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Global Classical Solutions of Nonlinear Wave Equations

Philip Brenner[1] and Wolf von Wahl[2]

[1] Department of Mathematics, Chalmers University of Technology and University of Göteborg, Fack, S-402 20 Göteborg, Sweden
[2] Lehrstuhl für Angewandte Mathematik, Universität Bayreuth, Postfach 3008, D-8580 Bayreuth, Federal Republic of Germany

0. Introduction. Some Basic Definitions

This paper deals with the existence of global classical and global strong solutions to nonlinear wave equations

$$u''+\sum_{|\alpha|\leqq 2m} a_\alpha(x)\, D^\alpha u+f(u^2)\, u=0 \tag{0.1}$$

where

$$A=\sum_{|\alpha|\leqq 2m} a_\alpha(x)\, D^\alpha, \qquad D^\alpha=\prod_{j=1}^{n}\left(\frac{1}{i}\frac{\partial}{\partial x_j}\right)^{\alpha_j} \qquad \text{for } \alpha=(\alpha_1,\ldots,\alpha_n),$$

is a formally selfadjoint positive elliptic operator and f is a real C^2-function with non-negative primitive function. We treat both the homogeneous initial-boundary value problem over $\mathbb{R}_+\times\Omega$ for a smooth open set $\Omega\subset\mathbb{R}^n$ and the Cauchy problem over $\mathbb{R}_+\times\mathbb{R}^n$. For the initial-boundary value problem we show that any strong solution of (0.1), known to exist according to Heinz and von Wahl [4] if

$$f^{(j)}(u^2)=O(|u|^{\frac{2m}{n-2m}-2j}), \qquad j=0,1,\ |u|\to\infty,$$

is not only in the domain of definition of A as a selfadjoint operator in $L^2(\Omega)$, namely in $H^{2m,2}(\Omega)\cap\mathring{H}^{m,2}(\Omega)$, but also in $H^{2m+\sigma,2}(\Omega)$ for any σ with $0<\sigma<\min\left(\frac{2m}{n-4m},1\right)$ if $n<\max(4m+4\sqrt{m}, 6m+2)$, and $0<\sigma<\frac{n-4m}{4}-\left[\left(\frac{n-4m}{4}\right)^2-m\right]^{\frac{1}{2}}$ otherwise. Notice that $\sigma>\frac{2m}{n-4m}$ for $n\geqq\max(4m+4\sqrt{m}, 6m+2)$. This enables us to prove that in fact

$$u\in C^0(\mathbb{R}_+, H^{4m,2}(\Omega)) \tag{0.2}$$

if the dimension n is at most $6m+1$. This means in particular that u is a classical global solution if f and the initial data fulfil some additional differentiability and compatibility conditions. The last is of course not needed in case $\Omega=\mathbb{R}^n$. In the

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important case $m=1$, the restriction of the dimension n is $n \leqq 7$. Concerning the initial-boundary value problem the existence of classical solutions was up till now only known for dimension $n \leqq 4m$ (i.e. 4 in case $m=1$); cf. [11]. An example for our theory is the nonlinearity $f(u^2)\,u = \sin u = \frac{\sin u}{u}\,u$.

In contrast to the above, there are a number of papers containing much sharper results for the Cauchy problem. For $n \leqq 9$ and $A = -\Delta$, Pecher [5, 6] proved that (0.1) has global classical solutions even if the growth assumptions $f^{(j)}(u) = O(|u|^{\frac{2}{n-2} - 2j})$, $j=0,1$, $|u| \to \infty$, mentioned above, are (essentially) replaced by the weaker conditions

$$|f^{(j)}(u^2)| \leqq c\,|u|^{\frac{4}{n-2} - \varepsilon - 2j}, \qquad |u| \geqq 1,\ j=0,1,\ n \leqq 6, \tag{0.3}$$

$$|f^{(j)}(u^2)| \leqq c\,|u|^{\frac{4}{n-2} \cdot \frac{n}{n+1} - \varepsilon - 2j}, \qquad |u| \geqq 1,\ j=0,1,\ n=7,8,9 \tag{0.4}$$

for some $\varepsilon > 0$. The result of Pecher was carried over by Brenner [2] to more general equations: It was assumed that

$$A = \sum_{|\alpha| \leqq 2} a_\alpha(x)\, D^\alpha$$

is a second order positive elliptic operator with C^∞-coefficients $a_\alpha(x)$ such that

$$a_\alpha(x) = a_\alpha(\infty) \qquad \text{for } |x| \text{ sufficiently large}, \tag{0.5}$$

and in addition, the dimension was restricted to $n \leqq 7$. Actually, the growth conditions on f's derivatives were also relaxed somewhat. Under the more restrictive growth conditions $f^{(j)}(u^2) = O(|u|^{\frac{2}{n-2} - 2j})$ mentioned above, von Wahl [9] proved the existence of classical solutions for $n \leqq 6$ without the asymptotic assumption (0.5).

It is the aim of this paper to improve also on the above results for the Cauchy problem: If $m=1$ and $n \leqq 9$ and (0.5) holds, we will prove that the Cauchy problem for (0.1) has global classical solutions if f (essentially) fulfills the growth conditions

$$|f^{(j)}(u^2)| \leqq c\,|u|^{\frac{4}{n-2} - \varepsilon - 2j}, \qquad |u| \geqq 1,\ j=0,1, \tag{0.6}$$

for some $\varepsilon > 0$. This means in particular that $f(u^2)\,u = O(|u|^{\frac{n+2}{n-2} - \varepsilon})$ as $|u| \to \infty$. Our conditions also include the nonlinearity $f(u^2)\,u = \sin u$.

As for the existence of strong global solutions of the Cauchy problem in case $m=1$, and (0.5) holds, we prove that such solutions exist under the growth conditions

$$|f^{(j)}(u^2)| \leqq c\,|u|^{\frac{2}{n-4} - \varepsilon - 2j}, \qquad |u| \geqq 1,\ j=0,1.$$

The dimension n is not restricted in this case, and we even prove that the strong solution is not only in $C^0(\mathbb{R}_+, H^{2,2})$ but actually in $C^0(\mathbb{R}_+, H^{2+\sigma,2})$ for any $\sigma < \frac{2}{n-4}$. For sufficiently small values of σ the growth restriction may even be relaxed a little. For the proof of these results it is not sufficient to use Banach's fixed point theorem, which seems to be the common usage in the field of non-linear wave equations. Instead we propose a different existence proof based on Tychonoff's fixed point theorem (applied in the weak topology on $L^2(\mathbb{R}_+ \times \mathbb{R}^n)$). This method may also give better results for the existence of locally strong solutions to abstract second order equations $u'' + Au + M(u) = 0$ in a Hilbert space H, since it then requires that M is a mapping from $D(A)$ into H instead of requiring M to be a mapping from $D(A^{\frac{1}{2}})$ into H as in [4].

In order to carry out the above program, we need estimates of the non-linearity in fractional order Besov spaces (cf. [2, 6]). Instead of using nonlinear interpolation (as in [6]) we will derive these estimates by direct calculations. In order to do so we use the definition of the norm on the Besov space $B_p^{s,q}$ in terms of difference quotients. Let

$$\omega_p(t,u) = \sup_{|h| \leqq t} \|u(\cdot + h) - u\|_p, \qquad \|\cdot\|_p := \|\cdot\|_{L^p(\mathbb{R}^n)}, \qquad p \geqq 1,$$

and write, for $s > 0$, $s = [s] + \bar{s}$ where $[s]$ is the largest integer $< s$ and $0 < \bar{s} < 1$. Then a norm on $B_p^{s,q}$ is given by

$$\|u\|_p + \left(\int_0^\infty \Big(t^{-\bar{s}} \sum_{|\alpha| = [s]} \omega_p(t, D^\alpha u)\Big)^q \frac{dt}{t} \right)^{1/q} \tag{0.7}$$

with the usual interpretation for $q = \infty$. The use of (0.7) together with Besov space inequalities for non-linearities $f(u^2)u$ and for the solution of

$$u'' + \sum_{|\alpha| \leqq 2} a_\alpha(x) D^\alpha u + F = 0 \tag{0.8}$$

(with zero Cauchy data say), is one of the main ideas of the present paper. Let us remind the reader that the inequalities for the solution of (0.8) reads

$$\|u(t)\|_{B_{p'}^{s',q}} \leqq c(T) \int_0^t |t - \tau|^{-\gamma} \|F\|_{B_p^{s,q}} d\tau, \qquad 0 \leqq t \leqq T, \tag{0.9}$$

where

$$\frac{1}{p} + \frac{1}{p'} = 1, \qquad \delta = \frac{1}{2} - \frac{1}{p'}, \qquad \delta(n+1) \leqq 1 + s - s',$$

$$\gamma = 1 + s - s' - 2n\delta \geqq -(n-1)\delta.$$

Before we proceed to describe the disposition of the contents of this paper, let us remind the reader of some useful facts, that will be used freely in the following: First, the inclusions between Besov and Sobolev spaces $B_p^{s,q}$, $H^{s,p}$ over $\mathbb{R}^n$:

$$B_p^{s,p} \subseteq H^{s,p} \subseteq B_p^{s,2}, \quad 1 < p \leqq 2, \qquad B_{p'}^{s,p'} \supseteq H^{s,p'} \supseteq B_{p'}^{s,2}, \quad 2 \leqq p' < \infty.$$

Next, the extension operator T: $H^{k,2}(\Omega) \to H^{k,2}(\mathbb{R}^n)$ has the properties: $\|Tu\|_{H^{s,2}} \leqq c \|u\|_{H^{s,2}(\Omega)}$ for $u \in H^{k,2}(\Omega)$, $0 \leqq s \leqq k$, and so T has an extension – also denoted by T – to $\mathscr{L}(H^{s,2}(\Omega), H^{s,2}(\mathbb{R}^n))$. For this and related results, see Triebel [8]. We here used the notation $\mathscr{L}(X, Y)$ for the set of bounded operators between two Banach spaces X and Y. We will also use $C^k(I, B)$ and $L^p(I, B)$ to denote the B-valued C^k- and L^p-functions over I with values in the Banach space

B, respectively. In the same way $C^{0,1}(I, B)$ denotes the Lipschitz continuous functions with values in B. In the following, c will denote a constant, not necessarily the same at each occurence.

The plan of this paper is as follows: In Chapters I and II we deal with the (abstract) initial-boundary value problem on a Hilbert space. In Chapter III the Cauchy problem is treated on the basis of the estimates for the Green's operator of the wave equation on $\mathbb{R}_+ \times \mathbb{R}^n$ derived in [2], and given in (0.9) above.

We finally want to mention that all our results can be carried over to the time dependent case, that is to the equation

$$u'' + \sum_{|\alpha| \leqq 2m} a_\alpha(t, x) D_u^\alpha + f(u^2) u = 0.$$

I. Abstract Nonlinear Wave Equations

In this section we want to give a generalization of a theorem on abstract nonlinear wave equations proved by Heinz - von Wahl [4]. Let H be a Hilbert space, let A be a positive selfadjoint operator, i.e. $(Au, u) \geqq c \|u\|^2$, $u \in D(A)$, with a positive constant c. Let k be any real number $\geqq 1$, let a mapping

$$M\colon D(A^{\frac{k}{2}}) \to D(A^{\frac{k-1}{2}})$$

be given which is locally Lipschitz continuous, i.e.

$$\|A^{\frac{k-1}{2}}(M(u) - M(v))\| \leqq L(C) \|A^{\frac{k}{2}}(u - v)\|, \qquad u, v \in D(A^{\frac{k}{2}}),$$

where $L(C)$ is a positive constant depending on $C \geqq \|A^{\frac{k}{2}} u\| + \|A^{\frac{k}{2}} v\|$. Then we have the following local existence theorem:

Theorem I.1. *Let $\phi \in D(A^{\frac{k+1}{2}})$, $\psi \in D(A^{\frac{k}{2}})$. Then there exists a positive number $T(\phi, \psi) = T(k, \phi, \psi)$ with the following properties: There is a unique*

$$u \in \bigcap_{0 < \tilde{T} < T(\phi, \psi)} C^2([0, \tilde{T}], H)$$

with $\dfrac{d^i}{dt^i} u(t) \in D(A^{\frac{k+1-i}{2}})$, $i = 0, 1, 2$,

$$A^{\frac{k+1-i}{2}} u \in \bigcap_{0 < \tilde{T} < T(\phi, \psi)} C^0([0, T], H),$$

$$u'' + Au + M(u) = 0,$$
$$u(0) = \phi,$$
$$u'(0) = \psi,$$

$$\lim_{t \uparrow T(\phi, \psi)} \|A^{\frac{k}{2}} u(t)\| = +\infty \qquad \text{if } T(\phi, \psi) < \infty.$$

Proof. The proof can be carried through exactly in the same way as that of Theorem 1 in Heinz - von Wahl [4]. Instead of considering the integral equation (7) in [4] one has to treat the integral equation

$$A^{\frac{k}{2}} u(t) = \cos A^{\frac{1}{2}} t A^{\frac{k}{2}} \phi + \sin A^{\frac{1}{2}} t A^{\frac{k-1}{2}} \psi - \int_0^t \sin A^{\frac{1}{2}}(t-s) A^{\frac{k-1}{2}} M(u(s))\, ds.$$

There is also a proof being contained in [12], Theorem IV.2.3; in order to see this one has to transform the wave equation into a first order differential equation in the direct sum $H \oplus H$, as it was done in [9] or [10]. ■

II. Initial-Boundary Value Problems for Nonlinear Wave Equations in $L^2(\Omega)$

Let Ω be an open set of $\mathbb{R}^n$ with boundary of class C^∞. Let m be a fixed integer $\geqq 1$. For every multiindex α of $\mathbb{R}^n$ with $|\alpha| \leqq 2m$ let there a function A_α: $\mathbb{R}^n \to \mathbb{R}$ be given with the following properties: The A_α are infinitely many times continuously differentiable, every derivative is bounded on $\mathbb{R}^n$; moreover

$$M|\xi|^{2m} \geqq \sum_{|\alpha|=2m} A_\alpha(x)\, \xi^\alpha \geqq M^{-1}|\xi|^{2m}, \qquad x \in \mathbb{R}^n,$$

with a positive constant M and the operator

$$A = \sum_{|\alpha| \leqq 2m} A_\alpha(x)\, D^\alpha$$

is assumed to be formally selfadjoint. As it is well known the unbounded operator A in $L^2(\Omega)$ given by

$$Au = \sum_{|\alpha| \leqq 2m} A_\alpha(x)\, D^\alpha u, \qquad u \in D(A) = H^{2m,2}(\Omega) \cap \mathring{H}^{m,2}(\Omega)$$

is then selfadjoint, see [3, Theorem 5(iii)]. For our purposes it is no loss of generality to assume that Gårdings inequality

$$(Au, u) \geqq c\, \|u\|_{m,2}, \qquad u \in D(A),$$

holds with a positive constant c. Thus A is positive selfadjoint. As it will be proved at the end of this chapter we have the relations

$$\mathring{H}^{2\rho m,2}(\Omega) \subset D(A^\rho) \subset H^{2\rho m,2}(\Omega), \qquad 0 \leqq \rho, \tag{II.1}$$

$$H^{2\rho m,2} = D(A^\rho), \qquad 0 \leqq \rho \leqq \frac{1}{4m}, \tag{II.2}$$

$$H^{2\rho m,2}(\Omega) \cap \mathring{H}^{m,2}(\Omega) = D(A^\rho), \qquad \tfrac{1}{2} \leqq \rho \leqq 1, \tag{II.3}$$

$$c\, \|u\|_{2\rho m,2} \leqq \|A^\rho u\| \leqq c\, \|u\|_{2\rho m,2}, \qquad u \in D(A^\rho),\ 0 \leqq \rho. \tag{II.4}$$

Here and in the sequel c is a positive constant depending on n, m, Ω, ρ, M and the $\|A_\alpha\|_{C^{2m([\rho]+1)}(\mathbb{R}^n)}$, but not on u. Observe that the order of derivatives of the A_α appearing in the c can be lowered. In what follows let $n \geqq 4m+1$.

For the rest of the paper the interpolation-technique used in the proof of the following lemma is decisive.

Lemma II.1. *Let F be a continuously differentiable mapping from $\mathbb{R}_+$ into $\mathbb{R}$ with* $|F'(u^2)u| \leqq c$,

$$|F(u^2)-F(v^2)| \leqq c|u-v|^\rho, \qquad u, v \in \mathbb{R},$$

where ρ is a fixed number with $0 \leqq \rho < \frac{2m}{n-4m} \wedge 1$. *Let* $\varepsilon \geqq 0$,

$$\frac{2m}{n-2m} \leqq \rho+\varepsilon < \frac{2m}{n-4m} \wedge 1.$$

Then for real $u \in H^{2m,2}(\Omega)$, $v \in H^{m+\rho+\varepsilon,2}(\Omega)$ the expression $F(u^2)v$ is in $H^{\rho+\varepsilon,2}(\Omega)$ and satisfies the estimate

$$\|F(u^2)v\|_{\rho+\varepsilon,2} \leqq \frac{1}{\sqrt{\eta}} c(\|u\|_{2m}^{\rho+\varepsilon+\eta}+1)\|v\|_{m+\rho+\varepsilon,2}, \qquad 1 \geqq \eta > 0.$$

Proof. Let u, v be extended to the whole of $\mathbb{R}^n$ as functions of $H^{2m+\rho+\varepsilon,2}(\mathbb{R}^n)$, $H^{m+\rho+\varepsilon,2}(\mathbb{R}^n)$ by means of the $H^{2m+\rho+\varepsilon,2}(\Omega)$-, $H^{m+\rho+\varepsilon,2}(\Omega)$-extension operator respectively (cf. 0.). These extensions are real and will also be denoted by u, v. Moreover we denote the resulting extension (as a pointwise defined function) of $F(u^2)v$ to $\mathbb{R}^n$ also by $F(u^2)v$. If $\|F(u^2)v\|_{H^{\rho+\varepsilon,2}(\mathbb{R}^n)}$ can be estimated in the desired way our lemma is proved. We have to estimate the expression

$$\begin{aligned}
&\int_0^1 \Big[\frac{1}{t^{\rho+\varepsilon}} \sup_{|h|\leqq t} \|(F(u^2(\cdot+h))-F(u^2(\cdot)))\,v(\cdot+h)\|_{L^2(\mathbb{R}^n)} \\
&\qquad + \frac{1}{t^{\rho+\varepsilon}} \sup_{|h|\leqq t} \|F(u^2(\cdot))\,(v(\cdot+h)-v(\cdot))\|_{L^2(\mathbb{R}^n)}\Big]^2 \frac{dt}{t} \\
&\leqq 2\int_0^1 \Big[\frac{1}{t^{\rho+\varepsilon}} \sup_{|h|\leqq t} \|(F(u^2(\cdot+h))-F(u^2(\cdot)))\,v(\cdot+h)\|_{L^2(\mathbb{R}^n)}\Big]^2 \frac{dt}{t} \\
&\qquad + 2\int_0^1 \Big[\frac{1}{t^{\rho+\varepsilon}} \sup_{|h|\leqq t} \|F(u^2(\cdot))\,(v(\cdot+h)-v(\cdot))\|_{L^2(\mathbb{R}^n)}\Big]^2 \frac{dt}{t}.
\end{aligned}$$

The first integral on the right hand side of the last inequality is denoted by I^2, the second one by II^2. With respect to I^2 we have if $\eta > 0$, $\rho+\varepsilon+\eta < \frac{2m}{n-4m} \wedge 1$:

$$\begin{aligned}
&|F(u^2(x+h))-F(u^2(x))|\,|v(x+h)| \\
&\leqq c \cdot |u(x+h)-u(x)|^{\rho+\varepsilon+\eta}\,|v(x+h)|, \\
&\leqq c \cdot \left|\int_0^1 \frac{\partial}{\partial\tau} u((1-\tau)x+\tau(x+h))\,d\tau\right|^{\rho+\varepsilon+\eta} |v(x+h)|, \\
&\leqq c \cdot \left|\int_0^1 h \cdot \nabla u((1-\tau)x+\tau(x+h))\,d\tau\right|^{\rho+\varepsilon+\eta} |v(x+h)|, \\
&\leqq t^{\rho+\varepsilon+\eta} \left(\int_0^1 |\nabla u(x+\tau h)|^{2p(\rho+\varepsilon+\eta)}\,d\tau\right)^{\frac{1}{2p}} |v(x+h)|,
\end{aligned}$$

and therefore

$$\mathrm{I}^2 \leqq c \int_0^1 t^{\eta-1} \left\| \left(\int_0^1 |\nabla u(x+\tau h)|^{2p(\rho+\varepsilon+\eta)} d\tau \right)^{\frac{1}{2p}} |v(x+h)| \right\|^2_{L^2(\mathbb{R}^n)} dt$$

$$\leqq c \int_0^1 t^{\eta-1} \left(\int_{\mathbb{R}^n} |\nabla u(x)|^{2p(\rho+\varepsilon+\eta)} dx \right)^{\frac{1}{p}} \|v\|^2_{L^{2q}(\mathbb{R}^n)} dt.$$

Here

$$\frac{1}{2q} = \frac{1}{2} - \frac{m+\rho+\varepsilon}{n},$$

$$\frac{1}{2p} = \frac{m+\rho+\varepsilon}{n}.$$

Moreover we assumed without loss of generality that u is continuously differentiable.

From the latter equality it follows that

$$\frac{1}{2p(\rho+\varepsilon+\eta)} = \frac{m+\rho+\varepsilon}{n(\rho+\varepsilon+\eta)}.$$

Let us consider the inequality

$$\frac{m+\rho+\varepsilon}{n(\rho+\varepsilon)} > \frac{1}{2} - \frac{2m-1}{n}$$

being equivalent with

$$m+\rho+\varepsilon > (\rho+\varepsilon)\left(\frac{n}{2} - 2m + 1\right),$$

$$\frac{m}{\frac{n}{2} - 2m} > \rho+\varepsilon,$$

$$\frac{2m}{n-4m} > \rho+\varepsilon.$$

Moreover $2p(\rho+\varepsilon) \geqq 2$ since $\rho+\varepsilon \geqq \dfrac{2m}{n-2m}$. Thus for sufficiently small $\eta>0$ we have

$$\frac{m+\rho+\varepsilon}{n(\rho+\varepsilon+\eta)} > \frac{1}{2} - \frac{2m-1}{n}$$

and by Sobolev we get

$$\mathrm{I}^2 \leqq \frac{1}{\eta} c \|u\|^{2(\rho+\varepsilon+\eta)}_{H^{2m,2}(\mathbb{R}^n)} \|v\|^2_{H^{m+\rho+\varepsilon,2}(\mathbb{R}^n)}.$$

Now we want to estimate II^2. We have

$$\begin{aligned}\mathrm{II}^2 &\leqq 2\int_0^1 \left(\int_{\mathbb{R}^n} |u(x)|^{2p(\rho+\varepsilon)}\,dx\right)^{\frac{1}{p}} \left[\sup_{|h|\leqq t} \frac{\|v(\cdot+h)-v(\cdot)\|_{L^{2q}(\mathbb{R}^n)}}{t^{\rho+\varepsilon}}\right]^2 \cdot \frac{dt}{t}\\ &\quad + 4|f(0)|^2 \|v\|^2_{H^{\rho+\varepsilon,2}(\mathbb{R}^n)},\\ &\leqq c\int_0^1 \left(\int_{\mathbb{R}^n} |u(x)|^{2p(\rho+\varepsilon)}\,dx\right)^{\frac{1}{p}} \left[\sup_{|h|\leqq t} \frac{\|v(\cdot+h)-v(\cdot)\|_{H^{m+\varepsilon+\rho,2}(\mathbb{R}^n)}}{t^{\rho+\varepsilon}}\right]^2 \cdot \frac{dt}{t}\\ &\quad + 4|f(0)|^2 \|v\|^2_{H^{m+\rho+\varepsilon,2}(\mathbb{R}^n)},\\ &\leqq c(\|u\|^{2(\rho+\varepsilon)}_{H^{2m,2}(\mathbb{R}^n)}+1)\,\|v\|^2_{H^{m+\rho+\varepsilon,2}(\mathbb{R}^n)}\end{aligned}$$

where $\frac{1}{2q}=\frac{1}{2}-\frac{m}{n}$, $\frac{1}{2p}=\frac{m}{n}$. The application of Sobolev's imbedding theorem is justified since

$$\frac{1}{2p(\rho+\varepsilon)}=\frac{m}{n(\rho+\varepsilon)}>\frac{1}{2}-\frac{2m}{n},$$
$$2p(\rho+\varepsilon)\geqq 2$$

as was just proved. Estimating $\|F(u^2)v\|_{L^2(\mathbb{R}^n)}$ by the same procedure gives the inequality

$$\|F(u^2)v\|_{L^2(\mathbb{R}^n)}\leqq c(\|u\|^{\rho+\varepsilon}_{H^{2m,2}(\mathbb{R}^n)}+1)\,\|v\|_{H^{m+\rho+\varepsilon,2}(\mathbb{R}^n)}. \quad \blacksquare$$

Now we want to prove a second nonlinear interpolation lemma, based upon a further restriction of the dimension. For this purpose we assume for a moment that $n\geqq(4m+4\sqrt{m})\wedge(6m+2)$. Then $\left(\frac{n-4m}{4}\right)^2\geqq m$. Thus the definition

$$\rho^*:=\frac{n-4m}{4}-\sqrt{\left(\frac{n-4m}{4}\right)^2-m}$$

makes sense. Set $\rho_1=0$,

$$\rho_2=\frac{2m}{n-4m},$$

$$\rho_3=\frac{2m}{n-4m-2\rho_2+\frac{1}{3}\rho_2},$$

and for arbitrary $\nu\in\mathbb{N}$, $\nu\geqq 3$,

$$\rho_\nu=\frac{2m}{n-4m-2\rho_{\nu-1}+\frac{1}{\nu}\rho_2}.$$

It is not difficult to show that all $\rho_\nu<1$ and that $\{\rho_\nu\}$ is monotonically increasing with the limit ρ^*. Moreover we have $\rho^*\leqq 1$,

$$\frac{m+\rho_\nu}{n\rho_\nu}=\frac{1}{2}-\frac{2m-1+\rho_{\nu-1}-\frac{1}{\nu}\rho_2}{n},$$

$$>\frac{1}{2}-\frac{2m-1+\rho_{\nu-1}-\frac{1}{2\nu}\rho_2}{n}.$$

Lemma II.2. *Let* $n\geqq(4m+4\sqrt{m})\wedge(6m+2)$. *Let* F *be a continuously differentiable mapping from* $\mathbb{R}_+$ *into* $\mathbb{R}$ *with* $|uF'(u^2)|\leqq c$,

$$|F(u^2)-F(v^2)|\leqq c\,|u-v|^\rho,\qquad u,v\in\mathbb{R},$$

where ρ *is a fixed number with* $0\leqq\rho<\rho^*$. *Then for real* $u\in H^{2m+\rho_{\nu-1},2}(\Omega)$, $v\in H^{2m+\rho_\nu,2}(\Omega)$ *the expression* $F(u^2)\,v$ *is in* $H^{\rho_\nu,2}(\Omega)$ *and satisfies the estimate*

$$\|F(u^2)\,v\|_{\rho_\nu,2}\leqq\frac{1}{\sqrt{\eta}}\,c(\|u\|_{2m+\rho_{\nu-1},2}^{\rho_\nu+\eta}+1)\,\|v\|_{m+\rho_\nu,2},\qquad 1\geqq\eta>0,$$

provided $\rho_\nu\geqq\rho$, *i.e.* $\nu\geqq\nu(\rho)$.

Proof. The proof is the same as that of Lemma II.1. We set

$$\frac{1}{2q}=\frac{1}{2}-\frac{m+\rho_\nu}{n},$$

$$\frac{1}{2p}=\frac{m+\rho_\nu}{n}$$

so that

$$\frac{1}{2p\rho_\nu}=\frac{m+\rho_\nu}{n\rho_\nu}>\frac{1}{2}-\frac{2m-1+\rho_{\nu-1}-\frac{\rho_2}{2\nu}}{n}.\qquad\blacksquare$$

Now we describe the nonlinearity in our equations. We assume that $M(u)$ is formally given by

$$M(u)=f((\mathrm{Re}\,u)^2)\,\mathrm{Re}\,u$$

where $f\colon\mathbb{R}_+\to\mathbb{R}$ is a C^2-function satisfying the following growth conditions: Let $u,v\in\mathbb{R}$, then

$$|f(u^2)-f(v^2)|\leqq c\,|u-v|^\rho,$$
$$|f'(u^2)\,u^2-f'(v^2)\,v^2|\leqq c\,|u-v|^\rho,\qquad |f''(u^2)|\,|u|^3\leqq c,\ |u|\geqq 1,$$

for a ρ, $0\leqq\rho<\frac{2m}{n-4m}\wedge 1$. The connection between the conditions above on f and conditions of the form $|f^{(j)}(u^2)|\leqq c\,|u|^{\rho_j}$, $|u|\geqq 1$, $j=1,2,3$ is discussed in detail in the beginning of Chapter III.

We want to prove a first a-priori estimate, namely

Proposition II.3. *Let* $\varepsilon>0$, $\frac{2m}{n-2m}\leqq\rho+\varepsilon<\frac{2m}{n-4m}\wedge 1$. *Let* $\varphi\in D(A^{1+(\rho+\varepsilon)/2m})$, $\psi\in D(A^{\frac{1}{2}+(\rho+\varepsilon)/2m})$ *be real. Let* $T>0$, *let* u *be real,*

$$u\in C^2([0,T],L^2(\Omega)),$$
$$u'\in C^0([0,T],D(A^{\frac{1}{2}+(\rho+\varepsilon)/2m})),$$
$$u\in C^0([0,T],D(A^{1+(\rho+\varepsilon)/2m})),$$
$$u''+Au+M(u)=0 \quad \text{in } [0,T],$$
$$u(0)=\varphi,$$
$$u'(0)=\psi.$$

Then the following a-priori estimate holds:

$$\sup_{0\leqq t\leqq T}\|A^{1+(\rho+\varepsilon)/2m}u(t)\|+\sup_{0\leqq t\leqq T}\|A^{\frac{1}{2}+(\rho+\varepsilon)/2m}u'(t)\|$$
$$\leqq c(T,\sup_{0\leqq t\leqq T}(\|Au(t)\|+\|A^{\frac{1}{2}}u'(t)\|),\|A^{1+(\rho+\varepsilon)/2m}\varphi\|+\|A^{\frac{1}{2}+(\rho+\varepsilon)/2m}\psi\|).$$

Proof. For our solution u under consideration we have: $M(u)\in C^0([0,T],L^2(\Omega))$. Therefore u fulfils the integral equation

$$u(t)=\sin A^{\frac{1}{2}}t\,A^{-\frac{1}{2}}\psi+\cos A^{\frac{1}{2}}t\,\varphi-\int_0^t\sin A^{\frac{1}{2}}(t-s)\,A^{-\frac{1}{2}}M(u(s))\,ds.$$

Formal differentiation with respect to t yields:

$$(M(u))'=(2f'(u^2)\,u^2+f(u^2))\,u'. \tag{II.5}$$

On using Sobolev's inequality we get

$$\|(M(u))'\|\leqq c(\||u|^{\rho+\varepsilon}\|^{\rho+\varepsilon}_{L^{2p}(\Omega)}+1)\,\|u'\|_{L^{2q}(\Omega)} \tag{II.6}$$

with $\frac{1}{2q}=\frac{1}{2}-\frac{m+\rho+\varepsilon}{n}$, $\frac{1}{2p}=\frac{m+\rho+\varepsilon}{n}$. Because of our restriction to ρ we can choose $\varepsilon>0$ in such a way that $\frac{1}{2p(\rho+\varepsilon)}\geqq\frac{1}{2}-\frac{2m}{n}$, $2p(\rho+\varepsilon)\geqq 2$, $\rho+\varepsilon<\frac{2m}{n-4m}\wedge 1$.

In regard of our conditions to f this means that $M(u)\in C^1([0,T],L^2(\Omega))$, and the derivative is given by the expression on the right hand side of (II.5). Thus we get by partial integration

$$u'(t)=\cos A^{\frac{1}{2}}t\,\psi-\sin A^{\frac{1}{2}}t\,A^{\frac{1}{2}}\varphi-\int_0^t\cos A^{\frac{1}{2}}(t-s)\,M(u(s))\,ds,$$
$$=\cos A^{\frac{1}{2}}t\,\psi-\sin A^{\frac{1}{2}}t\,A^{\frac{1}{2}}\varphi-\sin A^{\frac{1}{2}}t\,A^{-\frac{1}{2}}M(\varphi)$$
$$-\int_0^t\sin A^{\frac{1}{2}}(t-s)\,A^{-\frac{1}{2}}(M(u(s)))'\,ds,$$
$$=\cos A^{\frac{1}{2}}t\,\psi-\sin A^{\frac{1}{2}}t\,A^{\frac{1}{2}}\varphi-\sin A^{\frac{1}{2}}t\,A^{-\frac{1}{2}}M(\varphi)$$
$$-\int_0^t\sin A^{\frac{1}{2}}(t-s)\,A^{-\frac{1}{2}}(2f'(u^2(s))\,u^2(s)+f(u^2(s)))\,u'(s)\,ds.$$

Our Lemma II.1 then shows that for some $\eta>0$

$$\|M(u(s))'\|_{\rho+\varepsilon,2}\leqq c(\|u(s)\|_{2m,2}^{\rho+\varepsilon+\eta}+1)\|u'(s)\|_{m+\rho+\varepsilon,2}.$$

Let $\{v_n\}$ be a sequence of $C^m(\bar{\Omega})\cap \mathring{H}^{m,2}(\Omega)\cap H^{m+\rho+\varepsilon,2}(\Omega)$ converging to $u'(s)$ in $H^{m+\rho+\varepsilon,2}(\Omega)$. Because of our growth conditions on f', f'' we get that

$$f'(u^2(s))\,u^2(s)\,v_n\in \mathring{H}^{1,2}(\Omega),$$
$$f(u^2(s))\,v_n\in \mathring{H}^{1,2}(\Omega).$$

Therefore

$$(2f'(u^2(s))\,u^2(s)+f(u^2(s)))\,v_n\in D(A^{(\rho+\varepsilon)/2m}).$$

Because of our Lemma II.1

$$\begin{aligned}&\|A^{(\rho+\varepsilon)/2m}(2f'(u^2(s))\,u^2(s)+f(u^2(s)))\,v_n\|\\ &\leqq c\,\|(2f'(u^2(s))\,u^2(s)+f(u^2(s)))\,v_n\|_{\rho+\varepsilon,2}\\ &\leqq c(\|u(s)\|_{2m,2}^{\rho+\varepsilon+\eta}+1)\;\|v_n\|_{m+\rho+\varepsilon,2}.\end{aligned}$$

The $(2f'(u^2(s))\,u^2(s)+f(u^2(s)))\,v_n$ thus being weakly convergent to $(2f'(u^2(s))\,u^2+f(u^2(s)))\,u(s)$ in $L^2(\Omega)$ we see that

$$(M(u(s)))'\in D(A^{(\rho+\varepsilon)/2m}),$$
$$\|A^{(\rho+\varepsilon)/2m}(M(u(s)))'\|\leqq c(\|u(s)\|_{2m,2}^{\rho+\varepsilon+\eta}+1)\;\|u'(s)\|_{m+\rho+\varepsilon,2}.$$

On applying the same method we also get the relation: $M(u(s))\in D(A^{(\rho+\varepsilon)/2m})$, $0\leqq s\leqq T$, and

$$\|A^{(\rho+\varepsilon)/2m}\,M(u(s))\|\leqq c(\|u(s)\|_{2m,2}^{\rho+\varepsilon+\eta}+1)\;\|u(s)\|_{m+\rho+\varepsilon,2}.$$

Thus we arrive at the estimate for u'. By partial integration we get

$$\begin{aligned}u(t)=&\sin A^{\frac{1}{2}}t\,A^{-\frac{1}{2}}\psi+\cos A^{\frac{1}{2}}t\,\varphi+\int_0^t\cos A^{\frac{1}{2}}(t-s)\,A^{-1}(M(u(s)))'\,ds\\ &-A^{-1}M(u(t))+\cos A^{\frac{1}{2}}t\,A^{-1}M(\varphi)\end{aligned}$$

from which the estimate for u follows. ∎

The assumptions in Proposition II.3 can slightly be relaxed:

Theorem II.4. *Let* $\varepsilon>0$, $\dfrac{2m}{n-2m}\leqq\rho+\varepsilon<\dfrac{2m}{n-4m}\wedge 1$. *Let* $\varphi\in D(A^{1+(\rho+\varepsilon)/2m})$, $\psi\in D(A^{\frac{1}{2}+(\rho+\varepsilon)/2m})$ *be real. Let* $T>0$, *let* u *be real,*

$$u\in C^2([0,T],L^2(\Omega)),\qquad \varphi\in D(A^{\frac{k}{2}+\frac{1}{2}}),\qquad \psi\in D(A^{\frac{k}{2}})$$

and

$$u'\in C^0([0,T],D(A^{\frac{k}{2}})),$$
$$u\in C^0([0,T],D(A^{\frac{k}{2}+\frac{1}{2}}))\quad \textit{for a } k\in\mathbb{R},\ k\geqq 1,$$

$$u'' + Au + M(u) = 0 \quad \textit{in } [0, T],$$
$$u(0) = \varphi,$$
$$u'(0) = \psi.$$

Moreover let M be a Lipschitz continuous mapping from $D(A^{\frac{k}{2}})$ into $D(A^{\frac{k}{2}-\frac{1}{2}})$, i.e.

$$\|A^{\frac{k-1}{2}}(M(u) - M(v))\| \leqq c(C)\, \|A^{\frac{k}{2}}(u-v)\|$$

with a positive constant $c(C)$, $C \geqq \|A^{\frac{k}{2}} u\| + \|A^{\frac{k}{2}} v\|$. Then

$$u \in C^0([0, T], D(A^{1+(\rho+\varepsilon)/2m})),$$
$$u' \in C^0([0, T], D(A^{\frac{1}{2}+(\rho+\varepsilon)/2m})).$$

Proof. According to Theorem I.1 there exists a $T(\varphi, \psi)$, $\infty \geqq T(\varphi, \psi) > 0$, with the following properties: There is one and only one

$$u \in \bigcap_{0 < \tilde{T} < T(\varphi,\psi)} C^2([0, \tilde{T}], D(A^{\frac{k}{2}-\frac{1}{2}}))$$

with

$$u' \in \bigcap_{0 < \tilde{T} < T(\varphi,\psi)} C^1([0, \tilde{T}], D(A^{\frac{k}{2}})),$$
$$u \in \bigcap_{0 < \tilde{T} < T(\varphi,\psi)} C^0([0, \tilde{T}], D(A^{\frac{k}{2}+\frac{1}{2}})),$$
$$u'' + Au + M(u) = 0, \qquad 0 \leqq t < T(\varphi, \psi),$$
$$u(0) = \varphi,$$
$$u'(0) = \psi.$$

Moreover

$$\lim_{t \uparrow T(\varphi,\psi)} \|A^{\frac{k}{2}} u(t)\| = \infty$$

if $T(\varphi, \psi) < \infty$. We consider now the equation (the E_λ are the spectral resolution of A, $\lambda \in \mathbb{R}$, $\bar{m}$ is an element of $\mathbb{N}$):

$$u''_{\bar{m}} + Au_{\bar{m}} + E_{\bar{m}} M(E_{\bar{m}} u_{\bar{m}}) = 0, \tag{II.7}$$
$$u(0) = E_{\bar{m}} \varphi, \tag{II.8}$$
$$u_t(0) = E_{\bar{m}} \psi. \tag{II.9}$$

Let $T^{\bar{m}}(\varphi, \psi)$ the just mentioned quantity belonging to (II.7), (II.8) and (II.9). We have

$$T(\varphi, \psi),\, T^{\bar{m}}(\varphi, \psi) > c(\|A^{\frac{k}{2}+\frac{1}{2}} \varphi\| + \|A^{\frac{k}{2}} \psi\|) > 0,$$

where the constant c does not depend on $\bar{m}$. Moreover we have on $[0, c(\|A^{\frac{k}{2}+\frac{1}{2}} \varphi\| + \|A^{\frac{k}{2}} \psi\|)]$:

$$\left.\begin{matrix} \|A^{\frac{k}{2}+\frac{1}{2}} u(t)\| + \|A^{\frac{k}{2}} u'(t)\| \\ \|A^{\frac{k}{2}+\frac{1}{2}} u_{\bar{m}}(t)\| + \|A^{\frac{k}{2}} u'_{\bar{m}}(t)\| \end{matrix}\right\} \leqq c(\|A^{\frac{k}{2}+\frac{1}{2}} \varphi\| + \|A^{\frac{k}{2}} \psi\|),$$

where also $c(\|A^{\frac{k}{2}+\frac{1}{2}}\varphi\|+\|A^{\frac{k}{2}}\psi\|)$ does not depend on $\bar{m}$. As it immediately follows from the integral equation for u we have

$$u_{\bar{m}}\in C^0([0, c(\|A^{\frac{k}{2}+\frac{1}{2}}\varphi\|+\|A^{\frac{k}{2}}\psi\|)], D(A^{1+(\rho+\varepsilon)/2m})),$$
$$u'_{\bar{m}}\in C^0([0, c(\|A^{\frac{k}{2}+\frac{1}{2}}\varphi\|+\|A^{\frac{k}{2}}\psi\|)], D(A^{\frac{1}{2}+(\rho+\varepsilon)/2m})).$$

Thus the proof of Proposition II.3 gives an a-priori estimate

$$\begin{aligned}&\|A^{1+(\rho+\varepsilon)/2m}u_{\bar{m}}(t)\|+\|A^{\frac{1}{2}+(\rho+\varepsilon)/2m}u'_{\bar{m}}(t)\|\leqq c,\\ &0\leqq t\leqq c(\|A^{\frac{k}{2}+\frac{1}{2}}\varphi\|+\|A^{\frac{k}{2}}\psi\|),\end{aligned} \tag{II.10}$$

where c does not depend on $\bar{m}$ since

$$\begin{aligned}\|E_{\bar{m}}(M(E_{\bar{m}}u_{\bar{m}}))'\|_{\rho+\varepsilon,2}&\leqq c\|A^{(\rho+\varepsilon)/2m}E_{\bar{m}}(M(E_{\bar{m}}u_{\bar{m}}))'\|,\\ &\leqq c\|A^{(\rho+\varepsilon)/2m}(M(E_{\bar{m}}u_{\bar{m}}))'\|\end{aligned}$$

and also

$$\|E_{\bar{m}}M(E_{\bar{m}}u_{\bar{m}})\|_{\rho+\varepsilon,2}\leqq c\|A^{(\rho+\varepsilon)/2m}M(E_{\bar{m}}u_{\bar{m}})\|;$$

observe that

$$u'_{\bar{m}}, u_{\bar{m}}\in\bigcap_{\substack{l\in\mathbb{N}\\ l\geqq 1}} C^0([0, T^{\bar{m}}(\varphi,\psi)), D(A^l))$$

and therefore $u'_{\bar{m}}, u_{\bar{m}}\in C^0([0, T^m(\varphi,\psi)), \mathring{H}^{m,2}(\Omega)\cap C^m(\Omega))$. Now we want $\bar{m}$ to tend to ∞. Following the lines of the proof of Theorem 1 in [4] (as it has to be done for the proof of Theorem I.1) we see that

$$\sup_{\bar{m}'\geqq\bar{m}}(\|A^{\frac{k}{2}}(u_{\bar{m}'}(t)-u_{\bar{m}}(t))\|+\|A^{\frac{k}{2}+\frac{1}{2}}(u'_{\bar{m}'}(t)-u'_{\bar{m}}(t))\|)\to 0. \qquad \bar{m}\to\infty.$$
$$0\leqq t\leqq c(\|A^{\frac{k}{2}+\frac{1}{2}}\varphi\|+\|A^{\frac{k}{2}}\psi\|)$$

and that the limit in the norm above coincides with u on $[0, c(\|A^{\frac{k}{2}+\frac{1}{2}}\varphi\| + \|A^{\frac{k}{2}}\psi\|)]$. The a-priori estimate (II.10) shows that

$$\begin{aligned}&\|A^{1+(\rho+\varepsilon)/2m}u(t)\|+\|A^{\frac{1}{2}+(\rho+\varepsilon)/2m}u'(t)\|\leqq c,\\ &0\leqq t\leqq c(\|A\varphi\|+\|A^{\frac{1}{2}+(\rho+\varepsilon)/2m}\psi\|).\end{aligned}$$

The integral equalities for u, u' show that even

$$u\in C^0([0, c(\|A^{\frac{k}{2}+\frac{1}{2}}\varphi\|+\|A^{\frac{k}{2}}\psi\|)], D(A^{1+(\rho+\varepsilon)/2m})),$$
$$u'\in C^0([0, c(\|A^{\frac{k}{2}+\frac{1}{2}}\varphi\|+\|A^{\frac{k}{2}}\psi\|)], D(A^{\frac{1}{2}+(\rho+\varepsilon)/2m})).$$

From this Theorem II.4 follows, since we can continue the whole procedure on a second interval, then on a third one and so on, the length of these intervals being bounded from below by a constant depending only on

$$\sup_{0\leqq t\leqq T}(\|A^{\frac{k}{2}+\frac{1}{2}}u(t)\|+\|A^{\frac{k}{2}}u'(t)\|).$$

If the dimension n is restricted we get a sharper version of Theorem II.4, namely

Theorem II.5. *Let* $n \geqq (4m+4\sqrt{m}) \wedge (6m+2)$. *Let*

$$\varphi \in \bigcap_{0<\varepsilon \leqq \rho^*} D(A^{1+(\rho^*-\varepsilon)/2m}), \qquad \psi \in \bigcap_{0<\varepsilon \leqq \rho^*} D(A^{\frac{1}{2}+(\rho^*-\varepsilon)/2m})$$

be real. Let $T>0$, *let* u *be real,*

$$u \in C^2([0,T], L^2(\Omega)), \qquad \varphi \in D(A^{\frac{k}{2}+\frac{1}{2}}), \qquad \psi \in D(A^{\frac{k}{2}})$$

and

$$u' \in C^0([0,T], D(A^{\frac{k}{2}})),$$
$$u \in C^0([0,T], D(A^{\frac{k}{2}+\frac{1}{2}})) \quad \textit{for a } k \in \mathbb{R},\ k \geqq 1,$$
$$u'' + Au + M(u) = 0 \quad \textit{in } [0,T],$$
$$u(0) = \varphi,$$
$$u'(0) = \psi.$$

Let M *be a Lipschitz continuous mapping from* $D(A^{\frac{k}{2}})$ *into* $D(A^{\frac{k}{2}-\frac{1}{2}})$, *i.e.*

$$\|A^{\frac{k-1}{2}}(M(u)-M(v))\| \leqq c(C)\, \|A^{\frac{k}{2}}(u-v)\|$$

with a positive constant $c(C)$, $C \geqq \|A^{\frac{k}{2}} u\| + \|A^{\frac{k}{2}} v\|$. *Then*

$$u \in \bigcap_{0<\varepsilon \leqq \rho^*} C^0([0,T], D(A^{1+(\rho^*-\varepsilon)/2m})),$$
$$u' \in \bigcap_{0<\varepsilon \leqq \rho^*} C^0([0,T], D(A^{\frac{1}{2}+(\rho^*-\varepsilon)/2m})).$$

Proof. The argument is the same as in the proof of Theorem II.4. We carry it through first for ρ_1 instead of $\rho+\varepsilon$. Thus we see that

$$u \in C^0([0,T], D(A^{1+\rho_1/2m})),$$
$$u' \in C^0([0,T], D(A^{\frac{1}{2}+\rho_1/2m})).$$

Then we choose ρ_2 instead of ρ_1 and so on. ∎

It often happens that a nonlinear term M is not only a Lipschitz continuous mapping from $D(A^{\frac{k}{2}})$ into $D(A^{\frac{k}{2}-\frac{1}{2}})$ for one k but for a pair k_1, k_2. The following theorem deals with such a situation.

Theorem II.6. *Let* $k_1, k_2 \geqq 1$, $k_2 \geqq k_1$. *Let*

$$\varphi \in D(A^{\frac{k_2}{2}+\frac{1}{2}}) \cap D(A^{1+\tilde{\rho}/2m}) \cap D_1,$$
$$\psi \in D(A^{\frac{k_2}{2}-\frac{1}{2}}) \cap D(A^{\frac{1}{2}+\tilde{\rho}/2m}) \cap D_2,$$

where $\tilde{\rho}$ *is a positive number with*

$$\frac{2m}{n-2m} \leqq \tilde{\rho} < \frac{2m}{n-4m} \wedge 1.$$

Moreover

$$D_1 = \bigcap_{0<\varepsilon\leqq\rho^*} D(A^{1+(\rho^*-\varepsilon)/2m}), \qquad n\geqq(4m+4\sqrt{m})\wedge(6m+2),$$

$$D_1 = H, \quad (4m+4\sqrt{m})\vee(6m+2)>n\geqq 4m+1,$$

$$D_2 = \bigcap_{0<\varepsilon\leqq\rho^*} D(A^{\frac{1}{2}+(\rho^*-\varepsilon)/2m}), \qquad n\geqq(4m+4\sqrt{m})\wedge(6m+2),$$

$$D_2 = H, \quad (4m+4\sqrt{m})\vee(6m+2)>n\geqq 4m+1.$$

Let M be a Lipschitz continuous mapping from $D(A^{\frac{k_i}{2}})$ into $D(A^{\frac{k_i}{2}-\frac{1}{2}})$, $i=1,2$. For $u\in D(A^{\frac{k_2}{2}})$ let M satisfy an estimate

$$\|A^{\frac{k_2-1}{2}} M(u)\| \leqq g(\|u\|_{H^{2m+\tilde{\tilde{\rho}},2}(\Omega)})\, \|A^{\frac{k_2}{2}} u\|$$

where g is a continuous mapping from $\mathbb{R}_+$ into itself, and where $\tilde{\tilde{\rho}}<\frac{2m}{n-4m}\wedge 1$, $(4m+4\sqrt{m})\vee(6m+2)>n\geqq 4m+1$, $\tilde{\tilde{\rho}}<\rho^*$, $n\geqq(4m+4\sqrt{m})\wedge(6m+2)$. Then any solution u of $u''+Au+M(u)=0$ in $[0,T]$ with $u(0)=\varphi$, $u'(0)=\psi$,

$$u\in C^2([0,T], L^2(\Omega)),$$
$$u'\in C^1([0,T], D(A^{\frac{k_1}{2}})),$$
$$u\in C^0([0,T], D(A^{\frac{k_1}{2}+\frac{1}{2}}))$$

is in fact an element of $C^0([0,T], D(A^{\frac{k_2}{2}+\frac{1}{2}}))$ with $u'\in C^1([0,T], D(A^{\frac{k_2}{2}}))$.

Proof. Let $T(k_2)$ be the positive number $T(\varphi,\psi)$ which one gets by application of Theorem I.1 in the case $k=k_2$. The only thing we have to do is to derive an a-priori estimate for $\|A^{\frac{k_2}{2}} u(t)\|$. On using Theorem II.4 and II.5 we get that

$$\begin{aligned} &u\in C^0([0,T], D(A^{1+\tilde{\tilde{\rho}}/2m})),\\ &u'\in C^0([0,T], D(A^{\frac{1}{2}+\tilde{\tilde{\rho}}/2m})). \end{aligned} \tag{II.11}$$

The integral equation for u gives the estimate

$$\begin{aligned} &\|A^{\frac{k_2}{2}-\frac{1}{2}} u'(t)\| + \|A^{\frac{k_2}{2}} u(t)\|\\ &\quad\leqq \|A^{\frac{k_2}{2}}\varphi\| + \|A^{\frac{k_2}{2}-\frac{1}{2}}\psi\| + \int_0^t g(\|u(s)\|_{H^{2m+\tilde{\tilde{\rho}},2}(\Omega)})\, \|A^{\frac{k_2}{2}} u(s)\|\, ds. \end{aligned}$$

In view of (II.11) and (II.4) the desired a-priori estimate follows. ∎

Before proving the announced theorem on the domains of definition of the fractional powers of A we discuss some examples for our theorems.

Examples. 1. *For Theorems II.4 and II.5.* The assumptions of theorems II.4 and II.5 concerning the existence of a solution u of $u''+Au+M(u)=0$ with $u(0)=\varphi$,

$u'(0)=\psi$ are fulfilled for $k=1$ and all $T<T(\varphi,\psi)$ if f fulfills the following additional conditions:

$$|f(u^2)| \leqq c\,|u|^{\frac{2m}{n-2m}}, \qquad |u| \geqq 1,$$

$$|f'(u^2)| \leqq c\,|u|^{\frac{2m}{n-2m}-2}, \qquad |u| \geqq 1.$$

(Observe that $T(\varphi,\psi)$ is the positive number of Theorem I.1 in the case $k=1$.) For a proof cf. Heinz - von Wahl [4, § 2]. If $\int_0^r f(s)\,ds \geqq 0$, $r \geqq 0$, then $T(\varphi,\psi)= +\infty$, cf. also Heinz - von Wahl [4, § 2]. Another example is

$$M(u)=\sin \operatorname{Re} u = \frac{\sin \operatorname{Re} u}{\operatorname{Re} u} \operatorname{Re} u = f((\operatorname{Re} u)^2) \operatorname{Re} u.$$

2. *For Theorem II.6*

Let f now be of class C^3 too with $|f'''(u^2)\,u^4| \leqq c$, $|u| \geqq 1$.

If additionally we set $m=1$ and if we restrict the dimension to $n=5,6,7$ the assumptions of Theorem II.6 are fulfilled for $k_1=1$ (as mentioned before) and for $k_2=3$. To see this let us first estimate $\|A f(u^2)\,u\|$. By formal differentiation we get

$$\frac{\partial^2 f(u^2)\,u}{\partial x_i\,\partial x_j} = (f(u^2)+2f'(u^2)\,u^2)\frac{\partial^2 u}{\partial x_i\,\partial x_j} + (4f''(u^2)\,u^3+6f'(u^2)\,u)\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}. \tag{II.12}$$

On applying Hölders inequality and Sobolev we get for the first term on the right side:

$$\left\|(f(u^2)+2f'(u^2)\,u^2)\frac{\partial^2 u}{\partial x_i\,\partial x_j}\right\| \leqq c(\|u\|^{\rho+\varepsilon}_{H^{2,2}(\Omega)}+1)\,\|u\|_{H^{3,2}(\Omega)}$$

and for the second term

$$\left\|(4f''(u^2)\,u^3+6f'(u^2)\,u)\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\right\| \leqq c\left\|\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\right\| \leqq c\left\|\frac{\partial u}{\partial x_i}\right\|_{L^{2q_1}(\Omega)}\left\|\frac{\partial u}{\partial x_j}\right\|_{L^{2q_2}(\Omega)}$$

with $\frac{1}{2q_2}=\frac{1}{2}-\frac{2}{n}$, $\frac{1}{2q_1}=\frac{2}{n}$. n being restricted to 5, 6 or 7 we have

$$\frac{2}{n} \geqq \frac{1}{2} - \frac{1+\rho+\varepsilon}{n}$$

for a suitable ρ, $0<\rho<\frac{2}{n-4}$ and any ε with $\frac{2}{n-2} \leqq \rho+\varepsilon < \frac{2}{n-4}$. On using Sobolev we arrive at

$$\|f(u^2)\,u\|_{H^{2,2}(\Omega)} \leqq \tilde{g}(\|u\|_{H^{2+\rho+\varepsilon,2}(\Omega)}) \cdot \|u\|_{H^{3,2}(\Omega)} \tag{II.13}$$

with a continuous function $g\colon \mathbb{R}_+ \to \mathbb{R}_+$. For $u \in H^{3,2}(\Omega) \cap \mathring{H}^{1,2}(\Omega) \supset D(A^{\frac{3}{2}})$ we have

$$f_k(u^2)\, u \in \mathring{H}^{1,2}(\Omega)$$

and by Sobolev

$$f_k(u^2)\, u \to f(u^2)\, u \quad \text{ in } H^{1,2}(\Omega),$$

where

$$f_k(x^2) = \begin{cases} k, & f(x^2) \geqq k \\ f(x^2), & -k < f(x^2) < k. \\ -k, & f(x^2) \leqq -k \end{cases}$$

$f(u^2)u$ thus being in $\mathring{H}^{1,2}(\Omega)$ we have in fact proved that $f(u^2)u \in D(A)$ if $u \in D(A^{\frac{3}{2}})$. Now we deal with the Lipschitz continuity. As it concerns the first term on the right side of (II.12) we have for, $u, v \in D(A^{\frac{3}{2}})$ the estimates

$$\left\| (f'(u^2)\, u^2 - f'(v^2)\, v^2) \frac{\partial^2 u}{\partial x_i\, \partial x_j} \right\| \leqq c\, \|u - v\|_{L^{2q_1}(\Omega)}\, \|u\|_{H^{3,2}(\Omega)},$$

$$\left\| f'(v^2)\, v^2 \frac{\partial^2 (u-v)}{\partial x_i\, \partial x_j} \right\| \leqq c\, \|v\|_{L^{2q_1}(\Omega)}\, \|u - v\|_{H^{3,2}(\Omega)},$$

where $\frac{1}{2q_1} = \frac{1}{n}$. Since $\frac{1}{n} \geqq \frac{1}{2} - \frac{3}{n}$ we can use Sobolev to get the desired estimate. As for the term

$$\left\| f(u^2) \frac{\partial^2 u}{\partial x_i\, \partial x_j} - f(v^2) \frac{\partial^2 v}{\partial x_i\, \partial x_j} \right\|$$

it can be handled similarly. Dealing with the second term on the right side of (II.12) we only treat

$$f''(u^2)\, u^3 \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} - f''(v^2)\, v^3 \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j}.$$

We have

$$\left\| (f''(u^2)\, u^3 - f''(v^2)\, v^3) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \right\| \leqq c\, \|u - v\|_{L^{2q_1}(\Omega)}\, \|\nabla u\|^2_{L^{4q_2}(\Omega)}$$

where $\frac{1}{4q_2} = \frac{1}{2} - \frac{2}{n}$, i.e. $\frac{1}{2q_2} = 1 - \frac{4}{n}$ and consequently $\frac{1}{2q_1} = \frac{4}{n} - \frac{1}{2}$; hence it follows that $\frac{1}{2q_1} \geqq \frac{1}{2} - \frac{3}{n}$ for $n = 5, 6, 7$. Using the norm equivalence (II.4) the inequality (II.13) changes into

$$\|AM(u)\| \leqq g(\|u\|_{H^{2+\rho+\varepsilon,2}(\Omega)})\, \|A^{\frac{3}{2}} u\|$$

with a continuous function $g\colon \mathbb{R}^+ \to \mathbb{R}^+$, and moreover we get

$$\|A(M(u) - M(v))\| \leqq c(C)\, \|A^{\frac{3}{2}}(u - v)\|,$$

where $C \geqq \|A^{\frac{3}{2}} u\| + \|A^{\frac{3}{2}} v\|$. Thus Theorem II.6 furnishes

$$u \in C^0([0, T], D(A^2)) \subset C^0([0, T], H^{4,2}(\Omega)).$$

In particular u is Hölder continuous in $[0,T]\times\bar{\Omega}$. 4 being $>\frac{n}{2}$, $n=5,6,7$, we get for $\Omega=\mathbb{R}^n$ that u is in fact a classical solution on $[0,T]\times\mathbb{R}^n$ if f is of class C^5 (cf. von Wahl [9, p. 269]). A case where the growth condition on f''' is fulfilled is $M(u)=\frac{\sin \operatorname{Re} u}{\operatorname{Re} u}\operatorname{Re} u=f((\operatorname{Re} u)^2)\operatorname{Re} u$.

There is also a corresponding result for arbitrary m. We only want to sketch the proof.

We assume that $4m+1\leqq n\leqq 6m+1$. Differentiating the equation with respect to t we formally get

$$\begin{aligned}(u')''+Au'+(f(u^2)+2f'(u^2)u^2)u'&=0,\\ u'(0)&=\psi,\\ u''(0)&=-A\phi-f(\phi^2)\phi.\end{aligned}$$

Differentiating the nonlinearity a second time with respect to t gives

$$(f(u^2)u)''=(f(u^2)+2f'(u^2)u^2)u''+(4f''(u^2)u^3+6f'(u^2)u)u'^2.$$

Now we have

$$\|u'\|^2_{L^4(\Omega)}\leqq\|u'\|_{L^{2q_1}(\Omega)}\|u'\|_{L^{2q_2}(\Omega)}\leqq c\cdot\|u'\|_{L^{2q_1}(\Omega)}\|u'\|_{H^{2m,2}(\Omega)}$$

where $\frac{1}{2q_2}=\frac{1}{2}-\frac{2m}{n}$, $\frac{1}{2q_1}=\frac{2m}{n}\geqq\frac{1}{2}-\frac{m+\rho+\varepsilon}{n}$ if $4m+1\leqq n\leqq 6m+1$,

and

$$\|(f(u^2)+2f'(u^2)u^2)u''\|\leqq c(\|u\|^{\rho+\varepsilon}_{H^{2m,2}(\Omega)}+1)\|u''\|_{H^{m,2}(\Omega)}.$$

This combined with Lemma II.1 gives: $u\in C^0([0,T],H^{3m,2}(\Omega))$, $u'\in C^0([0,T],H^{2m,2}(\Omega))$, $u''\in C^0([0,T],H^{m,2}(\Omega))$, $T<T(\phi,\psi)$. Next considering the twice differentiated equation

$$\begin{aligned}&(u'')''+Au''+(f(u^2)+2f'(u^2)u^2)u''+(4f''(u^2)u^3+6f'(u^2)u)u'^2=0,\\ &u''(0)=-A\phi-f(\phi^2)\phi,\quad u'''(0)=-A\psi-(f(\phi^2)+2f'(\phi^2)\phi^2)\psi,\end{aligned}$$

we have to deal with the Lipschitz continuity of the nonlinearity. As in the case $m=1$ we get that for

$$\begin{aligned}u'''&\in C^0([0,T],H^{m,2}(\Omega)),\\ u''&\in C^0([0,T],H^{2m,2}(\Omega)),\\ u'&\in C^0([0,T],H^{3m,2}(\Omega)),\\ u&\in C^0([0,T],H^{4m,2}(\Omega))\end{aligned}$$

the expression $(f(u^2)+2f'(u^2)u^2)u''+(4f''(u^2)u^3+6f'(u^2)u)u'^2$ is in fact an element of $C^{0,1}([0,T],L^2(\Omega))$. Using the estimates needed for the latter result one can easily show that u has in fact the properties mentioned above.

As it concerns the initial values for u'',u''' they must be in $D(A)$, $D(A^{\frac{1}{2}})$ respectively. This of course is fulfilled in particular if

$$\phi \in D(A^2),$$
$$f(\phi^2)\phi \in D(A) = H^{2m,2}(\Omega) \cap \mathring{H}^{m,2}(\Omega),$$
$$\psi \in D(A^{\frac{3}{2}}),$$
$$(f(\phi^2) + 2f'(\phi^2)\phi^2)\psi \in D(A^{\frac{1}{2}}) = \mathring{H}^{m,2}(\Omega).$$

For $\phi \in D(A^2)$, $\psi \in D(A^{\frac{3}{2}})$ the expressions $f(\phi^2)\phi$, $(f(\phi^2)+2f'(\phi^2)\phi^2)\psi$ are in $D(A)$, $D(A^{\frac{1}{2}})$ respectively if f is of class C^{2m}. ■

We now prove the announced theorem concerning the domain of definition of the fractional powers of selfadjoint elliptic operators.

Theorem II.7. *We have*

$$D(A^\sigma) \subset H^{2\sigma m,2}(\Omega), \qquad 0 \leqq \sigma,$$

with a continuous imbedding. The following estimates hold:

$$\|A^\sigma u\| \leqq c(M, n, m, \Omega, \|\nabla^{|\alpha|([\sigma]+1)} A_\alpha\|_{C^0(\bar{\Omega})}, \|\nabla^{|\alpha|([\sigma]+1)-1} A_\alpha\|_{C^0(\bar{\Omega})}, \ldots, \|A_\alpha\|_{C^0(\bar{\Omega})}) \|u\|_{H^{2m\sigma,2}(\Omega)},$$

$u \in D(A^{[\sigma]+1})$, $0 \leqq \sigma$, *and*

$$\|u\|_{H^{2m\sigma,2}(\Omega)} \leqq c(M, n, m, \Omega, \|\nabla^{|\alpha|([\sigma]+1)} A_\alpha\|_{C^0(\bar{\Omega})}, \|\nabla^{|\alpha|([\sigma]+1)-1} A_\alpha\|_{C^0(\bar{\Omega})}, \ldots, \|A_\alpha\|_{C^0(\bar{\Omega})}) \|A^\sigma u\|,$$

$0 \leqq \sigma$, $u \in D(A^\sigma)$.

Proof. First we treat the case $\sigma \in [0,1]$ on which all others then will be reduced. Let T be the extension-operator from $H^{2m,2}(\Omega)$ into $H^{2m,2}(\mathbb{R}^n)$ (cf. introduction). In particular this means that for s, $0 \leqq s \leqq 2m$, space $T(H^{s,2}(\Omega))$ is contained in $H^{s,2}(\mathbb{R}^n) = D([(-\Delta+1)^m]^{\frac{s}{2m}})$; $(-\Delta+1)^m$ is the positive definite selfadjoint elliptic operator $\left(-\sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} + 1\right)^m$ with domain of definition $D((-\Delta+1)^m) = H^{2m,2}(\mathbb{R}^n)$. We have

$$\|[(-\Delta+1)^m]\, Tu\| = \|Tu\|_{H^{2m,2}(\mathbb{R}^n)} \leqq c\,\|u\|_{H^{2m,2}(\Omega)} \leqq c\,\|Au\|.$$

On applying the Heinz-Kato inequality [7, p. 145] we see that

$$T(D(A^\sigma)) \subset D([(-\Delta+1)^m]^\sigma),$$
$$\|[(-\Delta+1)^m]^\sigma\, Tu\| \leqq c\,\|A^\sigma u\|, \qquad 0 \leqq \sigma \leqq 1,$$

$u \in D(A^\sigma)$. On using the Fourier transformation we get

$$D([(-\Delta+1)^m]^\sigma) = H^{2\sigma m,2}(\mathbb{R}^n),$$
$$\|[(-\Delta+1)^m]^\sigma u\| = \|u\|_{H^{2\sigma m,2}(\mathbb{R}^n)},$$

whence it follows

$$\|Tu\|_{H^{2\sigma m,2}(\mathbb{R}^n)} \leqq c\,\|A^\sigma u\|.$$

T being the extension operator we have for $u \in \mathring{H}^{m,2}(\Omega) \cap H^{2m,2}(\Omega)$ the inequality

$$\|u\|_{H^{2\sigma m,2}(\Omega)} \leqq c\,\|A^\sigma u\|.$$

$\mathring{H}^{m,2}(\Omega) \cap H^{2m,2}(\Omega)$ being dense in $D(A^\sigma)$ with respect to the graph norm of A^σ the last inequality holds for all $u \in D(A^\sigma)$ and we have $D(A^\sigma) \subset H^{2\sigma m,2}(\Omega)$. For the second direction we need Hadamard's three lines theorem. Let $v \in H^{2m,2}(\mathbb{R}^n)$. We define

$$R_\Omega v := v - w \in \mathring{H}^{m,2}(\Omega) \cap H^{2m,2}(\Omega),$$

where w is the uniquely determined solution of Dirichlet's problem

$$(-\Delta+1)^m_\Omega w = 0,$$
$$w - v \in \mathring{H}^{m,2}(\Omega)$$

over Ω; instead of $v|\Omega$ we also write v, and we set

$$(-\Delta+1)^m_\Omega u = \left(-\sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} + 1\right)^m u,$$
$$u \in H^{2m,2}(\Omega) \cap \mathring{H}^{m,2}(\Omega),$$
$$(-\Delta+1)^m_{f,\Omega} g = \left(-\sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} + 1\right)^m g,$$
$$g \in H^{2m,2}(\Omega).$$

Then $(-\Delta+1)^m = (-\Delta+1)^m_{\mathbb{R}^n} = (-\Delta+1)^m_{f,\mathbb{R}^n}$. According to [3, Theorem 1] we have

$$\|v - w\|_{H^{2m,2}(\Omega)} \leqq c\,\|v\|_{H^{2m,2}(\Omega)}.$$

For $\tilde{u} \in H^{2m,2}(\mathbb{R}^n)$ we define

$$f(z) := A^{1/2+\sigma+it} R_\Omega (-\Delta+1)^{-(1/2+\sigma+it)}\tilde{u}, \qquad z = \sigma + it.$$

Then f is holomorphic in $0 < \sigma < \frac{1}{2}$, continuous in $0 \leqq \sigma \leqq \frac{1}{2}$. Moreover f satisfies the estimates

$$\|f(0+it)\| \leqq c\,\|(-\Delta+1)^{-(1/2+it)m}\tilde{u}\|_{H^{m,2}(\mathbb{R}^n)},$$
$$\leqq c\,\|\tilde{u}\|,$$
$$\|f(1+it)\| \leqq c\,\|(-\Delta+1)^{-(1+it)m}\tilde{u}\|_{H^{2m,2}(\mathbb{R}^n)},$$
$$\leqq c\,\|\tilde{u}\|.$$

Following Hadamard's three lines theorem we get

$$\|A^{\sigma+\frac{1}{2}} R_\Omega (-\Delta+1)^{-(\frac{1}{2}+\sigma)m}\tilde{u}\| \leqq c\,\|\tilde{u}\|,$$
$$\|A^{\sigma+\frac{1}{2}} R_\Omega \tilde{u}\| \leqq c\,\|(-\Delta+1)^{(\frac{1}{2}+\sigma)m}\tilde{u}\|.$$

Now set $v = \tilde{u}$; then $R_\Omega \tilde{u} = R_\Omega v = v - w \in D(A)$ and

$$\|A^{\sigma+\frac{1}{2}}(v-w)\| \leqq c\,\|\tilde{u}\|_{H^{2m(\sigma+1/2),2}(\mathbb{R}^n)}.$$

If $\tilde{u}$ is the extension of a function $u \in \mathring{H}^{m,2}(\Omega) \cap H^{2m,2}(\Omega)$ to the whole of $\mathbb{R}^n$, then $w=0$, and we get

$$\|A^{\sigma+\frac{1}{2}} u\| \leqq c \|\tilde{u}\|_{H^{2m(\sigma+1/2),2}(\mathbb{R}^n)},$$
$$\leqq c \|u\|_{H^{2m(\sigma+1/2),2}(\mathbb{R}^n)}.$$

For u as above we have $(-\Delta+1)^m_{f,\Omega} v = (-\Delta+1)^m_\Omega v$ and

$$\|v-w\|_{L^2(\Omega)} = \|v\|_{L^2(\Omega)}.$$

A calculation analogous to the foregoing one then shows that

$$\|A^\sigma u\| \leqq c \|u\|_{H^{2m\sigma,2}(\Omega)}, \qquad 0 \leqq \sigma \leqq \tfrac{1}{2}.$$

Thus we get the desired result for all σ, $0 \leqq \sigma \leqq 1$. Treating the case $\sigma > 1$ we set $\delta = \sigma - [\sigma]$. Then $1 \geqq \delta \geqq 0$ and we have

$$\|A^\sigma u\| \geqq c \|A^{[\sigma]} u\|_{H^{2\delta m,2}(\Omega)}.$$

By linear elliptic regularity theory (cf. [3, p. 52]) $A^{-[\sigma]}$ being a bounded linear operator from $L^2(\Omega)$ into $H^{2[\sigma]m,2}(\Omega)$ and from $H^{2m,2}(\Omega)$ into $H^{2[\sigma]m+2m,2}(\Omega)$ it is also a bounded linear operator from $H^{2\delta m,2}(\Omega)$ into $H^{2[\sigma]m+2\delta m,2}(\Omega)$, cf. [1, p. 77]. Thus we get

$$\|A^{[\sigma]} u\|_{H^{2\delta m,2}(\Omega)} \geqq c \|u\|_{H^{2[\sigma]m+2\delta m,2}(\Omega)}.$$

Moreover

$$\|A^\sigma u\| \leqq c \|A^{[\sigma]} u\|_{H^{2\delta m,2}(\Omega)}.$$

It is not difficult to prove that for a function $\phi \in C^\infty(\mathbb{R}^n)$ having all its derivatives bounded the expression ϕg fulfills the estimate:

$$\|\phi g\|_{H^{s,2}(\Omega)} \leqq c(s,n,m,\Omega,\phi) \|g\|_{H^{s,2}(\Omega)}, \qquad 0 \leqq s \leqq 2m, \ g \in H^{2sm,2}(\Omega).$$

Thus we get

$$\|A^{[\sigma]} u\|_{H^{2\delta m,2}(\Omega)} \leqq c \|u\|_{H^{2[\sigma]m+2\delta m,2}(\Omega)},$$

and our theorem is proved. ■

In what follows it is useful to define the notion of a weak solution for the wave equation in a way slightly different from that used in Heinz - von Wahl [4].

Definition II.8. Let $T>0$. Let M be a Lipschitz continuous mapping from $D(A^{\frac{1}{2}}) \cap L^p(\Omega)$ into $L^2(\Omega)$ for a certain $p \geqq 1$, i.e. *for $u, v \in D(A^{\frac{1}{2}}) \cap L^p(\Omega)$ we have*

$$\|M(u) - M(v)\| \leqq c(C) \|A^{\frac{1}{2}}(u-v)\|,$$

where $C \geqq \|u\|_{L^p(\Omega)} + \|v\|_{L^p(\Omega)} + \|A^{\frac{1}{2}} u\| + \|A^{\frac{1}{2}} v\|$. Let $\phi \in D(A^{\frac{1}{2}}) \cap L^p(\Omega)$, $\psi \in L^2(\Omega)$. An element

$$u \in C^0([0,T], D(A^{\frac{1}{2}})) \cap L^\infty((0,T); L^p(\Omega)) \cap C^1([0,T], L^2(\Omega))$$

with

$$u(t)=\cos A^{\frac{1}{2}} t\, \phi+\sin A^{\frac{1}{2}} t\, A^{-\frac{1}{2}} \psi - \int_0^t \sin A^{\frac{1}{2}}(t-s)\, A^{-\frac{1}{2}} M(u(s))\, ds$$

is called a weak solution over $[0, T]$ *of* $u''+Au+M(u)=0$ *with initial values* ϕ, ψ.

Proposition II.9. *For given initial data* ϕ, ψ *there exists at most one weak solution over* $[0, T]$ *of* $u''+Au+M(u)=0$.

Proof. On using Gronwall's inequality the proof immediately follows from the integral equation. ∎

We want now to dispense with the existence assumption in Proposition II.3. We get

Theorem II.10. *Let* $\varepsilon>0$,

$$\frac{2m}{n-2m} \leqq \rho+\varepsilon<\frac{2m}{n-4m} \wedge 1.$$

Let

$$\phi \in D(A^{1+(\rho+\varepsilon)/2m}), \qquad \phi \text{ real},$$
$$\psi \in D(A^{1/2+(\rho+\varepsilon)/2m}), \qquad \psi \text{ real},$$

let $T>0$.

We assume moreover: If u is a real valued, a.e. on $(0, \tilde{T}) \times \Omega$ *defined function with*

$$u \in C^2([0, \tilde{T}], L^2(\Omega)),$$
$$u' \in C^0([0, \tilde{T}], D(A^{1/2+(\rho+\varepsilon)/2m})),$$
$$u \in C^0([0, \tilde{T}], D(A^{1+(\rho+\varepsilon)/2m})),$$
$$u''+Au+M(u)=0 \quad \text{in } [0, \tilde{T}],$$
$$u(0)=\phi,$$
$$u'(0)=\psi$$

for some $\tilde{T}$, $0 \leqq \tilde{T} \leqq T$, *then*

$$\sup_{0 \leqq t \leqq \tilde{T}} (\|A^{1+(\rho+\varepsilon)/2m} u(t)\|+\|A^{1/2+(\rho+\varepsilon)/2m} u'(t)\|) \leqq h(T, \|A^{1+(\rho+\varepsilon)/2m} \phi\|+\|A^{1/2+(\rho+\varepsilon)/2m} \psi\|)$$

where h is a continuous function from $\mathbb{R}_+ \times \mathbb{R}_+$ *into* $\mathbb{R}_+$; *moreover h does not depend on* $\tilde{T}$. *Then there exists in fact a unique u with*

$$u \in C^2([0, T], L^2(\Omega)),$$
$$u \in C^0([0, T], D(A^{1+(\rho+\varepsilon)/2m},$$
$$u' \in C^0([0, T], D(A^{1/2+(\rho+\varepsilon)/2m})),$$

$$u''+Au+M(u)=0 \quad in\,[0,T],$$
$$u(0)=\phi,$$
$$u'(0)=\psi.$$

Proof. For all K, $\hat{T}$ with $K\geqq 0$, $\hat{T}>0$ the set

$$\begin{aligned}\Gamma(K,\hat{T}):=\{w\,|\,w\in L^\infty((0,\hat{T});D(A^{1+(\rho+\varepsilon)/2m})),\\ w'\in L^\infty((0,\hat{T});D(A^{1/2+(\rho+\varepsilon)/2m})),\\ w \text{ real},\quad w(0)=\phi,\\ \|\|w\|\|^{\rho+\varepsilon}_{(0,\hat{T})}:=\operatorname*{ess\,sup}_{t\in(0,\hat{T})}(\|A^{1+(\rho+\varepsilon)/2m}w(t)\|+\|A^{1/2+(\rho+\varepsilon)/2m}w'(t)\|)\leqq K\}\end{aligned}$$

is a compact convex set in the Hilbert space $L^2(Q_{\hat{T}})$, $Q_{\hat{T}}:=(0,\hat{T})\times\Omega$, endowed with the weak topology. Namely, if $\{w_\nu\}$ is a sequence being contained in $\Gamma(K,\hat{T})$ there is a subsequence $\{w_{j_\nu}\}$ with

$$\begin{aligned}&w_{j_\nu}\rightharpoonup w \quad \text{in } L^2((0,\hat{T}),D(A^{1+(\rho+\varepsilon)/2m})),\\ &w'_{j_\nu}\rightharpoonup w' \quad \text{in } L^2((0,\hat{T}),D(A^{\frac{1}{2}+(\rho+\varepsilon)/2m})),\\ &w(0)=\phi;\end{aligned}$$

we can assume that also

$$\begin{aligned}&w_{j_\nu}\to w \quad \text{weak star in } L^\infty((0,\hat{T});D(A^{1+(\rho+\varepsilon)/2m})),\\ &w'_{j_\nu}\to w' \quad \text{weak star in } L^\infty((0,\hat{T});D(A^{\frac{1}{2}+(\rho+\varepsilon)/2m})).\end{aligned}$$

We specialise K to

$$2\,\|A^{1+(\rho+\varepsilon)/2m}\phi\|+2\,\|A^{\frac{1}{2}+(\rho+\varepsilon)/2m}\psi\|+4\,\|A^{(\rho+\varepsilon)/2m}M(\phi)\|+1.$$

Let us define a mapping $\mathscr{F}$ by setting

$$\begin{aligned}(\mathscr{F}\,w)(t)=\sin A^{\frac{1}{2}}t\,A^{-\frac{1}{2}}\psi+\cos A^{\frac{1}{2}}t\,\phi\\ -\int_0^t \sin A^{\frac{1}{2}}(t-s)\,A^{-\frac{1}{2}}M(w(s))\,ds,\quad 0\leqq t\leqq\hat{T}.\end{aligned} \tag{II.14}$$

As it was done in the proof of Proposition II.3 one can show that

$$(M(w))'\in L^\infty((0,\hat{T});H^{\rho+\varepsilon,2}(\Omega))$$

and

$$(M(w(s)))'=(M(w))'(s)=(2f'(w^2(s))\,w^2(s)+f(w^2(s)))\,w'(s) \quad \text{a.e. in } (0,\hat{T}).$$

From this it follows that in fact

$$M(w),\quad (M(w))'\in L^\infty((0,\hat{T});D(A^{(\rho+\varepsilon)/2}))$$

(cf. the corresponding part in the proof of Proposition II.3). Forming $(\mathscr{F}\,w)'$ according to (II.14) and using partial integration as in the proof of Proposition II.3 we get the formulas

$$(\mathscr{F} w)'(t) = \cos A^{\frac{1}{2}} t\, \psi - \sin A^{\frac{1}{2}} t\, A^{\frac{1}{2}} \phi - \sin A^{\frac{1}{2}} t\, A^{-\frac{1}{2}} M(w(0)) - \int_0^t \sin A^{\frac{1}{2}}(t-s)\, A^{-1} M(w(s))'\, ds, \tag{II.15}$$

$$(\mathscr{F} w)(t) = \sin A^{\frac{1}{2}} t\, A^{-\frac{1}{2}} \psi + \cos A^{\frac{1}{2}} t\, \phi - A^{-1} M(w(t)) + \cos A^{\frac{1}{2}} t\, A^{-1} M(w(0)) + \int_0^t \cos A^{\frac{1}{2}}(t-s)\, A^{-1}(M(w(s)))'\, ds. \tag{II.16}$$

Now observe that

$$w(0) = \phi,$$
$$M(w(t)) = \int_0^t (M(w))'(s)\, ds + M(\phi),$$

and therefore

$$\|A^{\frac{1}{2}+(\rho+\varepsilon)/2m} A^{-\frac{1}{2}} M(w(t))\| \leqq t \operatorname*{ess\,sup}_{0<s<\hat{T}} \|A^{(\rho+\varepsilon)/2m}(M(w))'(s)\| + \|A^{(\rho+\varepsilon)/2m} M(\phi)\|,$$
$$\|A^{1+(\rho+\varepsilon)/2m} A^{-1} M(w(t))\| \leqq t \operatorname*{ess\,sup}_{0<s<\hat{T}} \|A^{(\rho+\varepsilon)/2m}(M(w))'(s)\| + \|A^{(\rho+\varepsilon)/2m} M(\phi)\|.$$

Thus we get

$$|||\mathscr{F} w|||_{(0,\hat{T})}^{\rho+\varepsilon} \leqq K$$

if $\hat{T} = \hat{T}_1$ is small enough, say

$$T_1 \leqq \delta = \delta(\|A^{1+(\rho+\varepsilon)/2m} \phi\| + \|A^{\frac{1}{2}+(\rho+\varepsilon)/2m} \psi\| + \|A^{(\rho+\varepsilon)/2m} M(\phi)\|, T),$$

the bound δ only depending on the quantities T,

$$\|A^{1+(\rho+\varepsilon)/2m} \phi\| + \|A^{\frac{1}{2}+(\rho+\varepsilon)/2m} \psi\| + \|A^{(\rho+\varepsilon)/2m} M(\phi)\|.$$

Moreover

$$\mathscr{F} w \in C^0([0, \hat{T}_1], D(A^{1+(\rho+\varepsilon)/2m})),$$
$$(\mathscr{F} w)' \in C^0([0, T_1], D(A^{\frac{1}{2}+(\rho+\varepsilon)/2m})).$$

Now we want to show that $\mathscr{F}$ is continuous with respect to the weak topology of $L^2(Q_{\hat{T}_1})$. If $\{w_\nu\}$ is a sequence contained in $\Gamma(K, \hat{T}_1)$ with accumulation point w in the weak topology of $L^2(Q_{\hat{T}_1})$ we choose a subseuqence $\{w_{j_\nu}\}$ as before. On applying Rellich's choice theorem we see that

$$w_{j_\nu} \to w$$

a.e. in $Q_{\hat{T}_1}$. Thus

$$M(w_{j_\nu}) \rightharpoonup M(w) \quad \text{in } L^2(Q_{\hat{T}_1}).$$

Let us consider the integral equation for $\mathscr{F} w_{j_\nu}$. We have ($v(t):=\sin A^{\frac{1}{2}} t A^{-\frac{1}{2}} \psi + \cos A^{\frac{1}{2}} t \phi$):

$$\int_0^{\hat{T}_1} (\mathscr{F} w_{j_\nu}(t), \psi(t))\, dt$$
$$= \int_0^{\hat{T}_1} (v(t), \psi(t)\, dt - \int_0^{\hat{T}_1} \int_0^t (M(w_{j_\nu}(s)), \sin A^{\frac{1}{2}}(t-s) A^{-\frac{1}{2}} \psi(t))\, ds\, dt, \qquad \psi \in L^2(Q_{\hat{T}}).$$

Of course $M(w_{j_\nu}) \rightharpoonup M(w)$ in $L^2((0,t)\times\Omega)$. The expressions $\|M(w_{j_\nu}(s))\|$ being essentially bounded on $(0, \hat{T}_1)$ (cf. the estimate for $\|(M(u(s)))'\|_{\rho+\varepsilon,2}$ in the proof of Proposition II.3) one gets by Lebesgue's theorem that

$$\int_0^{\hat{T}_1} (\mathscr{F} w_{j_\nu}(t), \psi(t))\, dt \to \int_0^{\hat{T}_1} (\mathscr{F} w(t), \psi(t))\, dt.$$

Thus $\mathscr{F}$ is also continuous. On applying Tychonoff's fixed point theorem one gets that $\mathscr{F}$ has a fixed point u with

$$u \in C^0([0, \hat{T}_1], D(A^{1+(\rho+\varepsilon)/2m})),$$
$$u' \in C^0([0, \hat{T}_1], D(A^{\frac{1}{2}+(\rho+\varepsilon)/2m})),$$
$$u(0) = \phi,$$
$$u'(0) = \psi.$$

On using the integral Eqs. (II.15) and (II.16) we see that $u \in C^2([0, \hat{T}_1], L^2(\Omega))$ and $u'' + Au + M(u) = 0$ in $[0, \hat{T}_1]$. Now we repeat the whole procedure with initial values

$$u(\hat{T}_1), \qquad u'(\hat{T}_1).$$

This gives as before a solution of $u'' + Au + M(u) = 0$ over $[\hat{T}_1, \hat{T}_2]$ and so on. Because of our a-priori estimate for the quantity

$$\sup_{0 \leqq t \leqq \tilde{T}} (\|A^{1+(\rho+\varepsilon)/2m} u(t)\| + \|A^{\frac{1}{2}+(\rho+\varepsilon)/2m} u'(t)\|)$$

we can choose $\hat{T}_\nu$ in such a way that

$$\hat{T}_{\nu+1} - \hat{T}_\nu \geqq \delta_2 = \delta_2(\|A^{1+(\rho+\varepsilon)/2m} \phi\| + \|A^{\frac{1}{2}+(\rho+\varepsilon)/2m} \psi\|, T) > 0$$

with a positive constant $\delta_2(\|A^{1+(\rho+\varepsilon)/2m} \phi\| + \|A^{\frac{1}{2}+(\rho+\varepsilon)/2m} \psi\|, T)$. So far we proved every part of Theorem II.10 with the exception of the uniqueness. One easily shows that u is also a weak solution over $[0, T]$ being in $L^\infty((0,T), L^{n\rho_0}(\Omega))$ with $\rho_0 = \min\left(\dfrac{2m}{n-4m}, 1\right)$. Thus Proposition II.9 shows the uniqueness.

III. On Classical and Strong Global Solutions of (H) $u'' + A(x, D) u + M(u) = 0$, $u(0) = \phi$, $u'(0) = \psi$

Where we for simplicity assume that ϕ, ψ are in $H^{1,k}$, for k large enough (this can be made much more precise!) and where

1° A is a real positive elliptic differential operator with C^∞-coefficients, which are constant for $|x|$ large,

2° $M\in C^\sigma$, for some $\sigma\geqq 3$ and $M(\mathbb{R})\subseteq\mathbb{R}$,

3° $|M^{(j)}(u)|\leqq c(1+|u|^{\rho_j})$, $j=0,1,\ldots,\sigma$; $\rho_j\geqq 0$ where $\rho_0=\rho+1$ and $\rho_1=\rho$, and where for simplicity we assume $\rho_j\leqq\max(0,\rho+1-j)$,

4° $M(u)=f(u^2)u$ and $\int_0^v f(u)\,du\geqq 0$ for $v\geqq 0$.

Notice that in [2], we used the notation ρ for ρ_0 instead of, as here, for ρ_1. Since the assumptions on A are invariant under addition of $f(0)u$, say, we may assume, as we also will, that $f(0)=0$, i.e. $M'(0)=0$. If we assume that $\rho<1$ and that $f\in C^\sigma$ as above, and then compare condition 3° with the growth conditions used in Chapter II, namely

$$|f(u^2)-f(v^2)|\leqq c|u-v|^\rho, \tag{i}$$

$$|f''(u^2)||u|^3\leqq c, \tag{ii}$$

$$|f'(u^2)u^2-f'(v^2)v^2|\leqq c|u-v|^\rho, \tag{iii}$$

$$|f'''(u^2)u^4|\leqq c \tag{iv}$$

we find that (i) and (ii) imply with (iii) for $v=0$ that 3° holds for $j=0,1,2$. The condition on M''' is implied by (iv) (which was used in the example for Theorem II.6). On the other hand, if condition 3° holds then also (i), (ii) and (iv) hold (and (iii) for $v=0$). Thus condition (iii) remains to be satisfied. However, this condition will only be needed in the application of Theorem II.10 in Theorem III.2 below. Straightforward computations show that (iii) is satisfied if in addition to 1° through 4° above, we assume that

5° $|M''(u)|\leqq c(1+|u|)^{\rho-1}$.

This extra condition will be added in Theorem III.2 below, but will not otherwise be assumed in this Chapter.

In the lemmas below, u will denote a solution of the integral equation corresponding to (H), and a statement „$u\in L^\infty_{\mathrm{loc}}(\mathbb{R}_+;X)$" will be short for "$u\in L^1_{\mathrm{loc}}(I;X)\Rightarrow u\in L^\infty(\bar I;X)$", any bounded interval $I\subseteq\mathbb{R}_+$.

Lemma III.1. (H) *has a weak solution in* $L^\infty_{\mathrm{loc}}(\mathbb{R}_+;H^{1,2})$. This is a well known consequence of 4° and the energy inequality.

Lemma III.2. (H) *has a classical solution for* $n\leqq 9$ *if* $u\in L^\infty_{\mathrm{loc}}(\mathbb{R}_+;L^r)$ *for some* $r\geqq n+1$. This was proved in [2].

Lemma III.3. *Let* $s=\bar s+\sigma$, $0<\sigma<1$ *and* $s\in\mathbb{Z}_+$. *Then an equivalent norm on* $B_p^{s,q}$ *is given by*

$$\|u\|^{\#}_{B_p^{s,q}}=\|u\|_p+\left(\int_0^1 t^{-\sigma q}\sup_{|h|\leqq t}\sum_{|\alpha|=\bar s}\|D^\alpha(u_h-u)\|_p^q\frac{dt}{t}\right)^{\frac{1}{q}}$$

where $u_h(x)=u(x+h)$.

For a proof, cf. Löfström and Bergh [1].

We will below prove inequalities of the type

$$\|M(u)\|^{\#}_{B^{s,2}_{p}} \leqq c(u)(\|u\|^{\#}_{B^{s',2}_{p'}}+1) \qquad (\#)$$

where $c(u) \in L^{\infty}_{\mathrm{loc}}$, for solutions $u=u(t)$ of (H). Sometimes we even prove the corresponding $L^p - L^{p'}$-inequalities. The $L^p - L^q$-inequalities (0.9) for the equation (H) then proves that if $\delta(n+1) \leqq 1+s-s'$ and $\delta(n-1)<1$ $\left(\text{where } 1<p\leqq 2, \frac{1}{p}+\frac{1}{p'}=1, \delta=\frac{1}{2}-\frac{1}{p'}\right)$, then $u \in L^{\infty}_{\mathrm{loc}}(\mathbb{R}_+; B^{s',2}_{p'})$ and so in particular (by [2], Lemma 1.1), $u \in L^{\infty}_{\mathrm{loc}}(\mathbb{R}_+; H^{s',p'})$. This argument will be applied time after time, without further reference.

The critical point in improving upon previous results [2, 6] is now:

Lemma III.4. *Let $n \geqq 6$, and assume that*

$$\rho < \frac{2(1+(n-1)\delta)}{n-2} - \frac{2}{n-2}(\rho - s), \qquad s<\rho. \qquad \text{(III.1)}$$

Then $u \in L^{\infty}_{\mathrm{loc}}(\mathbb{R}_+; H^{s',p'})$ for $\delta(n+1)=1+s-s'$.

Corollary III.4.1. *If $\rho < \frac{4}{n-2}$, then $u \in L^{\infty}_{\mathrm{loc}}(\mathbb{R}_+; L^r)$ for any r such that $\frac{1}{2} \geqq \frac{1}{r} > \frac{1}{2} - \frac{1+\rho-(n-1)^{-1}}{n}$.*

Proof of Lemma III.4. By 3° and 4°, and since we may assume that $M'(0)=0$, $M(u)=f(u^2)u$, where $|f(u^2)| \leqq c|u|^{\theta}$, $\rho \leqq \theta \leqq 1$. Thus

$$|M(u_h)-M(u)| \leqq c(|u-u_h|^{\bar{\theta}}|u-u_h|^{1-\bar{\theta}}|u|^{\rho}+|u-u_h|^{\theta}|u_h|),$$

where now $0 \leqq \bar{\theta} \leqq 1$ is, for the moment, arbitrary and $\rho \leqq \theta \leqq 1$. Hence for $1<p\leqq 2$, $\frac{1}{p}+\frac{1}{p'}=1$, $\delta(n+1)=1+s-s'$,

$$\begin{aligned} &t^{-s}\|M(u_h)-M(u)\|_p \\ &\leqq ct^{-s}\|u-u_h\|_2^{\bar{\theta}}\|u-u_h\|_{s',p'}^{1-\bar{\theta}}\|u\|_{s',p'}^{\rho}+ct^{-s}\|u-u_h\|_2^{\theta}\|u\|_{s',p'} \end{aligned} \qquad \text{(III.2)}$$

provided

$$\frac{1}{p} \geqq \frac{1}{2}\bar{\theta}+(1-\bar{\theta})\left(\frac{1}{p'}-\frac{s'}{n}\right)+\rho\left(\frac{1}{p'}-\frac{s'}{n}\right), \qquad \text{(III.2a)}$$

$$\frac{1}{p} \geqq \frac{1}{2}\theta+\left(\frac{1}{p'}-\frac{s}{n}\right). \qquad \text{(III.2b)}$$

We take $\bar{\theta}=\rho$, and $s<\theta=\rho=\bar{\theta}$. Then, in particular, (III.2a) implies (III.2b). In addition, since $\bar{\theta}=\rho$, (III.2a) is equivalent to

$$\frac{1}{2}\rho - \frac{1+s-(n+1)\delta}{n} \leqq 2\delta$$

which is equivalent to

$$\frac{n\rho}{n-2} < \frac{2(1+(n-1)\delta)}{n-2} + \frac{2}{n-2}s$$

which is equivalent to (III.1). Hence (III.2) holds, where ρ satisfies (III.1). From (III.2) and Lemma III.3 we have (since $\theta > s$ and $\bar{\theta} > s$!)

$$\|M(u)\|_{B^s_p,2} \leqq c\,\|u\|_{1,2}(\|u\|_{s',p'} + \|u\|^{\rho}_{s',p'}) \leqq c\,\|u\|_{1,2}(\|u\|_{B^{s'}_{p'},2} + 1)$$

and the lemma follows as indicated above. ■

Lemma III.5. *Assume that* $\rho < \frac{4}{n-2}$. *Then* $u \in L^{\infty}_{\mathrm{loc}}(\mathbb{R}_{+1}; H^{1,q'})$, $\delta_q(n+1) = 1$, *where* $\delta_q = \frac{1}{2} - \frac{1}{q'}$, *for* $6 \leqq n \leqq 10$. *If* $\rho < \frac{4}{n-2}\frac{n}{n+1}$ *this conclusion holds for all* n.

Proof. For the last statement we merely refer to [2]. Assume then that $u \in L^{\infty}_{\mathrm{loc}}(\mathbb{R}_+; L^r)$ with r as in Corollary III.4.1. Then

$$\|M(u)\|_{1,q'} \leqq c\,\|u\|^{\rho}_{r}\,\|u\|_{1,q'} \tag{III.3}$$

provided

$$\rho\frac{1}{r} \leqq 2\delta = \frac{2}{n+1}.$$

Since we may use any $r \geqq 2$ such that

$$\frac{1}{r} > \frac{1}{2} - \frac{n-2}{n(n-1)} - \frac{\rho}{n}$$

it is sufficient to require that

$$\rho\left(\frac{1}{2} - \frac{n-2}{n(n-1)} - \frac{\rho}{n}\right) < \frac{2}{n+1}$$

or, equivalently, that

$$\rho^2 - \left(\frac{n-2}{2} + \frac{1}{n-1}\right)\rho + 2\frac{n}{n+1} > 0. \tag{III.4}$$

This certainly holds for $\rho = 0$. Since the left hand side takes its minimal value at $\rho = (n-2)/4 + 1/(2n-2) > 4/(n-2)$ for $n \geqq 6$, we only have to check (III.4) for $\rho = 4/(n-2)$. Thus we obtain the condition

$$\frac{4}{n-2}\left(\frac{4}{n-2} - \frac{1}{n-1}\right) > \frac{2}{n+1} \Leftrightarrow \frac{4}{n-2} > \frac{n-2}{2(n+1)} + \frac{1}{n-1}$$

which is easy to verify for $6 \leqq n \leqq 10$. Thus (III.4), and hence (III.3) is proved for $\rho < 4/(n-2)$, $6 \leqq n \leqq 10$, where the r-value of Corollary III.4.1 is used. This completes the proof of Lemma III.5. ■

Remark. We may actually obtain a better bound for ρ for $n \geqq 11$, using (III.4). A simple approximation actually shows that (III.4) holds for

$$\rho < \frac{4}{n-2} \frac{n}{n+1-\frac{2}{n-2}\frac{n+3}{n-1}}$$

and as a conclusion, $u \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+; H^{1,q'})$ for such values of ρ, all $n \geqq 11$. ■

Lemma III.6. *Let* $\frac{1}{2} - \frac{1}{q'} = \delta_q(n+1) = 1 - \mu$. *Assume that*

$$\rho < \frac{4}{n-2-2\varepsilon} \cdot \frac{n - \frac{1}{2}(n-1)\mu}{n+1} \tag{III.5}$$

and that $u \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+; H^{1,r})$ *where* $\frac{1}{r} = \frac{1}{2} - \frac{\varepsilon}{n}$, $\varepsilon \geqq 0$. *Then* $u \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+; H^{1+\sigma+\mu,q'})$, *where*

$$\sigma < \frac{4}{n-2-2\varepsilon} \cdot \frac{n - \frac{1}{2}(n-1)\mu}{n+1}. \tag{III.6}$$

Proof. For $|\alpha| = 1$ we have, $|h| \leqq t$ and $u_h(x) = u(x+h)$ as before,

$$\begin{aligned} D^\alpha(M(u_h) - M(u)) &= M'(u_h) D^\alpha u_h - M'(u) D^\alpha u \\ &= M(u_h)(D^\alpha u_h - D^\alpha u) + (M'(u_h) - M'(u)) D^\alpha u. \end{aligned}$$

By assumption

$$|M'(u_h)| \leqq c|u_h|^\rho \quad \text{and} \quad |M'(u_h) - M'(u)| \leqq c|u_h - u|^\theta, \qquad \rho \leqq \theta \leqq 1.$$

Hence for $\rho \leqq \theta \leqq 1$, $0 \leqq \bar\theta \leqq \theta$, $\sigma < \bar\theta$,

$$\begin{aligned} &t^{-\sigma} \|D^\alpha(M(u_h) - M(u))\|_q \\ &\quad \leqq c t^{-\sigma} \|D^\alpha u_h - D^\alpha u\|_{\mu,q'} \|u_h\|^\rho_{1,r} + c t^{-\sigma} t^{\bar\theta} \|u\|^{\bar\theta}_{1,r} \|u - u_h\|^{\theta-\bar\theta}_{1,r} \|D^\alpha u\|_{\sigma+\mu,q'} \end{aligned} \tag{III.7}$$

which holds by Hölder's inequality if

$$\frac{1}{q} \geqq \frac{1}{q'} - \frac{\mu}{n} + \rho\left(\frac{1}{r} - \frac{1}{n}\right), \tag{III.8a}$$

$$\frac{1}{q} \geqq \frac{1}{q'} - \frac{\sigma+\mu}{n} + \bar\theta\frac{1}{r} + (\theta - \bar\theta)\left(\frac{1}{r} - \frac{1}{n}\right). \tag{III.8b}$$

Since (III.5) holds, and since (III.8a) is equivalent with

$$\rho\frac{n-2-2\varepsilon}{2n} \leqq 2\delta_q + \frac{\mu}{n} = \frac{2(1-\mu)}{n+1} + \frac{\mu}{n} = \frac{2n-(n-1)\mu}{n(n+1)}$$

(III.8a) holds. In addition, if $\sigma \geqq \rho$, $\bar\theta = \theta - \sigma > 0$ sufficiently small, then (III.8b) holds if (III.6) is satisfied. On the other hand, if $\sigma < \rho$, we may take $\theta = \rho$ and $\bar\theta$

$-\sigma>0$ so small that the bound

$$\rho<\frac{4}{n-2-2\varepsilon}\cdot\frac{n-\frac{1}{2}(n-1)\mu}{n+1}$$

implies that

$$\frac{1}{q}\geqq\frac{1}{q'}-\frac{\mu}{n}+\rho\left(\frac{1}{r}-\frac{1}{n}\right)+(\bar{\theta}-\sigma)\frac{1}{n}$$

so that (III.8b) holds. Thus, squaring (III.7) and integrating over (0,1) against dt/t we obtain by Lemma 3 and the usual inclusions between Besov- and Sobolev spaces

$$\|M(u)\|_{B_q^{\frac{1}{2}+\sigma,2}}\leqq c\,\|u\|_{B_{q'}^{\frac{1}{2}+\mu+\sigma,2}}[\|u\|_{1,r}^{\rho}+\|u\|_{1,r}].$$

The L^p-L^q-estimates for the solution of (H) now implies the statement of Lemma III.6. ■

Corollary III.6.1. *Assume that* $u\in L^{\infty}_{\mathrm{loc}}(\mathbb{R}_+;H^{1,q'})$ *and that* $\delta_q(n+1)=1$. *If* $\rho<4/(n-2)$, *then* $u\in L^{\infty}_{\mathrm{loc}}(\mathbb{R}_+;H^{1+\sigma,q'})$ *for any* $\sigma<\sigma_0$, *where* σ_0 *is the first positive zero of*

$$x^2-x\left(\frac{n-4}{2}+\frac{1}{n+1}\right)+2\frac{1}{n+1}\geqq 0.$$

Proof. Since we may take $\varepsilon=n/(n+1)$ and $\mu=0$ in Lemma III.6, we find that

$$\kappa_n=\frac{4}{n-2-2\varepsilon}\frac{n}{n+1}=\frac{4}{n-2}\frac{1}{1-\frac{2}{n-2}\frac{n}{n+1}}\frac{n}{n+1}=\frac{4}{n-2}\frac{1}{1+\frac{1}{n}-\frac{2}{n-2}}>\frac{4}{n-2}.$$

Hence the bound $\rho<4/(n-2)$ implies the conclusion of Lemma III.6. Since $\kappa_n>\rho$ we may choose $\sigma<x_0$ in Lemma III.6 such that $x_0\geqq\rho$, and such that

$$x_0\leqq\frac{4}{n-2-2\varepsilon}\frac{n}{n+1},\qquad \varepsilon\geqq\frac{n}{n+1}.$$

Replacing $H^{1,r}$ by $H^{1+x_\nu,q'}$ and putting $\varepsilon=\varepsilon_\nu=n/(n+1)+x_\nu$, we obtain the iterative formula

$$x_{\nu+1}=\frac{4}{n-2-2\varepsilon_\nu}\frac{n}{n+1},\qquad \varepsilon_\nu=\frac{n}{n+1}+x_\nu,\qquad \nu\geqq 0,$$

and so the corollary follows form the monotonicity of the map $x_\nu\to x_{\nu+1}$.

Corollary III.6.2. *If* $6\leqq n\leqq 9$ *and if* $\rho<4/(n-2)$, *then* $u\in L^{\infty}_{\mathrm{loc}}(\mathbb{R}_+;H^{2,q'})$.

Proof. Since $\sigma_0=B-\sqrt{B^2-C}$, where $B=(n-4)/4+\frac{1}{2}/(n+1)$ and $C=2n/(n+1)$ in Corollary III.6.1, we find that $\sigma_0=1$ for the values of n between 6 and 9. In addition, $u\in L^{\infty}_{\mathrm{loc}}(\mathbb{R}_+;H^{2,q'})$ if $u\in L^{\infty}_{\mathrm{loc}}(\mathbb{R}_+;H^{1,r})$ where $1/r-1/n\leqq 2/(n+1)$, by Lemma 6.1 in [2]. By the above consequence of Corollary III.6.1, we only have to check that $\frac{1}{2}-(2+n/(n+1))/n<2/(n+1)$, i.e. $n\leqq 10-\frac{6}{n+1}$, which holds for $n\leqq 9$. ■

Lemma III.7. *Let* $6 \leqq n \leqq 9$. *Assume that* $u \in L^{\infty}_{\mathrm{loc}}(\mathbb{R}_+; H^{2+\varepsilon,q'})$, $\varepsilon \geqq 0$, *and that* $\rho < 4/(n-2)$. *If in addition*

$$s < \frac{4}{n-(5+2\varepsilon)-(4+2\varepsilon)\frac{1}{n}} - 1, \tag{III.9}$$

then $u \in L^{\infty}_{\mathrm{loc}}(\mathbb{R}_+; H^{2+s,q'})$, *where* $\delta_q(n+1)=1$, $\delta_q = 1/2 - 1/q'$.

Remark. As proved below, $\varepsilon = 0$ implies that we can choose $s>0$ in (III.9) for $6 \leqq n \leqq 9$. In fact, for these values of n, and for $0 \leqq \varepsilon \leqq 1$, ε is smaller than the right hand side of (III.9), that is we may for any ε, $0 \leqq \varepsilon \leqq 1$, choose $s > \varepsilon$ satisfying (III.9). ■

Proof. As in the proof of Lemma III.4, we first prove an $L^p - L^q$-inequality for $M(u)$. We have, $|\alpha| = |\beta| = 1$,

$$\begin{aligned} &D^{\alpha+\beta}(M(u_h) - M(u)) \\ &\quad = (M''(u_h) D^{\alpha} u_h D^{\beta} u_h - M''(u) D^{\alpha} u D^{\beta} u) + (M'(u_h) D^{\alpha+\beta} u_h - M'(u) D^{\alpha+\beta} u) \\ &\quad = I + II. \end{aligned}$$

By assumption

$$|M''(u_h)| \leqq c \quad \text{and} \quad |M''(u_h) - M''(u)| \leqq c|u-u_h|^{\bar{\theta}}, \qquad 0 \leqq \bar{\theta} \leqq 1$$

and hence, if we estimate I by

$$\begin{aligned} |I| &\leqq |(M''(u_h) - M''(u)) D^{\alpha} u_h D^{\beta} u_h| + |M''(u)| \, |D^{\alpha} u_h D^{\beta} u_h - D^{\alpha} u D^{\beta} u| \\ &\leqq c|u-u_h|^{\theta} |D^{\alpha} u_h| \, |D^{\beta} u_h| + c|D^{\alpha}(u_h - u)| \, |D^{\beta} u_h| + |D^{\alpha} u| \, |D^{\beta}(u_h - u)|, \end{aligned}$$

we get by Hölder's inequality

$$\begin{aligned} & t^{-s} \|D^{\alpha+\beta}(M(u_h) - M(u))\|_q \\ &\quad \leqq c t^{-s} \|u_h - u\|^{\bar{\theta}}_{1+\varepsilon,q'} \|D^{\alpha} u\|_{1+s,q'} \|D^{\beta} u\|_{1+\varepsilon,q'} \\ &\qquad + c t^{-s} \|D^{\alpha}(u_h - u)\|_{1,q'} \|D^{\beta} u_h\|_{1+\varepsilon,q'} \\ &\qquad + c t^{-s} \|D^{\beta}(u_h - u)\|_{1,q'} \|D^{\alpha} u\|_{1+\varepsilon,q'} \\ &\qquad + c t^{-s} \|u_h - u\|^{\theta}_{1+\varepsilon,q'} \|D^{\alpha+\beta} u\|_{s,q'} \\ &\qquad + c t^{-s} \|u_h\|^{\rho}_{1+\varepsilon,q'} \|D^{\alpha+\beta}(u_h - u)\|_{q'} \end{aligned} \tag{III.10}$$

provided

$$\frac{1}{q} \geqq \bar{\theta}\left(\frac{1}{q} - \frac{1+\varepsilon}{n}\right) + \frac{1}{q'} - \frac{1+\varepsilon}{n} + \frac{1}{q'} - \frac{1+s}{n} \tag{III.11a}$$

$$\frac{1}{q} \geqq \frac{1}{q'} - \frac{1}{n} + \frac{1}{q'} - \frac{1+\varepsilon}{n} \tag{III.11b}$$

$$\frac{1}{q} \geqq \theta\left(\frac{1}{q'} - \frac{1+\varepsilon}{n}\right) + \frac{1}{q'} - \frac{s}{n'}, \qquad 1 \geqq \theta \geqq \rho, \tag{III.11c}$$

$$\frac{1}{q} \geqq \rho\left(\frac{1}{q'} - \frac{1+\varepsilon}{n}\right) + \frac{1}{q'}. \tag{III.11d}$$

Now, (III.11 a) is equivalent with

$$\left(\frac{1}{q'}-\frac{2+\varepsilon}{n}\right)\leqq 2\delta-\left(\frac{1}{q'}-\frac{2+\varepsilon}{n}\right)-(\bar{\theta}-s)\left(\frac{1}{q'}-\frac{1+\varepsilon}{n}\right),$$

that is, equivalent with (notice that $\delta(n+1)=1$!)

$$s+1\leqq\frac{4}{n-(5+2\varepsilon)-(4+2\varepsilon)/n}-(\bar{\theta}-s)\left[1+\frac{2}{n-(5-2\varepsilon)-(4+2\varepsilon)/n}\cdot\frac{n+1}{n}\right].$$

Thus, if $\bar{\theta}>s$ and $\bar{\theta}-s$ is sufficiently small, (III.9) implies that (III.11 a) holds. In addition, the right hand side of (III.9) is decreasing in n, and for $n=9$,

$$\varepsilon<\frac{4}{n-(5+2\varepsilon)-(4+2\varepsilon)/n}-1,\quad 0\leqq\varepsilon\leqq 1,$$

as is easily checked by straightforward computations. We find in particular that the right hand side in (III.9) is positive for $n\in[6,9]$ and $\varepsilon\in[0,1]$. Next, condition (III.11 b) means that

$$\frac{\varepsilon}{n}\geqq\frac{1}{2}-3\delta-\frac{2}{n}=\frac{1}{2}-\frac{3}{n+1}-\frac{2}{n}=\frac{n^2-9n-4}{2n(n+1)}\qquad\text{(III.11 b)}'$$

and here the right hand side of (III.11 b)′ is <0 for $6\leqq n\leqq 9$. Hence (III.11 b) holds for all $\varepsilon\geqq 0$. Since $0\leqq s<\theta$ and $\rho<\theta\leqq 1$, (III.11 c) follows from the fact that for $6\leqq n\leqq 9$, strict inequality holds in (III.11 b) and from (III.11 d). Merely observe that by (III.11 b) and (III.11 d)

$$\frac{1}{q}\geqq\theta\left[\frac{1}{q'}-\frac{1+\varepsilon}{n}\right]+\frac{1}{q'}-\frac{\theta}{n},\qquad\rho\leqq\theta\leqq 1$$

where we choose $\theta=\rho$ if $s<\rho$ and $\theta-s$ small if $s\geqq\rho$. Finally we recognize (III.11 d) as (III.8 a) with $\mu=0$ and $1/r-1/n\geqq 1/q'-(1+\varepsilon)/n$ and this inequality certainly holds for $\varepsilon\geqq 0$ if $\rho<4/(n-2)$, as proved in Lemma III.6. Thus, as in Lemma III.6, we obtain from (III.10)

$$\|M(u)\|_{B_q^{2+s,2}}\leqq c(\|u\|_{2+\varepsilon,q'}+1)\|u\|_{B_{q'}^{2+s,2}}$$

and so Lemma III.7 follows from the L^p-L^q-estimates for the wave-equation. ■

Corollary III.7.1. *Let $6\leqq n\leqq 9$. Assume that $\rho<4/(n-2)$ and that $u\in L^\infty_{\mathrm{loc}}(\mathbb{R}_+;H^{2,q'})$. Then $u\in L^\infty_{\mathrm{loc}}(\mathbb{R}_+;H^{2+\sigma,q'})$ for any $\sigma<\sigma_1$, where σ_1 is the first positive zero of*

$$x^2-x\frac{n}{2(n+1)}\left(n-7-\frac{6}{n}\right)+\frac{1}{2}\frac{n}{n+1}\left(9+\frac{4}{n}-n\right)\geqq 0.$$

In particular, $\sigma_1=1$ for $n=6$, 7, 8 and 9.

Proof. As showed in the proof of Lemma III.7, the right hand side of (III.9) is $>\varepsilon$ and so applying the argument of Lemma III.7 recursively, replacing ε by s in each step, we obtain the corollary. ■

As an immediate consequence of Corollary III.7.1 and Corollary III.6.2, we now obtain:

Theorem III.1. *If $n \leqq 9$ and $\rho < 4/(n-2)$, then (H) has a global classical solution.*

Proof. By Lemma III.2, it is enough to prove that $u \in L^{\infty}_{\mathrm{loc}}(\mathbb{R}_+; L^r)$ for some $r \geqq n+1$. We may also restrict ourselves to the case $n \geqq 6$, since the result is already known for $n \leqq 6$ (see [2], and also [5] in the case of $A = -\Delta$). But then Corollaries III.6.1 and III.7.1 imply that we only have to prove that

$$\frac{1}{n+1} > \frac{1}{2} - \delta_q \frac{3}{n} = \frac{1}{q'} - \frac{3}{n},$$

which, since $\delta_q(n+1) = 1$, is equivalent to

$$n < 10 - \frac{4}{n+1}$$

which certainly holds for $n \leqq 9$. The proof of the theorem is then completed. ■

If we apply Lemma III.5, 6 and Corollary III.6.1 we find that for $n \geqq 10$ the solution u of (H) belongs to $L^{\infty}_{\mathrm{loc}}(\mathbb{R}_+; H^{1+\sigma, q'})$ for some $\sigma > 4n/(n-2)(n+1) > \rho$. Then Lemma III.6 with $q = q' = 2$ and $\mu = 1$ (Notice that we may take $\varepsilon > 1$ in (III.5), by the above) implies the following result:

Lemma III.8. *Let $\rho \leqq 2/(n-4)$ and assume that $\sigma < \sigma_2$, where σ_2 is the first positive zero of*

$$x^2 - \frac{n-4}{2} x + 1 \geqq 0.$$

Then $u \in L^{\infty}_{\mathrm{loc}}(\mathbb{R}_+; H^{2+\sigma, 2})$.

The proof should by now be obvious. As a consequence we thus obtain the following result by an application of Theorem II.10:

Theorem III.2. *Assume that $\rho < 2/(n-4)$ and $\sigma < 2/(n-4)$. If in addition to 1° to 4° we assume that*

5° $|M''(u)| \leqq c(1+|u|)^{\rho-1}$,

then (H) *has a global strong solution which belongs to $L^{\infty}_{\mathrm{loc}}(\mathbb{R}_+; H^{2+\sigma, 2})$.*

Remark. The result of Theorem III.2 applies also to certain initial-boundary value problems where the boundary is convex with respect to the operator A. This is a consequence of the fact that the $L^p - L^q$-estimates hold also for the solution of such a problem. ■

If in the proof of Lemma III.6 we assume that $u \in L^{\infty}_{\mathrm{loc}}(\mathbb{R}_+; H^{1+\bar{\sigma}, r})$ where $1/r - (1+\bar{\sigma})/n = 1/2 - (2+\varepsilon)/n$, and that $\sigma < \bar{\sigma}$ where $\sigma < \varepsilon$ is small, then we obtain as in the proof of (III.7),

$$\begin{aligned} &t^{-\sigma} \| D^{\alpha}(M(u_h) - M(u)) \|_2 \\ &\leqq c t^{-\sigma} \| D^{\alpha}(u_h - u) \|_{\bar{\sigma}-\sigma, r} \| u_h \|^{\rho}_{1+\bar{\sigma}, r} + c t^{-\sigma} t^{\bar{\theta}} \| u \|^{\bar{\theta}}_{1+\bar{\sigma}, r} \| u \|^{\theta-\bar{\theta}}_{1+\bar{\sigma}, r} \| D^{\alpha} u \|_{\bar{\sigma}, r} \end{aligned}$$

which holds provided $\bar{\sigma}-\tilde{\sigma}\geqq 0$ is small and

$$\frac{1+\varepsilon-\sigma}{n}>\rho\left(\frac{1}{r}-\frac{1}{n}\right)=\rho\left(\frac{1}{2}-\frac{2+\varepsilon}{n}\right),$$

$$\frac{1+\varepsilon}{n}\geqq\bar{\theta}\frac{1}{r}+(\theta-\bar{\theta})\left(\frac{1}{r}-\frac{1}{n}\right).$$

If we choose $\bar{\theta}-\sigma$ small and $\theta=\rho$ we find that if

$$\rho<\frac{2(1+\varepsilon-\sigma)}{n-4-2\varepsilon} \tag{III.12}$$

then these inequalities are valid. As in Lemma III.6, we may draw the conclusion that $u\in L^{\infty}_{\mathrm{loc}}(\mathbb{R}_+;H^{2+\sigma,2})$ for some $\sigma>0$ if (III.12) holds. In addition, we obtain easily the following table of values of ε using Lemma III.5 and Corollary III.6.1. In this table we have also computed the bound for ρ given by (III.12).

n	$\varepsilon>$	ρ-bound	$\frac{4}{n-2}\cdot\frac{n}{n+1}$	$\frac{4}{n-2}$
10	0,699	0,74	0,45	0,50
11	0,52	0,50	0,41	0,44
12	0,40	0,39	0,37	0,40
13	0,34	0,32	0,33	0,36

Thus we may draw the following additional conclusion:

Theorem III.3. *Assume that 5° holds and that* $\rho<\frac{4}{n-2}$ *for* $n=10$ *and* $\rho<\frac{4}{n-2}\frac{n}{n+1}$ *for* $n=11$, 12. *Then* $u\in L^{\infty}_{\mathrm{loc}}(\mathbb{R}_+;H^{2+\sigma,2})$ *for some* $\sigma>0$. *In general, this conclusion is valid for all n provided* (III.12) *holds where* $\varepsilon<\sigma_0-1/(n+1)$ *and* σ_0 *is the first positive zero of*

$$x^2-x\left(\frac{n-4}{2}+\frac{1}{n+1}\right)+2\frac{n}{n+1}.$$

In particular, this is the case if $\rho<\frac{2}{n-4}\left(1+\frac{3}{n-2}\right)$.

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