat{g} \in C^{N-d,-N}(0)$$

**Proof.** Since $h \in \mathcal{S}(\mathbb{R}^d) \iff \widehat{h} \in \mathcal{S}(\mathbb{R}^d)$, we can suppose $g$ to be oscillating in the strict sense, writing

$$g(x) = \sum_{|\alpha|=N} \partial^\alpha g_\alpha(x)$$

with each $g_\alpha \in L^\infty(\mathbb{R}^d)$. Hence, the wavelet coefficients of $\widehat{g}$:

$$\begin{aligned} C_{ab}(\widehat{g}) &= \langle g, \widehat{\psi_{ab}} \rangle \\ &= \int_{\mathbb{R}^d} g(x) e^{ibx} \overline{\widehat{\psi}(ax)} dx \\ &= \sum_{|\alpha|=N} \int_{\mathbb{R}^d} \partial^\alpha g_\alpha(x) e^{ibx} \overline{\widehat{\psi}(ax)} dx \end{aligned}$$

and integrating $N$ times by parts,

$$\begin{aligned} &= (-1)^N \sum_{|\alpha|=N} \int_{\mathbb{R}^d} g_\alpha(x) \partial^\alpha \left(e^{ibx} \overline{\widehat{\psi}(ax)}\right) dx \\ |C_{ab}(\widehat{g})| &\leq \sum_{|\alpha|=N} \|g_\alpha\|_{L^\infty} \sum_{k \leq \alpha} \binom{\alpha}{k} |b|^{|k|} a^{N-|k|} \left\| x \mapsto \widehat{\psi}^{(\alpha-k)}(ax) \right\|_{L^1} \\ &\leq \sum_{|\alpha|=N} C(\alpha) \|g_\alpha\|_{L^\infty} a^{-d} \sup_{k \leq \alpha} \left\| \widehat{\psi}^{(\alpha-k)} \right\|_{L^1} (a + |b|)^N \\ &\leq Ca^{N-d} \left(1 + \frac{|b|}{a}\right)^N \end{aligned}$$

□

## 3. The Periodic Case

In this section we use the following notations: $\lfloor x \rfloor$ is the largest integer $\leq x$ and $\lceil x \rceil$ is the smallest integer $\geq x$. If $A \subset \mathbb{R}^q$, $\mathcal{L}(A)$ is its $q$-Lebesgue measure. For $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ (or $\mathbb{Z}^d$), $\|x\| = \max_i |x_i|$.

The simplest examples of oscillating functions are periodic, of general form

$$f(x) = \sum_{m \in \mathbb{Z}^d} a_m e^{im.x}$$

where $(a_m) \in l^1(\mathbb{Z}^d)$. It is easy to see that $f$ is indefinitely oscillating in the strict sense if and only if $a_0 = 0$ (else $f$ has no oscillation at all), and the trace can be written

$$f_\Delta(t) = \sum_{m \in \mathbb{Z}^d} a_m e^{im.\mu.t}$$

Note that this is in general no longer a periodic trigonometric series. The frequency of the $m$th term is the vector $m.\mu \in \mathbb{R}^s$. Of course, each term of the series is oscillating but $m.\mu$ can become very small, so that $f_\Delta$ may have an accumulation of terms of arbitrary low frequency and may not be oscillating at all; therefore, to see if the sum oscillates we need to know how fast $m.\mu$ can get close to the 0, and compare it to the decay of $a_m$. This is a problem related to diophantine approximation.

## 3.1 Some Recalls on Diophantine Approximation

Let us denote by $\langle x \rangle$ the symmetrical fractional part of $x$, that is

$$\langle x \rangle = (x_1 + 1 - \lceil x_1 + 1/2 \rceil, \dots, x_s + 1 - \lceil x_s + 1/2 \rceil) \in (-1/2, 1/2]^s$$

Given an $n \times s$ real matrix $\nu$, the central question in simultaneous diophantine approximation consists of finding $q \in \mathbb{Z}^n$ such that $\langle q.\nu \rangle$ is close to 0 or, which is equivalent, such that $q.\nu$ is close to a vector $p \in \mathbb{Z}^s$. The first answer to this problem was given in 1842 by Dirichlet.

***Theorem 1 (Dirichlet).***
*There exists a constant $C < \infty$ such that for all $\nu \in (\mathbb{R}^n)^s$, the inequality*

$$\|\langle q.\nu \rangle\| \le \frac{C}{\|q\|^{\frac{n}{s}}} \tag{3.1}$$

*has infinitely many solutions $q \in \mathbb{Z}^n$.*

Our problem seems to be slightly different, since it is a matter of making $m.\mu$ close to the fixed vector 0. Actually, they are equivalent, by the following transformation. Since $\mu$ is a basis of $\Delta$, we can extract $s$ rows of $\mu$ such that the extracted matrix is invertible. Suppose to simplify that these rows are the $s$ last ones: $\mu = \begin{bmatrix} H \\ B \end{bmatrix}$ where $B$ is an $s \times s$ invertible matrix. We make a diophantine approximation of $\tilde{\mu} = -HB^{-1}$, that is, we find $q \in \mathbb{Z}^{d-s}$ and $p \in \mathbb{Z}^s$ such that $\|q.\tilde{\mu} - p\| \le \epsilon$. This can also be written $\|m.\mu.(-B)^{-1}\| \le \epsilon$, with $m = (q_1, \dots, q_{d-s}, p_1, \dots, p_s) \in \mathbb{Z}^d$, or $\|m.\mu\| \le C\epsilon$ where $C$ depends only on $B$.

We also notice that this application $\Phi : \mu \mapsto \tilde{\mu}$ preserves zero measures (that is, if $\mathcal{L}(A) = 0$ in $\mathbb{R}^{(d-s)\times s}$ then $\mathcal{L}(\Phi^{-1}(A)) = 0$ in $\mathbb{R}^{d\times s}$), so any result valid for almost every $\tilde{\mu}$ is also valid for almost every $\mu$.

Dirichlet's theorem can easily be transformed this way. Instead we directly give the following more subtle result of Khinchin [9], generalized by Groshev [5, 6].

***Theorem 2 (Khinchin–Groshev).***
*Let $\theta : \mathbb{N} \to \mathbb{R}^+$ be such that the function $r \mapsto r^{d-s-1}\theta(r)^s$ is decreasing.*[1] *For almost every $\mu \in (\mathbb{R}^d)^s$, the inequality*

$$\|m.\mu\| \le \theta(\|m\|) \tag{3.2}$$

*has infinitely many solutions $m \in \mathbb{Z}^d$ if and only if the sum $\sum_{r=1}^{\infty} r^{d-s-1}\theta(r)^s$ diverges.*

[1] Actually, this condition is only necessary for the "if" way, and for $d = s + 1$ or $s + 2$, and when $r \to \infty$.

## 3.2 Oscillation Tests

In the case of periodic functions, Proposition 1 can be rewritten as follows:

***Proposition 3.***

*Let $\phi \in \mathcal{S}(\mathbb{R}^s)$ be such that $\phi(0) \neq 0$. If there exists a constant $C < \infty$ such that for every $\epsilon \in (0, 1)$,*

$$\sum_{m\in\mathbb{Z}^d} \left|a_m \phi\left(\frac{m.\mu}{\epsilon}\right)\right| \leq C\epsilon^M \log(\epsilon)^{-2} \tag{3.3}$$

*then $f_\Delta$ has (at least) $M$ oscillations in the strict sense.*

**Proof.** Let $\widehat{\psi} = \phi$. We estimate for every $x \in \mathbb{R}^s$

$$\begin{aligned}
\left|\int_{\mathbb{R}^s} f_\Delta(t)\psi(\epsilon t - x)dt\right| &= \left|\sum_{m\in\mathbb{Z}^d} a_m \int_{\mathbb{R}^s} e^{im.\mu.(t+\frac{x}{\epsilon})}\psi(\epsilon t)dt\right| \\
&= \left|\sum_{m\in\mathbb{Z}^d} \frac{a_m}{\epsilon^s} e^{i\frac{m.\mu.x}{\epsilon}} \phi\left(\frac{m.\mu}{\epsilon}\right)\right| \\
&\leq \frac{1}{\epsilon^s} \sum_{m\in\mathbb{Z}^d} \left|a_m \phi\left(\frac{m.\mu}{\epsilon}\right)\right| \\
&\leq C\epsilon^{M-s} \log(\epsilon)^{-2}
\end{aligned}$$

and conclude with Proposition 1 (in $\mathbb{R}^s$). □

There is a (partial) converse result:

***Proposition 4.***

*Let $\phi \in \mathcal{S}(\mathbb{R}^s)$. If $f_\Delta$ has $M$ oscillations in the broad sense, then there exists a constant $C < \infty$ such that $\forall\epsilon \in (0, 1)$,*

$$\left|\sum_{m\in\mathbb{Z}^d} a_m \phi\left(\frac{m.\mu}{\epsilon}\right)\right| \leq C\epsilon^M \tag{3.4}$$

It is only partial because of course (3.4) does not imply (3.3). This means that we cannot exclude the possibility that, because of some cancellations between the $a_m$, $f_\Delta$ has strictly more than $M$ oscillation even if $\left|\sum_{m\in\mathbb{Z}^d} a_m \phi\left(\frac{m.\mu}{\epsilon}\right)\right| > C\epsilon^M$.

The hypothesis above is of course weaker than supposing oscillations in the strict sense; it would be equivalent if $f_\Delta$ were periodic, which is generally not true, but not far.

***Lemma 2.***

*For all $\alpha > 0$ and $\forall K \in \mathbb{N}$, $\exists\beta \in \mathbb{R}^s$, $|\beta| \geq K$ such that*

$$\sup_{t\in\mathbb{R}^s} |f_\Delta(t) - f_\Delta(t+\beta)| < \alpha$$

**Proof.** Since $\sum |a_m|$ converges, $\exists k \in \mathbb{N}$ such that $\sum_{\|m\|>k} |a_m| < \frac{\alpha}{4}$, hence

$$\left|\sum_{\|m\|>k} a_m \left(e^{im.\mu.t} - e^{im.\mu.(t+\beta)}\right)\right| < \frac{\alpha}{2}$$

for all $\beta \in \mathbb{R}^s$.

Thanks to Dirichlet's theorem, there exists an integer $\gamma > \frac{K}{2\pi}$ such that

$$\left\|\left\langle \gamma \mu^1 \right\rangle\right\| < \frac{\alpha}{4\pi dk \sum_{\|m\|\leq k} |a_m|} ;$$

thus, for $\|m\| \geq k$,

$$\left\|\left\langle \gamma m.\mu^1 \right\rangle\right\| < \frac{\alpha}{4\pi \sum_{\|m\|\leq k} |a_m|} .$$

Now we take $\beta = (2\pi\gamma, 0, \dots, 0)$: then $e^{im.\mu.\beta} = e^{i2\pi\langle \gamma m.\mu^1\rangle}$ and

$$\left|e^{im.\mu.\beta} - 1\right| < \frac{\alpha}{2\sum_{\|m\|\leq k} |a_m|}$$

$$\left|\sum_{\|m\|\leq k} a_m \left(e^{im.\mu.t} - e^{im.\mu.(t+\beta)}\right)\right| < \frac{\alpha}{2} ;$$

hence, the conclusion. □

**Proof of Proposition 4.** Let $\widehat{\psi} = \phi$. By hypothesis, $f_\Delta = g + h$, $g \in L^\infty(\mathbb{R}^s)$ having $M$ oscillations in the strict sense and $h \in \mathcal{S}(\mathbb{R}^s)$. Let $\|x\| \geq 2$ and $\epsilon \in (0, 1)$. For the oscillating part we have

$$\left|\int_{\mathbb{R}^s} g\left(t + \frac{x}{\epsilon^2}\right) \psi(\epsilon t) dt\right| \leq \sum_{|\alpha|=M} \left|\int_{\mathbb{R}^s} \partial^\alpha g_\alpha \left(t + \frac{x}{\epsilon^2}\right) \psi(\epsilon t) dt\right|$$

integrating $M$ times by part,

$$\begin{aligned}
&\leq \sum_{|\alpha|=M} \left|\int_{\mathbb{R}^s} g_\alpha \left(t + \frac{x}{\epsilon^2}\right) \epsilon^M \partial^\alpha \psi(\epsilon t) dt\right| \\
&\leq \epsilon^M \epsilon^{-s} \sum_{|\alpha|=M} \|g_\alpha\|_{L^\infty} \left\|\partial^\alpha \psi\right\|_{L^1} \\
&\leq C\epsilon^{M-s}
\end{aligned}$$

For the other part:

$$\begin{aligned}
\left|\int_{\mathbb{R}^s} h\left(t + \frac{x}{\epsilon^2}\right) \psi(\epsilon t) dt\right| \leq & \int_{\|t\|<\epsilon^{-\frac{3}{2}}} \left|h\left(t + \frac{x}{\epsilon^2}\right) \psi(\epsilon t)\right| dt \\
& + \int_{\|t\|>\epsilon^{-\frac{3}{2}}} \left|h\left(t + \frac{x}{\epsilon^2}\right) \psi(\epsilon t)\right| dt
\end{aligned}$$

and the first integral is dominated by any power of $\epsilon$ because of the decreasing of $h$; the second one idem because of the decreasing of $\psi$.

To sum up, we found a constant $C$ such that $\forall \epsilon \in (0, 1)$,

$$\sup_{\|x\|\geq 2} \left|\int f_\Delta \left(t + \frac{x}{\epsilon^2}\right) \psi(\epsilon t) dt\right| \leq \frac{C}{2} \epsilon^{M-s} \tag{3.5}$$

We now get rid of the $\frac{x}{\epsilon^2}$ term thanks to Lemma 2: there exists $\beta$, $\|\beta\| \geq \frac{2}{\epsilon^2}$ such that, taking $x = \beta\epsilon^2$,

$$\int_{\mathbb{R}^s} \left|f_\Delta \left(t + \frac{x}{\epsilon^2}\right) - f_\Delta(t)\right| \psi(\epsilon t) dt \leq \frac{C}{2} \epsilon^{M-s}$$

which implies, with (3.5),

$$\left|\int_{\mathbb{R}^s} f_\Delta(t)\psi(\epsilon t)dt\right| \le C\epsilon^{M-s} .$$

On the other hand, we have

$$\begin{aligned}\int_{\mathbb{R}^s} f_\Delta(t)\psi(\epsilon t)dt &= \sum_{m\in\mathbb{Z}^d} a_m \int_{\mathbb{R}^s} e^{im.\mu.t}\psi(\epsilon t)dt \\ &= \sum_{m\in\mathbb{Z}^d} \frac{a_m}{\epsilon^s}\phi\left(\frac{m.\mu}{\epsilon}\right) ;\end{aligned}$$

hence, (3.4). □

## 3.3 Applications

We now use the previous results to estimate the number of oscillations of the function $f_\Delta$.

***Proposition 5.***

*Suppose $\nu > d - s$. If $a_0 = 0$ and if there exists a constant $C < \infty$ such that*

$$\forall m \in \mathbb{Z}^d,\ |a_m| \le \frac{C}{\|m\|^\nu} \tag{3.6}$$

*then for almost every $\Delta$, $f_\Delta$ has at least $\left\lceil \frac{\nu s}{d-s} \right\rceil - 1$ oscillations in the strict sense.*

**Proof.** According to Theorem 2, for almost every $\mu$,

$$\|m.\mu\| \ge \theta(m) = \frac{\|m\|^{\frac{s-d}{s}}}{\log(\|m\|)^{\frac{2}{s}}} \tag{3.7}$$

except for a finite set of $m$s, because

$$r^{d-s-1}\theta(r)^s = \frac{1}{r\log(r)^2}$$

is summable at infinity.

Let $\phi \in \mathcal{D}(\mathbb{R}^s)$ have non-zero integral and support in $\left[-\frac{1}{2}, \frac{1}{2}\right]^s$, and let $\epsilon \in (0, 1)$: only the $m$s such that $\|m.\mu\| \le \frac{\epsilon}{2}$ remains in the left-hand member of (3.3). Note $H(\epsilon)$ the set of these points ($\subset \mathbb{Z}^s$) which we cut in disjoint annuli of width $\theta^{-1}(\epsilon)$:

$$H(\epsilon) = \bigcup_{k\in\mathbb{N}} H_k(\epsilon)$$

with

$$H_k(\epsilon) = \left\{m \in H(\epsilon), k\theta^{-1}(\epsilon) \le \|m\| < (k+1)\theta^{-1}(\epsilon)\right\}$$

and by the way, for $\epsilon$ small,

$$\theta^{-1}(\epsilon) \sim \left(\frac{s}{d-s}\right)^{\frac{2}{s-d}} \epsilon^{\frac{s}{s-d}} |\log(\epsilon)|^{\frac{2}{s-d}} .$$

Let us admit for the moment the following lemma:

***Lemma 3.***

*There exist $\epsilon_0 > 0$ and $C < \infty$ (depending only on $\mu$) such that for $0 < \epsilon < \epsilon_0$, each $H_k(\epsilon)$, $k \geq 1$ does not contain more than $Ck^{d-s-1}$ points. Moreover, $H_0(\epsilon)$ contains only the point 0.*

Since by hypothesis $a_0 = 0$, we get

$$\begin{aligned}\sum_{m\in\mathbb{Z}^d}\left|a_m\phi\left(\frac{m.\mu}{\epsilon}\right)\right| &\leq C\sum_{k=1}^{\infty}\sum_{m\in H_k(\epsilon)}\|m\|^{-\nu}\\ &\leq C\sum_{k=1}^{\infty}k^{d-s-1}\left(k\theta^{-1}(\epsilon)\right)^{-\nu}\\ &\leq C\theta^{-1}(\epsilon)^{-\nu}\sum_{k=1}^{\infty}k^{d-s-\nu-1}\\ &\leq C\epsilon^{\frac{\nu s}{d-s}}\,|\log(\epsilon)|^{\frac{2\nu}{d-s}}\\ &\leq C\epsilon^{\left\lceil\frac{\nu s}{d-s}\right\rceil-1}\log(\epsilon)^{-2}\end{aligned}$$

because in every case (integer or not), $\frac{\nu s}{d-s} > \left\lceil\frac{\nu s}{d-s}\right\rceil - 1$ and the difference compensates for the logarithm.

We just need to apply Proposition 3 to conclude. □

**Proof of Lemma 3.** In order to get rid of the exceptions, let

$$\epsilon_0 = \inf\left\{\|m.\mu\|, m\in\mathbb{Z}^d, m\neq 0 \text{ and does not satisfy (3.7)}\right\}$$

which is certainly $> 0$ because 1° if there were $m_0 \in \mathbb{Z}^d, m_0 \neq 0$ such that $m_0.\mu = 0$ then $\forall n \in \mathbb{N}, nm_0$ would not satisfy (3.7), whereas there can be only a finite number of such cases; 2° the infimum is reached, since this set is finite.

Thus, as soon as $\epsilon < \epsilon_0$, one has for *every* pair $(m, m') \in H(\epsilon)^2$, $m \neq m'$, $\theta(\|m - m'\|) \leq |(m - m').\mu| \leq \epsilon$ which implies

$$\|m - m'\| \geq \theta^{-1}(\epsilon)\,. \tag{3.8}$$

Note for $k \geq 1$

$$R_k(\epsilon) = \bigcup_{m\in H_k(\epsilon)} Q\left(m, \frac{\theta^{-1}(\epsilon)}{2}\right)$$

$Q(m, r)$ being the $d$-hypercube of center $m$ and side $2r$. It follows from (3.8) that these hypercubes have disjoint interiors. Each one has a $d$-volume equal to $\theta^{-1}(\epsilon)^d$, while

$$\begin{aligned}R_{k(\epsilon)} \subset & \left[-\epsilon - \frac{\theta^{-1}(\epsilon)}{2}, \epsilon + \frac{\theta^{-1}(\epsilon)}{2}\right]^s\\ & \times\left\{x\in\Delta^{\perp}, \left(k-\frac{1}{2}\right)\theta^{-1}(\epsilon) \leq x < \left(k+\frac{3}{2}\right)\theta^{-1}(\epsilon)\right\}\end{aligned}$$

whose $d$-volume is less than $C\theta^{-1}(\epsilon)k^{d-s-1}$, $C$ being some constant. Hence each $H_k(\epsilon)$ cannot contain more than $Ck^{d-s-1}$ points.

For the same reason, since $H_0(\epsilon)$ contains the point 0, it does not contain any other point. □

The fact that the number of oscillations depends on the decay of the $a_m$, hence on the global regularity of $f$, may seem surprising since the regularity appears nowhere in Definition 1 and following; but it is only a consequence of the "concurrence" between the decreasing of the frequencies $m.\mu$ and the decreasing of the Fourier coefficients $a_m$, as intuited in the beginning of this section.

Indeed this is unavoidable, as stated in the following (partial) converse:

***Proposition 6.***

*Suppose that there exists a constant $C > 0$ such that*

$$\forall m \in \mathbb{Z}^d, a_m \geq \frac{C}{\|m\|^\nu} \tag{3.9}$$

*except maybe for a finite number of coefficients. Then $f_\Delta$ has more than $\left\lceil \frac{\nu s}{d-s} \right\rceil - 1$ oscillations in the broad sense only on a negligible set of $\Delta$.*

**Proof.** We use again Theorem 2 with now

$$\|m.\mu\| \leq \theta(\|m\|) = \frac{\|m\|^{\frac{s-d}{s}}}{\log(\|m\|)^{\frac{1}{s}}} \tag{3.10}$$

which has for almost every $\mu$ infinitely many solutions $m(k)$, $k \in \mathbb{N}$ because

$$r^{d-s-1}\theta(r)^s = \frac{1}{r\log(r)}$$

is not summable. We can thus, construct a sequence

$$\epsilon(k) = \theta(\|m(k)\|) \to_{k\to\infty} 0 \,.$$

Let $\phi \in \mathcal{S}(\mathbb{R}^s)$ positive be such that $\phi(x) \geq 1$ when $\|x\| \leq 1$.

$$\begin{aligned}
\sum_{m\in\mathbb{Z}^d} a_m \phi\left(\frac{m.\mu}{\epsilon(k)}\right) &\geq a_{m(k)}\phi\left(\frac{m(k).\mu}{\epsilon(k)}\right) \\
&\geq C\|m(k)\|^{-\nu} \\
&\geq C\epsilon(k)^{\frac{\nu s}{d-s}} \log(\|m(k)\|)^{\frac{\nu}{d-s}} \\
&\gg_{k\to\infty} C\epsilon(k)^{\frac{\nu s}{d-s}} \\
&\gg_{k\to\infty} C\epsilon(k)^{\left\lceil \frac{\nu s}{d-s} \right\rceil}
\end{aligned}$$

because $\frac{\nu s}{d-s} \leq \left\lceil \frac{\nu s}{d-s} \right\rceil$; which proves, with Proposition 4, that supposing "$f_\Delta$ has $\left\lceil \frac{\nu s}{d-s} \right\rceil$ oscillations" is wrong. □

We see, in conjunction with Proposition 5, that this result is optimal.

A consequence of this is that, in the general case, to ensure that $f_\Delta$ is indefinitely oscillating, one needs to suppose $f \in C^\infty(\mathbb{R}^d)$. We see now that this is not even enough.

## 4. Counter-Example in the General Case

***Proposition 7.***

*There exists a non-zero, entire, (hence $C^\infty$) and indefinitely oscillating function $f$ on $\mathbb{R}^d$ such that for almost no subspace $\Delta$ of dimension $s$, $f_\Delta$ has more than $\left\lfloor \frac{d+s+1}{2} \right\rfloor$ oscillations in the broad sense.*

**Proof.** We construct $f$ by its Fourier transform. Let $0 < \alpha < \frac{1}{2}$, $u \in \mathbb{R}^d$, and

$$\widehat{f}(\xi) = \begin{cases} 0 \text{ if } |\xi| < 1 \text{or} |\xi| > 2\,; \\ \sum_{n\in\mathbb{N}} 2^{-n\alpha} e^{i2^n u.\xi} \text{ elsewhere}\,. \end{cases}$$

where $|\xi| = \sqrt{\xi_1^2, \ldots, \xi_d^2}$ is now the Euclidean norm.

$\widehat{f}$ being of compact support, by the theorem of Paley–Wiener, $f$ is entire. Since $\widehat{f} \in L^1(\mathbb{R}^d)$, $f \in L^\infty(\mathbb{R}^d)$. And since $\widehat{f}$ is zero in a neighborhood of 0, by Lemma 1, $f$ is indefinitely oscillating in the strict sense.

$\mu$ is still an orthonormal basis of $\Delta$ a subspace of $\mathbb{R}^d$. We suppose that $u \in \mathbb{R}^d$ is neither in $\Delta$ nor in $\Delta^\perp$—this happens for almost every $\Delta$. We write

$$\begin{aligned} f_\Delta(t) &= \langle f, \delta_{\mu.t} \rangle \\ &= (2\pi)^{-d} \left\langle \widehat{f}, \xi \mapsto e^{-i\xi.\mu.t} \right\rangle \end{aligned}$$

by Parseval's formula.

Let $\xi_\Delta$ and $\xi_{\Delta^\perp}$ be the orthogonal projections of $\xi$, respectively, on $\Delta$ and $\Delta^\perp$:

$$\begin{aligned} f_\Delta(t) &= (2\pi)^{-d} \int_{\mathbb{R}^d} \widehat{f}(\xi_\Delta + \xi_{\Delta^\perp})\, e^{i(\xi_\Delta+\xi_{\Delta^\perp}).\mu.t} d\xi \\ &= (2\pi)^{-d} \int_{\mathbb{R}^d} \widehat{f}(\xi_\Delta + \xi_{\Delta^\perp})\, e^{i\xi_\Delta.\mu.t} d\xi \end{aligned}$$

because $\xi_{\Delta^\perp}.\mu = 0$.

Let $\tilde{\mu} = (\mu^{s+1}, \ldots, \mu^d)$ be an orthonormal basis of $\Delta^\perp$. By the change of variables: $\xi \to (\zeta, \zeta')$, where $\zeta = \xi_\Delta.\mu$ and $\zeta' = \xi_{\Delta^\perp}.\mu$ we get

$$f_\Delta(t) = (2\pi)^{-d} \int_{\mathbb{R}^s} e^{i\zeta.t} \int_{\mathbb{R}^{d-s}} \widehat{f}(\mu.\zeta + \tilde{\mu}.\zeta') d\zeta' d\zeta$$

hence the Fourier transform of $f_\Delta$:

$$\widehat{f_\Delta}(\zeta) = (2\pi)^{s-d} \int_{\mathbb{R}^{d-s}} \widehat{f}(\mu.\zeta + \tilde{\mu}.\zeta') d\zeta'$$

which can be explicitly calculated:

$$\begin{aligned} \widehat{f_\Delta}(\zeta) &= (2\pi)^{s-d} \int\limits_{1-|\zeta|^2<|\zeta'|^2<4-|\zeta|^2} \sum_{n\in\mathbb{N}} 2^{-n\alpha} e^{i2^n u.(\mu.\zeta+\tilde{\mu}.\zeta')} d\zeta' \\ &= (2\pi)^{s-d} \sum_{n\in\mathbb{N}} 2^{-n\alpha} e^{i2^n u.\mu.\zeta} \int\limits_{1-|\zeta|^2<|\zeta'|^2<4-|\zeta|^2} e^{i2^n u.\tilde{\mu}.\zeta'} d\zeta' \\ &= (2\pi)^{s-d} \sum_{n\in\mathbb{N}} e^{i2^n u.\mu.\zeta} \underbrace{2^{-n\alpha} \int_{\sqrt{1-|\zeta|^2}}^{\sqrt{4-|\zeta|^2}} r^{d-s-1} \int_{S^{d-s-1}} e^{i2^n r u.\tilde{\mu}.\theta} d\theta dr}_{\lambda_n(\zeta)} \end{aligned}$$

Let $a = \sqrt{1-|\zeta|^2}$, $b = \sqrt{4-|\zeta|^2}$, $\beta = |u.\tilde{\mu}|$, $\rho = 2^n \beta r$, $m = d-s-1$ and $v = \beta^{-1} u.\tilde{\mu}$; so

$$\begin{aligned} \lambda_n(\zeta) &= 2^{-n\alpha} \int_a^b r^m \int_{S^m} e^{i\rho v.\theta} d\theta dr \\ &= 2^{-n\alpha} \int_a^b r^m g_m(\rho) dr \end{aligned}$$

where

$$g_m(\rho) = c_m \rho^{\frac{1-m}{2}} J_{\frac{m-1}{2}}(\rho)$$

$J_{\frac{m-1}{2}}$ being the Bessel function and $c_m = (2\pi)^{\frac{m+1}{2}}$.

We use the following asymptotic formula as $x \to \infty$:

$$J_q(x) = \sqrt{\frac{2}{\pi}} x^{-\frac{1}{2}} \cos\left(x - \frac{\pi q}{2} - \frac{\pi}{4}\right) + O\left(x^{-\frac{3}{2}}\right)$$

to finish calculating $\lambda_n(\zeta)$. First the main part:

$$\begin{aligned}
\lambda_n(0) &= c_m \beta^{\frac{1-m}{2}} 2^{n(\frac{1-m}{2}-\alpha)} \int_1^2 r^{\frac{1+m}{2}} J_{\frac{m-1}{2}}\left(2^n \beta r\right) dr \\
&= c_m \beta^{\frac{1-m}{2}} 2^{n(\frac{1-m}{2}-\alpha)} \left[2^{-n}\beta^{-1} r^{\frac{m+1}{2}} J_{\frac{m+1}{2}}\left(2^n \beta r\right)\right]_1^2 \\
&= c_m \beta^{-\frac{1+m}{2}} 2^{-n\left(\frac{1+m}{2}+\alpha\right)} \left(2^{\frac{m+1}{2}} J_{\frac{m+1}{2}}\left(2^{n+1}\beta\right) - J_{\frac{m+1}{2}}\left(2^n\beta\right)\right) \\
&= c_m \beta^{-\frac{1+m}{2}} \sqrt{\frac{2}{\pi}} 2^{-n(\frac{2+m}{2}+\alpha)} \\
&\quad \times \left(2^{\frac{m}{2}} \sin\left(2^{n+1}\beta - \frac{\pi m}{4}\right) - \sin\left(2^n\beta - \frac{\pi m}{4}\right) + O\left(2^{-n}\right)\right) \\
&= \gamma_n 2^{-n(\alpha+\frac{d-s+1}{2})}
\end{aligned}$$

with $0 < \limsup_{n\to\infty} |\gamma_n| < \infty$; as for the increment ($\zeta \to 0$):

$$\begin{aligned}
\lambda_n(\zeta) - \lambda_n(0) &= c_m \beta^{\frac{1-m}{2}} 2^{n(\frac{1-m}{2}-\alpha)} \int_{(a,1)\cup(b,2)} r^{\frac{1+m}{2}} J_{\frac{m-1}{2}}\left(2^n \beta r\right) dr \\
&= 2^{-n(\alpha+\frac{m-1}{2})} 2^{-\frac{n}{2}} O(\zeta^2) \\
&= 2^{-n(\alpha+\frac{d-s-1}{2})} O\left(\zeta^2\right) .
\end{aligned}$$

Remember that

$$(2\pi)^{d-s} \widehat{f_\Delta}(\zeta) = \underbrace{\sum_{n\in\mathbb{N}} \lambda_n(0) e^{i2^n u.\mu.\zeta}}_{f_1(\zeta)} + \underbrace{\sum_{n\in\mathbb{N}} \left(\lambda_n(\zeta) - \lambda_n(0)\right) e^{i2^n u.\mu.\zeta}}_{f_2(\zeta)}$$

and the following classical lemma on lacunary Fourier series:

***Lemma 4.***

*Let $h > 0$ and $w \in \mathbb{R}^s$. If there exists a constant $C < \infty$ such that $\forall n \in \mathbb{N}$, $\nu_n \le C2^{-hn}$, then*

$$\phi(\zeta) = \sum_{n\in\mathbb{N}} \nu_n e^{i2^n w.\zeta}$$

*is globally Hölder with exponent $h$.*

We thus, see that $f_1$ is at least $C^{\alpha+\frac{d-s+1}{2}}$ and $f_2$ is at least $C^{\alpha+\frac{d-s+3}{2}}(0)$ thanks to the $O(\zeta^2)$.

On the other hand, let $\psi \in \mathcal{S}(\mathbb{R}^s)$ be an admissible wavelet whose Fourier transform has a support in $\left\{\frac{2|u.\mu|\log(2)}{4} \le |x| \le \frac{2|u.\mu|\log(2)}{2}\right\}$. The wavelet coefficients of $f_1$ at scale $a = 2^{-j}$ are

$$\begin{aligned}
C_{2^{-j},b}(f_1) &= \langle \widehat{f_1}, \widehat{\psi_{2^{-j},b}} \rangle \\
&= \sum_{n\in\mathbb{N}} \lambda_n(0) \left\langle \delta_{2^n u.\mu \log(2)}, x \mapsto e^{-ibx} \widehat{\psi}\left(2^{-j}x\right)\right\rangle
\end{aligned}$$

because of the supports,

$$\left|C_{2^{-j},b}(f_1)\right| = \left|\gamma_j\right| 2^{-j(\alpha+\frac{d-s+1}{2})}$$

This implies that there exist a sequence $a_j \to 0$ and a constant $C_1 > 0$ such that $\left|C_{a_j0}(\widehat{f_\Delta})\right| \geq C_1 a_j^{\alpha+\frac{d-s+1}{2}}$. Suppose now that $\widehat{f_\Delta} \in C^{N-s,-N}(0)$: by definition there exists a constant $C_2 < \infty$ such that $\left|C_{a_j0}(\widehat{f_\Delta})\right| \leq C_2 a_j^{N-s}$. Then necessarily $N \leq \alpha + \frac{d+s+1}{2}$; if $\alpha$ is small enough ($< 1 + \left\lfloor \frac{d+s+1}{2} \right\rfloor - \frac{d+s+1}{2}$), we conclude with Proposition 2 that $f_\Delta$ has not more than $\left\lfloor \frac{d+s+1}{2} \right\rfloor$ oscillations in the broad sense. □

## 5. Further Remarks and Improvements

If $s = \dim(\Delta) = 1$, Lemma 2 expresses nothing but the fact that the trace of a periodic function is almost-periodic in the sense of Bohr [3]. We recall one of his most famous results on this subject.

***Theorem 3.***
*If $f$ is almost-periodic and $F : t \mapsto \int_0^t f(s)ds$ is bounded on $\mathbb{R}$, then $F$ is also almost-periodic.*

Starting from this remark, Meyer proposed the following characterization.

***Proposition 8.***
*Let $N \in \mathbb{N}$. If $f$ is an almost-periodic function on $\mathbb{R}$, the three following properties are equivalent.*

1. *$f$ has $N$ oscillations in the broad sense.*
2. *$f$ has $N$ oscillations in the strict sense.*
3. *$f = \partial^N f_N$, where $f_N$ is almost-periodic.*

**Proof.** Clearly, $2 \Longrightarrow 1$, and $3 \Longrightarrow 2$ is trivial since an almost-periodic function is bounded.

To prove $1 \Longrightarrow 3$, we proceed by induction: it is obvious for $N = 0$. Suppose that the equivalence is true for $N$, and that $f$ has $N+1$ oscillations in the broad sense:

$$f(t) = \partial^{N+1} a_{N+1}(t) + b_{N+1}(t)$$

where $a_{N+1} \in L^\infty(\mathbb{R})$ and $b_{N+1} \in \mathcal{S}(\mathbb{R})$. Since $\partial^{N+1} a_{N+1} \in L^\infty(\mathbb{R})$, we also have $\partial^k a_{N+1} \in L^\infty(\mathbb{R})$ for $0 \leq k \leq N+1$.

Consider $B_{N+1} : t \mapsto \int_{-\infty}^t b_{N+1}(t)dt$: it is bounded, and it suffices to show that $B_{N+1}(+\infty) = 0$ to have $B_{N+1} \in \mathcal{S}(\mathbb{R})$.

In other words, $S_{N+1} = \partial^N a_{N+1} + B_{N+1}(t)$ is a bounded primitive of an almost-periodic function, thus, by Theorem 3, almost-periodic itself. Another well-known property of almost-periodic functions is that $\frac{1}{T}\int_x^{x+T} S_{N+1}(t)dt$ converges, uniformly in $x$, to a constant $c_N$ as $T \to \infty$.

First $\left|\frac{1}{T}\int_x^{x+T} \partial^N a_{N+1}(t)dt\right| \leq \frac{1}{T}\left|L^\infty\right|_{\partial^{N-1}a_{N+1}} \to 0$ as $T \to \infty$. As for $\frac{1}{T}\int_x^{x+T} B_{N+1}(t)dt$, observe that it tends to $B_{N+1}(-\infty) = 0$ when $x \to -\infty$, and to $B_{N+1}(+\infty)$ when $x \to +\infty$. By uniformity of the limit, this implies that $B_{N+1}(+\infty) = 0$.

The function $S_{N+1}$ thus, has $N$ oscillations in the broad sense, so by the induction hypothesis, $S_{N+1} = \partial^N f_{N+1}$ where $f_{N+1}$ is almost-periodic. □

***Corollary 1.***
*If $a_m \geq \frac{1}{|m|^\nu}$, $s = 1$, and $\frac{\nu}{d-1}$ is an even integer, then for <u>no</u> line $\Delta$, $f_\Delta$ has more than $\frac{\nu}{d-1} - 1$ oscillations in the broad sense.*

This is to compare with Proposition 6.

**Proof.** As previously, $\mu$ is a unit vector of $\Delta$, and $f_\Delta(t) = f(\mu.t)$.

Suppose that $f_\Delta$ has $2q = \frac{\nu}{d-1}$ oscillations: by Proposition 8, there is an almost-periodic function $F$ such that $f_\Delta = \partial^{2q} F$. By the Uniqueness Theorem, necessarily $F(t) = \sum \frac{a_m}{(m.\mu)^{2q}} e^{im.\mu t}$.

By the Theorem of Dirichlet, there are a constant $C < \infty$ and infinitely many $m$'s such that $|m.\mu| \leq C\,|m|^{1-d}$. As a consequence, $\sum \frac{a_m}{(m.\mu)^{2q}}$ diverges, which yields a contradiction. □

Meyer also found a better counter-example than Proposition 7 if $d = 2$ and $s = 1$.

***Proposition 9.***

*There exists a $C^\infty$, indefinitely oscillating function $f$ on $\mathbb{R}^2$ such that for no line $\Delta$, $f_\Delta$ is 2-oscillating.*

**Proof.** The function is

$$f(x) = \sum_{n=2}^{\infty} \frac{\exp\left(i\left(\cos\left(\sqrt{n}\right)x_1 + \sin\left(\sqrt{n}\right)x_2\right)\right)}{n\log(n)^2}$$

It is entire and indefinitely oscillating because its Fourier transform has support on the unit circle.

Now if $\mu = \begin{bmatrix} \cos(\phi) \\ \sin(\phi) \end{bmatrix}$ is a unit vector of $\Delta$, we have

$$f_\Delta(t) = \sum_{n=2}^{\infty} \frac{\exp\left(i\cos\left(\sqrt{n} - \phi\right)t\right)}{n\log(n)^2}$$

With $n_j = \lfloor(\phi + \frac{\pi}{2} + \pi j)^2\rfloor$, remark that $n_j \sim \pi j^2$ and that $\sqrt{n_j} - \phi = \frac{\pi}{2} + \pi j + O(j^{-1})$, and thus, $\cos(\sqrt{n} - \phi) = O(j^{-1})$. Then $\sum_{n=2}^{\infty} \frac{\cos(\sqrt{n}-\phi)^{-2}}{n\log(n)^2} = +\infty$, which proves that $f_\Delta$ can't have 2 oscillations. □

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