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Schrödinger Operators with Singular Magnetic Vector Potentials

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Dedicated to our teacher Ernst Wienholtz on his 50th birthday

1. Introduction

In the present paper we consider formal Schrödinger operators

$$\mathscr{H} = -(\nabla - i\mathbf{a})^2 + q$$

with real-valued a_j and q, and various selfadjoint realizations of $\mathscr{H}$ in $L^2(\mathbb{R}^m)$. We are mainly concerned with the proof of two facts.

First, if we assume

$$\text{(C.1)} \qquad \mathbf{a} = (a_1, \ldots, a_m) \in L^2_{\text{loc}}(\mathbb{R}^m)^m, \quad 0 \leqq q \in L^1_{\text{loc}}(\mathbb{R}^m),$$

we give a new proof (Theorem 1) of the fact, that $C_0^\infty(\mathbb{R}^m)$ is a core of the maximal form associated with $\mathscr{H}$.

Second, if in addition

$$\text{(C.2)} \qquad \mathbf{a} = (a_1, \ldots, a_m) \in L^4_{\text{loc}}(\mathbb{R}^m)^m, \quad \operatorname{div} \mathbf{a} \in L^2_{\text{loc}}(\mathbb{R}^m), \quad 0 \leqq q \in L^2_{\text{loc}}(\mathbb{R}^m)$$

is required, then we prove (Theorem 2) the essential selfadjointness of $\mathscr{H}$ on $C_0^\infty(\mathbb{R}^m)$. Observe that Condition (C.2) is minimal (with respect to $\mathbf{a}$) to assure that $\mathscr{H}$ defines an operator from $C_0^\infty(\mathbb{R}^m)$ to $L^2(\mathbb{R}^m)$.

In a recent paper Kato [5] proved that the minimal operator associated with the form corresponding to $\mathscr{H}$ and a certain intermediate operator coincide. If in addition $\mathbf{a} \in L^p_{\text{loc}}(\mathbb{R}^m)^m$ for some $p > m$, then the minimal and the maximal operator coincide, too ([5, p. 106, Theorem III]). By using a comparison theorem for semigroups, based on a generalization of Kato's inequality, Simon [19] was able to prove last statement assuming only (C.1). In our proof of Theorem 1 we use cut-off arguments (as in [16]) to show that $\mathscr{D}(\mathscr{h}) \cap L^\infty(\mathbb{R}^m)$ is a form core of the maximal form $\mathscr{h}$ corresponding to $\mathscr{H}$. Again with truncation methods we prove an L^∞-a-priori estimate (Lemma 4) as well as a comparison theorem (Lemma 6; an alternative proof is given in Lemma 10).

Concerning the essential selfadjointness of $\mathscr{H} \mid C_0^\infty(\mathbb{R}^m)$ the vector potential $\mathbf{a}$ was assumed to be C^1 uptil recently (compare [3, 16, 20]). To our knowledge

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Jörgens [4] was the first who considered potentials $\mathbf{a}$ satisfying certain Stummel-type conditions. More sophisticated conditions of this type had been studied extensively by Schechter [14, 15].

Further results concerning singular vector potentials $\mathbf{a}$ are due to Simon [18, 19]; he assumes (C.2) for $m \leqq 4$, but in addition for $m > 4$ he has to require either $\mathbf{a} \in L^p_{\text{loc}}(\mathbb{R}^m)^m$ with $p > m$ or $\mathbf{a} \in L^p_{\text{loc}}(\mathbb{R}^m)^m$, $\operatorname{div} \mathbf{a} + q \in L^{p/2}_{\text{loc}}(\mathbb{R}^m)$ with $p = \frac{6m}{m+2}$ (see also [12]).

This dependance on dimension comes in by the use of certain Sobolev inequalities and elliptic regularity theory, not allowing to show the essential selfadjointness of $\mathscr{H}$ on $C_0^\infty(\mathbb{R}^m)$ under condition (C.2). But it was Simon [19, Conjecture, p. 38] who suggested that condition (C.2) is sufficient (and of course necessary) for the essential selfadjointness of $\mathscr{H} \mid C_0^\infty(\mathbb{R}^m)$. Our proof of Simon's conjecture avoids those methods mentioned above and is mainly based on the observation of $u \in W^2 \cap L^\infty$ implies $\nabla u \in (L^4)^m$ (Lemma 7), which turned out to be a special case of a more general Gagliardo-Nirenberg inequality [1, 11]. Exactly this observation enables us to show $(H+1)^{-1}(L^2 \cap L^\infty) \subset W^2_{\text{loc}}$ (Lemma 9). Finally we prove the essential selfadjointness of $\mathscr{H}$ on $C_0^\infty(\mathbb{R}^m)$ if (C.2) holds with respect to $\mathbf{a}$ and if $q = q_1 + q_2$, $q_1, q_2 \in L^2_{\text{loc}}(\mathbb{R}^m)$, $q_1(x) \geqq -c|x|^2$ and $0 \geqq q_2$ is Δ-bounded with relative bound smaller than one. In the proof we follow the ideas of [8, 16].

Since truncation methods (with respect to the range of a function) seem to be non-standard in studying selfadjointness problems, we sketch those proofs in the appendix. Finally we like to mention, that Kato's famous distributional inequality is a special case of a more general distributional inequality which may be derived by using the chain rule (compare [17]).

2. Preliminaries

Let $\mathbb{R}^m$ be the m-dimensional Euclidean space, represent points of $\mathbb{R}^m$ by $\mathbf{x} = (x_1, \ldots, x_m)$ and let $|\mathbf{x}| = \left(\sum_{j=1}^m x_j^2\right)^{\frac{1}{2}}$. For $1 \leqq p \leqq \infty$ let $L^p(\mathbb{R}^m)$ stand for the space of (equivalence classes of) complex-valued functions u which are measurable and satisfy $\int |u|^p < \infty$ if $p < \infty$ and $\|u\|_\infty = \operatorname{ess\,sup} |u| < \infty$ if $p = \infty$. In case $p = 2$, $L^2(\mathbb{R}^m)$ is a complex Hilbert space with scalar product $(u, v) = \int \bar{u} v$ and corresponding norm $\|u\| = (u, u)^{\frac{1}{2}}$.

Similarily $L^2(\mathbb{R}^m)^m$, the m-fold cartesian product of $L^2(\mathbb{R}^m)$, is equipped with the scalar product $(\mathbf{u}, \mathbf{v}) = \sum_{j=1}^m (u_j, v_j)$ and the norm $\|\mathbf{u}\| = (\mathbf{u}, \mathbf{u})^{\frac{1}{2}}$. Let Ω be a measurable subset of $\mathbb{R}^m$ and let $L^4(\Omega)^m$ be equipped with the norm $\|\mathbf{u}\|_{L^4(\Omega)} = (\int_\Omega |\mathbf{u}|^4)^{\frac{1}{4}}$. For $A \subset \mathbb{R}^m$ let us denote by χ_A the characteristic function of the set A.

The space of infinitely differentiable complex-valued functions with compact support will be denoted by $C_0^\infty(\mathbb{R}^m)$ or $\mathscr{D}(\mathbb{R}^m)$. $\mathscr{D}'(\mathbb{R}^m)$ is the space of distributions on $\mathbb{R}^m$. For $1 \leqq j \leqq m$ let $\partial_j = \partial/\partial x_j$ be the j-th partial derivative, each acting on $\mathscr{D}'(\mathbb{R}^m)$. For $n \in \mathbb{N}$, $1 \leqq p < \infty$, $\Omega \subset \mathbb{R}^m$ open, the Sobolev space $W^{n,p}(\Omega)$

is defined as the space of those $u \in L^p(\Omega)$, for which all partial derivatives up to order n are in $L^p(\Omega)$. $W^{n,p}(\Omega)$ is a Banach space with the norm $\|u\|_{n,p} = (\sum_{|\alpha| \leqq n} \int_\Omega |\nabla^\alpha u|^p)^{1/p}$, where $\alpha \in \mathbb{N}_0^m$, $|\alpha| = \sum_{j=1}^m \alpha_j$, $\nabla = (\partial_1, \ldots, \partial_m)$ and $\nabla^\alpha = \prod_{j=1}^m \partial_j^{\alpha_j}$. If $p = 2$, we always omit the index p, so e.g. $W^{n,2}(\Omega) = W^n(\Omega)$, $\|u\|_{n,2} = \|u\|_n$.

We call a linear subspace $F \subset \mathscr{D}'(\mathbb{R}^m)$ semi-local, if $\varphi u \in F$ for all $\varphi \in C_0^\infty(\mathbb{R}^m)$, $u \in F$ and define F_{loc} as the space of those $u \in \mathscr{D}'(\mathbb{R}^m)$ such that $\varphi u \in F$ when $\varphi \in C_0^\infty(\mathbb{R}^m)$. If in addition F is normed, then $u_n \to u$ in F_{loc} means $\varphi u_n \to \varphi u$ in F when $\varphi \in C_0^\infty(\mathbb{R}^m)$. Let $\Delta = \sum_{j=1}^m \partial_j^2$ be the Laplacian, acting on $\mathscr{D}'(\mathbb{R}^m)$, and let $\mathbf{D}u = \nabla u - i\mathbf{a}u \in \mathscr{D}'(\mathbb{R}^m)^m$, $\operatorname{div} \mathbf{b} = \sum_{j=1}^m \partial_j b_j \in \mathscr{D}'(\mathbb{R}^m)$ with $i = \sqrt{-1}$, $\mathbf{a} \in L^2_{\mathrm{loc}}(\mathbb{R}^m)^m$, $\mathbf{b} \in L^1_{\mathrm{loc}}(\mathbb{R}^m)^m$ and $u \in L^2_{\mathrm{loc}}(\mathbb{R}^m)$.

Concerning notations and results in the theory of linear operators in Hilbert space we refer to [6].

3. Uniqueness of Schrödinger Forms

Throughout this section we assume Condition (C.1), that is

$$\mathbf{a} \in L^2_{\mathrm{loc}}(\mathbb{R}^m)^m, \quad 0 \leqq q \in L^1_{\mathrm{loc}}(\mathbb{R}^m).$$

We consider the maximal form

$$\mathscr{Q}(\mathscr{h}) = \{u \in L^2(\mathbb{R}^m) \mid \mathbf{D}u \in L^2(\mathbb{R}^m)^m,\ q^{\frac{1}{2}} u \in L^2(\mathbb{R}^m)\} \tag{3.1}$$

$$\begin{aligned} \mathscr{h}(u,v) &= (\mathbf{D}u, \mathbf{D}v) + (q^{\frac{1}{2}}u, q^{\frac{1}{2}}v) \\ &= \sum_{j=1}^m (\partial_j u - i a_j u, \partial_j v - i a_j v) + (q^{\frac{1}{2}}u, q^{\frac{1}{2}}v) \end{aligned} \tag{3.2}$$

associated to the formal Schrödinger operator

$$\mathscr{H} = -\mathbf{D}^2 + q = -(\nabla - i\mathbf{a})^2 + q = -\sum_{j=1}^m (\partial_j - i a_j)^2 + q. \tag{3.3}$$

Clearly $C_0^\infty(\mathbb{R}^m) \subset \mathscr{Q}(\mathscr{h})$, but since [5, 19] it is also known, that $C_0^\infty(\mathbb{R}^m)$ is dense with respect to $|||u||| = [\mathscr{h}(u,u) + (u,u)]^{\frac{1}{2}}$.

This means, that the minimal form $\mathscr{h}_{\min}$, defined as the form closure of $\mathscr{h} \mid C_0^\infty(\mathbb{R}^m) \times C_0^\infty(\mathbb{R}^m)$, is the same as the maximal form $\mathscr{h}$.

Here we like to give a new proof of this fact using only well known truncation methods in $W^1(\mathbb{R}^m)$ ([2], see also the appendix). In Lemma 1 let us first summarize some simple facts about $\mathscr{h}$.

Lemma 1. *$\mathscr{h}$ is a symmetric closed form; hence there exists a unique selfadjoint operator H satisfying*

$$\mathscr{D}(H) = \{u \in \mathscr{Q}(\mathscr{h}) \mid \mathscr{h}(u, \cdot) \in L^2(\mathbb{R}^m)'\} \tag{3.4}$$

$$(Hu, v) = \mathscr{h}(u,v) \quad \textit{for } u \in \mathscr{D}(H),\ v \in \mathscr{Q}(\mathscr{h}). \tag{3.5}$$

Proof. Suppose $(u_n) \subset \mathcal{D}(\mathscr{h})$ and $|||u_n - u_l||| \to 0$ as $n, l \to \infty$. Then there exist $u, v \in L^2(\mathbb{R}^m)$, $\mathbf{w} \in L^2(\mathbb{R}^m)^m$ satisfying

$$u_n \to u, \quad q^{\frac{1}{2}} u_n \to v, \quad D_j u_n \to w_j \quad \text{in } L^2(\mathbb{R}^m) \quad (1 \leqq j \leqq m).$$

Clearly $v = q^{\frac{1}{2}} u$ and since $D_j u_n \to D_j u$ in $\mathscr{D}'(\mathbb{R}^m)$, we have $\mathbf{D}u = \mathbf{w}$, hence $u \in \mathcal{D}(\mathscr{h})$ and $|||u_n - u||| \to 0$ as $n \to \infty$. Thus $\mathscr{h}$ is a closed form, which is obviously symmetric. Applying the first representation theorem for symmetric forms [6; VI, Theorem 2.6], Lemma 1 is shown. □

Lemma 2. *$\mathcal{D}(\mathscr{h}) \cap L^\infty(\mathbb{R}^m)$ is dense in $\mathcal{D}(\mathscr{h})$ with respect to $|||\cdot|||$.*

Proof. Take $u \in \mathcal{D}(\mathscr{h})$, then due to $\mathbf{a}u \in L^1_{\text{loc}}(\mathbb{R}^m)^m$, $\mathbf{D}u \in L^2(\mathbb{R}^m)^m$, we have $u \in W^{1,1}_{\text{loc}}(\mathbb{R}^m)$, hence [7, 13] (see also appendix, Corollary 1)

$$\partial_j |u| = \operatorname{Re}\left(\frac{\bar{u}}{|u|} D_j u\right) \quad \text{in } \mathscr{D}'(\mathbb{R}^m) \quad (1 \leqq j \leqq m). \tag{3.6}$$

This implies $|\nabla |u|| \leqq |\mathbf{D}u|$ a.e. in $\mathbb{R}^m$, so $|u| \in W^1(\mathbb{R}^m)$. For an integer n consider now the Lipschitz-continuous function

$$\varphi_n(t) = \begin{cases} 1, & t \leqq n \\ \dfrac{n}{t}, & t > n \end{cases}$$

Clearly

$$\varphi_n'(t) = 0 \quad \text{if } t < n, \qquad 0 \leqq \varphi_n \leqq 1$$
$$|t\,\varphi_n(t)| \leqq n, \qquad |t\,\varphi_n'(t)| \leqq 1.$$

Since $|u| \in W^1(\mathbb{R}^m)$, we have ([2; Theorem 7.8], see appendix) $\varphi_n(|u|) \in W^1(\mathbb{R}^m)$ and $\nabla[\varphi_n(|u|)] = \varphi_n'(|u|)\, \nabla |u|$ a.e. in $\mathbb{R}^m$.

Define $u_n = u\,\varphi_n(|u|)$, then $u_n \in L^2(\mathbb{R}^m) \cap L^\infty(\mathbb{R}^m)$ and

$$\nabla u_n = \nabla u\, \varphi_n(|u|) + u\, \varphi_n'(|u|)\, \nabla |u| \quad \text{in } \mathscr{D}'(\mathbb{R}^m)^m,$$

hence $\mathbf{D}u_n = \mathbf{D}u\, \varphi_n(|u|) + u\, \varphi_n'(|u|)\, \nabla |u|$ in $\mathscr{D}'(\mathbb{R}^m)^m$.

Now we have pointwise a.e. in $\mathbb{R}^m$

$$|\mathbf{D}u_n - \mathbf{D}u| \leqq \chi_{\{|u| \geqq n\}} (|\mathbf{D}u| + |\nabla |u||)$$
$$|q^{\frac{1}{2}} u_n - q^{\frac{1}{2}} u| \leqq \chi_{\{|u| \geqq n\}} |q^{\frac{1}{2}} u|$$
$$|u_n - u| \leqq \chi_{\{|u| \geqq n\}} |u|.$$

Since $u \in \mathcal{D}(\mathscr{h})$, $\nabla |u| \in L^2(\mathbb{R}^m)^m$, we conclude $u_n \in \mathcal{D}(\mathscr{h})$ and $|||u_n - u||| \to 0$ as $n \to \infty$. □

Whereas Lemma 2 covers the main work of our proof of "$\mathcal{D}(\mathscr{h})$ is a core of $\mathscr{h}$" we follow now well known arguments in [5, 19]. We repeat these arguments (Lemma 3, Theorem 1) in order to keep our paper self-contained.

Lemma 3. *$\mathcal{D}(\mathscr{h})$ is semi-local, that is $\varphi u \in \mathcal{D}(\mathscr{h})$ when $\varphi \in C_0^\infty(\mathbb{R}^m)$, $u \in \mathcal{D}(\mathscr{h})$. The linear subspace $\mathscr{C}_0 = \{\varphi u \mid \varphi \in C_0^\infty(\mathbb{R}^m),\ u \in \mathcal{D}(\mathscr{h}) \cap L^\infty\}$ of $\mathcal{D}(\mathscr{h})$ is dense in $\mathcal{D}(\mathscr{h})$ with respect to $|||\cdot|||$.*

Proof. Let $\varphi \in C_0^\infty(\mathbb{R}^m)$ and $u \in \mathcal{D}(\mathscr{h})$, then $u \in W_{\mathrm{loc}}^{1,1}(\mathbb{R}^m)$ and so $\varphi u \in W^{1,1}(\mathbb{R}^m)$. Now we have in the distributional sense

$$\mathbf{D}(\varphi u) = \mathbf{D}u\,\varphi + u\nabla\varphi \in L^2(\mathbb{R}^m)^m. \tag{3.7}$$

This together with $\|q^{\frac{1}{2}} u\varphi\| \leqq \|q^{\frac{1}{2}} u\| \cdot \|\varphi\|_\infty$ implies $\varphi u \in \mathcal{D}(\mathscr{h})$.

In addition let us now suppose $u \in L^\infty(\mathbb{R}^m)$, $\varphi = 1$ in a neighbourhood of the origin and put $\varphi_n = \varphi\left(\frac{\cdot}{n}\right)$. Then $q^{\frac{1}{2}}\varphi_n u \to q^{\frac{1}{2}} u$ and (due to (3.7)) $\mathbf{D}(\varphi_n u) \to \mathbf{D}u$ in L^2, hence $|||\varphi_n u - u||| \to 0$ as $n \to \infty$. In view of Lemma 2 this shows that $\mathscr{C}_0$ is dense in $\mathcal{D}(\mathscr{h})$. □

Theorem 1. *$C_0^\infty(\mathbb{R}^m)$ is dense in $\mathcal{D}(\mathscr{h})$ with respect to $|||\cdot|||$, i.e. $C_0^\infty(\mathbb{R}^m)$ is a form core of H.*

Proof. In view of Lemma 3, $\mathscr{C}_0$ is dense in $\mathcal{D}(\mathscr{h})$. Take $u \in \mathscr{C}_0$, then $u \in W^1(\mathbb{R}^m)$, since $\mathbf{a} \in L^2_{\mathrm{loc}}(\mathbb{R}^m)^m$, $u \in L^\infty(\mathbb{R}^m)$ and $\operatorname{supp} u$ compact. Let $u_\varepsilon = J_\varepsilon u$, where J_ε is the Friedrichs mollifier, so that $u_\varepsilon \in C_0^\infty(\mathbb{R}^m)$ with a common support, $\|u_\varepsilon\|_\infty \leqq \|u\|_\infty$ and $u_\varepsilon \to u$ in $W^1(\mathbb{R}^m)$ as $\varepsilon \to 0$. We have also $u_\varepsilon \to u$ a.e. pointwise along some subsequence $\varepsilon = \varepsilon_n \to 0$. Thus

$$\mathbf{D}u_\varepsilon = \nabla u_\varepsilon - i\mathbf{a}u_\varepsilon \to \nabla u - i\mathbf{a}u = \mathbf{D}u \quad \text{in } L^2(\mathbb{R}^m)^m$$

and $q^{\frac{1}{2}} u_\varepsilon \to q^{\frac{1}{2}} u$ in $L^2(\mathbb{R}^m)$ along the sequence. Hence $C_0^\infty(\mathbb{R}^m)$ is dense in $\mathscr{C}_0$ with respect to $|||\cdot|||$. □

Lemma 4. *For each $\lambda > 0$ the equation $(H+\lambda)u = f \in L^\infty(\mathbb{R}^m)$ implies $u \in L^\infty(\mathbb{R}^m)$ and*

$$\|u\|_\infty \leqq \frac{1}{\lambda}\|f\|_\infty. \tag{3.8}$$

Proof. For $r > \frac{1}{\lambda}\|f\|_\infty$ fixed, let us consider the Lipschitz-continuous function $\varphi: \mathbb{R} \to \mathbb{R}$

$$\varphi(t) = \begin{cases} 0, & t \leqq r \\ \dfrac{t-r}{t}, & t > r. \end{cases}$$

Clearly

$$0 \leqq \varphi \leqq 1, \quad |t\varphi'(t)| \leqq 1 \quad \text{and} \quad \varphi'(t) = 0 \quad \text{if } t < r. \tag{3.9}$$

Since $|u| \in W^1(\mathbb{R}^m)$ (see (3.6)), we have ([2, Theorem 7.8], see appendix).

$$\varphi(|u|) \in W^1(\mathbb{R}^m), \qquad \nabla\varphi(|u|) = \varphi'(|u|)\nabla|u|$$

and so

$$\mathbf{D}[u\varphi(|u|)] = \mathbf{D}u\,\varphi(|u|) + u\,\varphi'(|u|)\nabla|u| \quad \text{in } \mathscr{D}'(\mathbb{R}^m)^m. \tag{3.10}$$

In view of (3.9), (3.10), $u\varphi(|u|) \in \mathcal{D}(\mathscr{h})$ follows. Now

$$\mathscr{h}(u, u\varphi(|u|)) + (u, u\varphi(|u|)) = ((H+\lambda)u, u\varphi(|u|)) = (f, u\varphi(|u|))$$

is valid, which gives

$$\int |\mathbf{D}u|^2\varphi(|u|) + \sum_{j=1}^m \int \overline{D_j u}\, u\varphi'(|u|)\,\partial_j|u| + \int (q+\lambda)|u|^2\varphi(|u|) = \int \bar{f}u\,\varphi(|u|).$$

Taking the real part of both sides of this equation and using (see (3.6)) $|u|\,\partial_j|u| = \mathrm{Re}(\overline{D_j u}\,u)$ we obtain the equation

$$\int |\mathbf{D}u|^2\,\varphi(|u|)+\int |\nabla|u||^2\,\varphi'(|u|)\,|u|+\int (q+\lambda)\,|u|^2\,\varphi(|u|)=\mathrm{Re}\int \bar{f}u\,\varphi(|u|).$$

Since $\varphi\geqq 0$, $\varphi'\geqq 0$, $q\geqq 0$ we have

$$\lambda\int |u|^2\,\varphi(|u|)\leqq \mathrm{Re}\int \bar{f}u\,\varphi(|u|). \tag{3.11}$$

Now observe that $\{|u|>r\}$ has finite measure. Thus (3.11) yields

$$\lambda\int_{\{|u|>r\}} |u|^2\,\varphi(|u|)\leqq \int_{\{|u|>r\}} \|f\|_\infty\cdot|u|\,\varphi(|u|),$$

hence

$$\int_{\{|u|>r\}} |u|\,\varphi(|u|)\cdot\left(|u|-\frac{1}{\lambda}\|f\|_\infty\right)\leqq 0.$$

Since $r>\frac{1}{\lambda}\|f\|_\infty$ and $\varphi(t)>0$ if $t>r$ we conclude $|u|=0$ a.e. on $\{|u|>r\}$. This means $|u|\leqq r$ a.e. in $\mathbb{R}^m$.

Thus we get $\|u\|_\infty\leqq\frac{1}{\lambda}\|f\|_\infty$, since $r>\frac{1}{\lambda}\|f\|_\infty$ was arbitrary. □

For any $u\in\mathcal{Q}(\hbar)$ we define $\hat{H}u\in\mathcal{D}'(\mathbb{R}^m)$ by (remind (C.1))

$$\begin{aligned}\hat{H}u&=-\Delta u+i\,\mathrm{div}(\mathbf{a}u)+i\mathbf{a}\cdot\mathbf{D}u+qu\\&=-\Delta u+2i\cdot\mathrm{div}(\mathbf{a}u)+(-i\cdot\mathrm{div}\,\mathbf{a}+\mathbf{a}^2+q)\,u\end{aligned} \tag{3.12}$$

(last equation being valid if (C.2) is satisfied)

and use the notations $H=H(\mathbf{a},q)$ resp. $H_n=H(\mathbf{a}_n,q_n)$ (to be understood in the sense of (3.4–5)) in order to indicate the dependence on the potentials $\mathbf{a}$, q resp. $\mathbf{a}_n$, q_n.

The next lemma summarizes more or less well known facts [5, 19].

Lemma 5. *Let $H=H(\mathbf{a},q)$ resp. $H_n=H(\mathbf{a}_n,q_n)$ be the selfadjoint operators in Lemma* 1. *Then*

(3.13) $\mathcal{D}(H)=\{u\in\mathcal{Q}(\hbar)\,|\,\hat{H}u\in L^2(\mathbb{R}^m)\}$ *and* $Hu=\hat{H}u$ *if* $u\in\mathcal{D}(H)$,

(3.14) $\varphi u\in\mathcal{D}(H)$ *for* $\varphi\in C_0^\infty(\mathbb{R}^m)$, $u\in\mathcal{D}(H)$ *and*
$H(\varphi u)=\varphi Hu-2\nabla\varphi\cdot\mathbf{D}u-(\Delta\varphi)\,u$,

(3.15) $(Hu,u)=\|\mathbf{D}u\|^2+\|q^{\frac{1}{2}}u\|^2$ *if* $u\in\mathcal{D}(H)$,

(3.16) $\mathscr{C}=\{\varphi u\,|\,\varphi\in C_0^\infty(\mathbb{R}^m),\ u\in(H+1)^{-1}(L^2\cap L^\infty)\}$ *is an operator core of* H,

(3.17) *If* $\mathbf{a}_n\to\mathbf{a}$ *in* $L^2_{\mathrm{loc}}(\mathbb{R}^m)^m$, $q_n\to q$ *in* $L^1_{\mathrm{loc}}(\mathbb{R}^m)$, *then* $H_n\to H$ *in the strong resolvent sense.*

Proof. (3.13): In view of Theorem 1 and Lemma 1 we have

$$\mathcal{D}(H)=\{u\in\mathcal{Q}(\hbar)\,|\,f\in L^2(\mathbb{R}^m):\ \hbar(u,\varphi)=(f,\varphi)\ \text{if}\ \varphi\in C_0^\infty(\mathbb{R}^m)\}.$$

But $\hbar(u,\varphi)=(\hat{H}u,\varphi)$ when $\varphi\in C_0^\infty(\mathbb{R}^m)$, hence the result.

(3.14): If $\varphi \in C_0^\infty(\mathbb{R}^m)$ and $u \in \mathscr{D}(H)$, then $\hat{H}(\varphi u) = \varphi \hat{H} u - 2\nabla\varphi \cdot \mathbf{D} u - (\Delta\varphi) u$. Since $\varphi u \in \mathscr{Q}(\hbar)$ by Lemma 3 and $\hat{H} u = H u \in L^2(\mathbb{R}^m)$, we get $\hat{H}(\varphi u) \in L^2(\mathbb{R}^m)$. Thus $\varphi u \in \mathscr{D}(H)$ and $\hat{H}(\varphi u) = H(\varphi u)$.

(3.15): This is clear, since $(Hu, u) = \hbar(u, u)$ if $u \in \mathscr{D}(H)$.

(3.16): In view of (3.15) and Lemma 1 $(H+1)^{-1}$ exists on $L^2(\mathbb{R}^m)$ and is bounded. Since $L^2 \cap L^\infty$ is dense in $L^2(\mathbb{R}^m)$, $(H+1)^{-1}(L^2 \cap L^\infty)$ is an operator core of H. Now let $u \in (H+1)^{-1}(L^2 \cap L^\infty)$, $\varphi \in C_0^\infty(\mathbb{R}^m)$ such that $\varphi = 1$ in a neighbourhood of the origin and put $\varphi_n = \varphi\left(\frac{\cdot}{n}\right)$. Thus $\varphi_n u \in \mathscr{D}(H)$, $\varphi_n u \to u$ in $L^2(\mathbb{R}^m)$ and due to (3.14)

$$H(\varphi_n u) = \varphi_n H u - 2\nabla\varphi_n \cdot \mathbf{D} u - (\Delta\varphi_n) u \to H u \quad \text{in } L^2(\mathbb{R}^m).$$

Therefore $\mathscr{C}$ is an operator core of H.

(3.17): Though (3.17) is shown in [19, Theorem 4.1], we like to prove it here since our proof does not need [19, Lemma 2.5]. Of course, our arguments are closely related to those in [19].

Let $f \in L^2(\mathbb{R}^m)$ and let $u_n = (H_n + i)^{-1} f$. Then $\|u_n\| \leqq \|f\|$ and

$$\|\mathbf{D}_n u_n\|^2 + \|q_n^{\frac{1}{2}} u_n\|^2 = \operatorname{Re}(f, u_n) \leqq \|f\|^2.$$

Thus (u_n) contains a subsequence (hereafter denoted again by (u_n)) such that with suitable $u, v \in L^2(\mathbb{R}^m)$, $\mathbf{w} \in L^2(\mathbb{R}^m)^m$

$$u_n \to u, \quad q_n^{\frac{1}{2}} u_n \to v, \quad \mathbf{D}_n u_n \to \mathbf{w} \qquad \text{weakly in } L^2.$$

Since $q_n^{\frac{1}{2}} u_n \to q^{\frac{1}{2}} u$, $\mathbf{D}_n u_n \to \mathbf{D} u$ in $\mathscr{D}'$, we see that $v = q^{\frac{1}{2}} u$, $\mathbf{D} u = \mathbf{w}$ and conclude $u \in \mathscr{Q}(\hbar)$. By definition of u_n and since $\mathbf{D}_n \varphi = \nabla\varphi - i\mathbf{a}_n \varphi \to \mathbf{D}\varphi$, $q_n^{\frac{1}{2}} \varphi \to q^{\frac{1}{2}} \varphi$ strongly in L^2 when $\varphi \in C_0^\infty(\mathbb{R}^m)$, we get from (3.2)

$$\hbar(u, \varphi) = \lim_{n\to\infty} \hbar_n(u_n, \varphi) = (f - iu, \varphi).$$

In view of Theorem 1 and (3.4) it follows $u \in \mathscr{D}(H)$ and $(H+i)u = f$, thus $u = (H+i)^{-1} f$. Since we could have started with an arbitrary subsequence of (u_n) the arguments above show, that $(H_n + i)^{-1} \to (H+i)^{-1}$ weakly.

Similarly $(H_n - i)^{-1} \to (H-i)^{-1}$ weakly, so by the resolvent formula

$$\|(H_n + i)^{-1} f\|^2 = \tfrac{1}{2} i (f, (H_n + i)^{-1} f - (H_n - i)^{-1} f) \to \|(H+i)^{-1} f\|^2$$

and thus the resolvent converges strongly. □

Next we provide a new proof for $|(H+\lambda) f| \leqq (-\Delta + \lambda)^{-1} |f|$ using again truncation methods. An alternative proof is given in Lemma 10.

Lemma 6. *Let* $\lambda > 0$, $f \in L^2(\mathbb{R}^m)$ *and assume* (C.1). *Then*

$$|(H+\lambda)^{-1} f| \leqq (-\Delta + \lambda)^{-1} |f|.$$

Proof. Of course, using an approximation argument, we may assume $f \in L^2 \cap L^\infty$. Let $u = (H+\lambda)^{-1} f$, then due to Lemma 4 we have $u \in L^\infty(\mathbb{R}^m)$ and thus

$u \in W^1_{\mathrm{loc}}(\mathbb{R}^m)$. For any $\varepsilon > 0$ let $w = w_\varepsilon = \dfrac{u}{\varepsilon + |u|}$, then $w \in W^1_{\mathrm{loc}}(\mathbb{R}^m)$ and $|w| \leqq 1$ a.e. in $\mathbb{R}^m$. Let $0 \leqq \varphi \in C_0(\mathbb{R}^m)$, then

$$\mathbf{D}(w\varphi) = \frac{\mathbf{D}u - w\nabla|u|}{\varepsilon + |u|}\,\varphi + w\nabla\varphi \in L^2(\mathbb{R}^m). \tag{3.18}$$

In view of $q^{\frac{1}{2}}(w\varphi) \in L^2(\mathbb{R}^m)$ and Lemma 3 we have $w\varphi \in \mathcal{D}(\hbar)$. By definition of u

$$\hbar(u, w\varphi) + \lambda(u, w\varphi) = (f, w\varphi).$$

This and (3.18) means (taking real parts)

$$\int \mathrm{Re}(\overline{\mathbf{D}u} \cdot \mathbf{D}(w\varphi)) + \int (q+\lambda)\,|u|\,|w|\,\varphi = \int \mathrm{Re}(fw)\,\varphi. \tag{3.18'}$$

Since $\mathrm{Re}(\bar{u} D_j u) = |u|\,\partial_j |u|$, $|\nabla|u|| \leqq |\mathbf{D}u|$ a.e. in $\mathbb{R}^m$ we conclude

$$\begin{aligned}\mathrm{Re}(\overline{\mathbf{D}u} \cdot \mathbf{D}(w\varphi)) &= \frac{|\mathbf{D}u|^2 - |w|\,|\nabla|u||^2}{\varepsilon + |u|}\,\varphi + |w|\,\nabla|u| \cdot \nabla\varphi \\ &\geqq |\nabla|u||^2\,\frac{1-|w|}{\varepsilon + |u|}\,\varphi + |w|\,\nabla|u| \cdot \nabla\varphi \geqq |w|\,\nabla|u| \cdot \nabla\varphi\end{aligned}$$

and thus using $q \geqq 0$, $|w| \leqq 1$

$$\int |w|\,\nabla|u| \cdot \nabla\varphi + \lambda \int |w|\,|u|\,\varphi \leqq \int |f|\,\varphi. \tag{3.19}$$

Remind $w = w_\varepsilon$, so that

$$\hbar_0(|u|, \varphi) := (\nabla|u|, \nabla\varphi) + \lambda(|u|, \varphi) \leqq (|f|, \varphi)$$

as $\varepsilon \to 0$. Let $v = (-\Delta + \lambda)^{-1}|f|$, then $\hbar_0(v, \varphi) = (|f|, \varphi)$ and thus $\hbar_0(|u| - v, \varphi) \leqq 0$ for all $0 \leqq \varphi \in C_0^\infty(\mathbb{R}^m)$.

By approximation (mollify and cut off $0 \leqq \varphi \in W^1(\mathbb{R}^m)$) we get

$$\hbar_0(|u| - v, \varphi) \leqq 0 \quad \text{for } 0 \leqq \varphi \in W^1(\mathbb{R}^m). \tag{3.20}$$

Let $\psi = |u| - v$, $\psi_+ = \max(\psi, 0)$. Then $\psi_+ \in W^1(\mathbb{R}^m)$ and $\psi\psi_+ = \psi_+^2$, $\nabla\psi \cdot \nabla\psi_+ = |\nabla\psi_+|^2$ (see appendix).

Taking $\varphi = \psi_+$ (3.20) implies

$$\lambda\|\psi_+\|^2 \leqq \hbar_0(\psi_+, \psi_+) = \hbar_0(\psi, \psi_+) \leqq 0.$$

Thus $\psi_+ = 0$, that is $|u| \leqq v$. □

Remark 1. In particular Lemma 6 implies the well known fact, that $(-\Delta + \lambda)^{-1}$ is positivity preserving. But this may be seen more directly as follows:

Let $0 \leqq f \in L^2(\mathbb{R}^m)$, $u = (-\Delta + \lambda)^{-1} f$. Then u is real-valued (since $-\Delta + \lambda$ is a real operator) and

$$\hbar_0(u, \varphi) = (f, \varphi) \qquad (\varphi \in W^1(\mathbb{R}^m) = \mathcal{D}(-\Delta)).$$

Let $u_- = \min(u, 0)$, then $u_- \in W^1(\mathbb{R}^m)$ and again

$$\lambda \|u_-\|^2 \leqq h_0(u_-, u_-) = h_0(u, u_-) = (f, u_-) \leqq 0.$$

Thus $u_- = 0$, that is $u = u_+ \geqq 0$.

We mention furthermore, that our method of proof may be applied to formally selfadjoint elliptic operators with variable coefficients.

Finally we remark, that the above method also provides a proof of (if q is not dropped in (3.18′))

$$|(H(\mathbf{a})+\lambda)^{-1} f| \leqq (H(\mathbf{0})+\lambda)^{-1} |f|.$$

4. Uniqueness of Schrödinger Operators

Throughout this section (except in Theorem 3 and 4) we assume Condition (C.2), that is

$$\mathbf{a} \in L^4_{\mathrm{loc}}(\mathbb{R}^m)^m, \quad \operatorname{div} \mathbf{a} \in L^2_{\mathrm{loc}}(\mathbb{R}^m), \qquad 0 \leqq q \in L^2_{\mathrm{loc}}(\mathbb{R}^m).$$

Observe, that (C.2) implies (C.1), so that all results of Sect. 3 are valid. In particular, see (3.16) and Lemma 4, we know that

$$\mathscr{C} = \{\varphi u \mid \varphi \in C_0^\infty(\mathbb{R}^m),\ u \in (H+1)^{-1}(L^2 \cap L^\infty)\}$$

is an operator core of H contained in L^∞. Thus, mollifying $u \in \mathscr{C}$, at first sight one believes to get a core $\hat{\mathscr{C}} \subset C_0^\infty(\mathbb{R}^m)$ of H. But unfortunately $u \in \mathscr{C}$ yields only $-\Delta u + 2i\mathbf{a} \cdot \nabla u \in L^2(\mathbb{R}^m)$, so that $-\Delta u_\varepsilon + 2i\mathbf{a} \cdot \nabla u_\varepsilon \to -\Delta u + 2i\mathbf{a} \cdot \nabla u$ is by no means clear. It is exactly this point, where up till now additional assumptions [5, 18, 19] were needed to obtain Δu, $\mathbf{a} \cdot \nabla u \in L^2(\mathbb{R}^m)$ via Sobolev inequalities. Since $u \to -\Delta u + 2i\mathbf{a} \cdot \nabla u$ is obviously continuous on $W^2(\Omega) \cap W^{1,4}(\Omega)$ for any bounded domain $\Omega \subset \mathbb{R}^m$, we overcome the difficulty mentioned above by showing $\mathscr{C} \subset \mathscr{D}(-\Delta) \cap W^{1,4}(\mathbb{R}^m)$. This is done in Lemma 7–9, being the crucial steps of this section.

Lemma 7 [1, 11]. *If $u \in \mathscr{D}(-\Delta) \cap L^\infty(\mathbb{R}^m)$, then $\nabla u \in L^4(\mathbb{R}^m)$. Moreover for any $\varepsilon > 0$ there exist constants c, $c(\varepsilon) > 0$ such that*

$$\|\nabla u\|_{L^4(\mathbb{R}^m)^m} \leqq \varepsilon \|\Delta u\| + c(\varepsilon) \|u\|_\infty \tag{4.1}$$

$$\|\nabla u\|^2_{L^4(\mathbb{R}^m)^m} \leqq c \|u\|_\infty \|\Delta u\| \tag{4.2}$$

for all $u \in \mathscr{D}(-\Delta) \cap L^\infty(\mathbb{R}^m)$.

Proof. We begin with the case $u \in C_0^\infty(\mathbb{R}^m)$ and we may assume throughout our proof u to be real-valued (otherwise consider $\operatorname{Re} u$ and $\operatorname{Im} u$). For $1 \leqq j \leqq m$ we have

$$\int (\partial_j u)^4 = \int \partial_j u (\partial_j u)^3 = -\int u\, \partial_j [(\partial_j u)^3] = -\int u\, 3(\partial_j u)^2 \partial_j^2 u.$$

Thus $\|\partial_j u\|^4_{L^4} \leqq 3 \|u\|_\infty \|\partial_j u\|^2_{L^4} \|\partial_j^2 u\|$, that is $\|\partial_j u\|^2_{L^4} \leqq 3 \|u\|_\infty \|\partial_j^2 u\|$.

By partial integration (since ∂_j and ∂_k commute) we have

$$\int (\partial_j \partial_k u)^2 = \int \partial_j^2 u \, \partial_k^2 u \qquad (1 \leqq j,\ k \leqq m)$$

which gives $\sum_{j,k=1}^{m} \|\partial_j \partial_k u\|^2 = \|\Delta u\|^2$. Thus

$$\|\partial_j u\|_{L^4}^2 \leqq 3 \|u\|_\infty \|\Delta u\| \qquad (1 \leqq j \leqq m). \tag{4.2'}$$

Next consider the case $u \in \mathscr{D}(-\Delta) \cap L^\infty$, $\operatorname{supp} u$ compact.

Let $u_n = J_{1/n} u$, where $J_{1/n}$ is the Friedrichs mollifier, so that $u_n \in C_0^\infty(\mathbb{R}^m)$, $u_n \to u$, $\Delta u_n \to \Delta u$ and $\|u_n\|_\infty \leqq \|u\|_\infty$.

Now (4.2′) shows, that $\partial_j u_n$ is a Cauchy sequence in $L^4(\mathbb{R}^m)$. Thus $\partial_j u_n \to \partial_j u$ in $L^4(\mathbb{R}^m)$ and (4.2′) holds good by a limiting process.

Finally let us consider $u \in \mathscr{D}(-\Delta) \cap L^\infty$. Then we choose a function $\varphi \in C_0^\infty(\mathbb{R}^m)$, satisfying $0 \leqq \varphi \leqq 1$ on $\mathbb{R}^m$ and $\varphi \equiv 1$ near the origin and put $\varphi_n = \varphi\left(\frac{\cdot}{n}\right)$. Clearly $u_n := \varphi u_n \in \mathscr{D}(-\Delta)$ has compact support, satisfies $\|u_n\|_\infty \leqq \|u\|_\infty$ and $u_n \to u$, $\Delta u_n \to \Delta u$ in $L^2(\mathbb{R}^m)$. As in the preceeding step we conclude $\partial_j u \in L^4(\mathbb{R}^m)$ and see that (4.2′) (implying (4.2)) holds true.

To obtain (4.1) let $\varepsilon > 0$. Then in view of (4.2)

$$\|\nabla u\|_{L^4(\mathbb{R}^m)^m} \leqq c^{\frac{1}{2}} \|u\|_\infty^{\frac{1}{2}} \|\Delta u\|^{\frac{1}{2}} \leqq \varepsilon \|\Delta u\| + \frac{c}{4\varepsilon} \|u\|_\infty$$

with a constant depending only on the dimension m. □

Lemma 8. *Let $H(\mathbf{a})$ be the selfadjoint operator of Lemma* 1. *Suppose Ω is a bounded subset of $\mathbb{R}^m$ and c is a positive number. Then there exists a constant $d > 0$ such that*

$$\|\Delta u\| \leqq 2 \|H(\mathbf{a}) u\| + d \|u\|_\infty \tag{4.3}$$

for all $u \in \mathscr{D}(H(\mathbf{a})) \cap \mathscr{D}(-\Delta) \cap L^\infty(\mathbb{R}^m)$ with $\operatorname{supp} u \subset \Omega$ and all vector potentials $\mathbf{a}$ *satisfying $\|\operatorname{div} \mathbf{a}\|_{L^2(\Omega)} + \|\mathbf{a}^2\|_{L^2(\Omega)} \leqq c$.*

Proof. Let $\mathbf{a}$ be a vector potential (satisfying (C.2)) with

$$\|\operatorname{div} \mathbf{a}\|_{L^2(\Omega)} + \|\mathbf{a}^2\|_{L^2(\Omega)} \leqq c$$

and let $u \in \mathscr{D}(H(\mathbf{a})) \cap \mathscr{D}(-\Delta) \cap L^\infty$ with $\operatorname{supp} u \subset \Omega$. In view of $\operatorname{div}(\mathbf{a} u) = (\operatorname{div} \mathbf{a}) u + \mathbf{a} \cdot \nabla u$ and formula (3.12), (3.13) we have

$$H(\mathbf{a}) u = -\Delta u + 2i \mathbf{a} \cdot \nabla u + (i \cdot \operatorname{div} \mathbf{a} + \mathbf{a}^2 + q) u.$$

Thus using (4.1) and $\|\mathbf{a}^2\|_{L^2(\Omega)} = \|\mathbf{a}\|_{L^4(\Omega)^m}^2$ we obtain

$$\begin{aligned} \|\Delta u\| &\leqq \|H(\mathbf{a}) u\| + 2 \|\mathbf{a}\|_{L^4(\Omega)^m} \|\nabla u\|_{L^4(\Omega)^m} + [\|\operatorname{div} \mathbf{a}\|_{L^2(\Omega)} \\ &\qquad + \|\mathbf{a}^2\|_{L^2(\Omega)} + \|q\|_{L^2(\Omega)}] \, \|u\|_\infty \\ &\leqq \|H(\mathbf{a}) u\| + 2\varepsilon \sqrt{c} \|\Delta u\| + [2c(\varepsilon)\sqrt{c} + c + \|q\|_{L^2(\Omega)}] \, \|u\|_\infty. \end{aligned}$$

Therefore

$$(1-2\varepsilon\sqrt{c})\,\|\Delta u\| \leqq \|H(\mathbf{a})\,u\| + [2c(\varepsilon)\sqrt{c}+c+\|q\|_{L^2(\Omega)}]\,\|u\|_\infty.$$

Choosing $\varepsilon^{-1}=4\sqrt{c}$, we obtain estimate (4.3). □

Lemma 9. *Let H be the selfadjoint operator of Lemma* 1. *Then*

$$\mathscr{C}=\{\varphi u \mid \varphi\in C_0^\infty(\mathbb{R}^m),\ u\in(H+1)^{-1}(L^2\cap L^\infty)\}$$

satisfies

$$\mathscr{C}\subset\mathscr{D}(-\Delta)\cap L^\infty(\mathbb{R}^m)\cap W^{1,4}(\mathbb{R}^m). \tag{4.4}$$

Proof. Let $\varphi\in C_0^\infty(\mathbb{R}^m)$, $u\in(H+1)^{-1}(L^2\cap L^\infty)$. Choose $\mathbf{a}_n\in C^\infty$ such that $\mathbf{a}_n\to\mathbf{a}$, $\operatorname{div}\mathbf{a}_n\to\operatorname{div}\mathbf{a}$ in L^2_{loc}(mollify **a**!). Let us denote by $H=H(\mathbf{a})$ resp. $H_n=H(\mathbf{a}_n)$ the selfadjoint operators of Lemma 1 corresponding to **a**, q resp. $\mathbf{a}_n$, q. Define $u_n=(H_n+1)^{-1}(H+1)\,u$, then $u_n\to(H+1)^{-1}(H+1)\,u=u$ by (3.17) and by (3.15) we have (remind $\mathbf{D}_n=\nabla-i\mathbf{a}_n$)

$$\|\mathbf{D}_n u_n\|^2+\|u_n\|^2\leqq((H_n+1)\,u_n,u_n)=((H+1)\,u,u_n)\leqq\|(H+1)\,u\|\,\|u_n\|.$$

This gives $\|u_n\|\leqq\|(H+1)\,u\|$ and $\|\mathbf{D}_n u_n\|\leqq\|(H+1)\,u\|$.

Let us now put $v_n=\varphi u_n$, then in view of (3.13), (3.14) we see that $v_n\in\mathscr{D}(H_n)$ and

$$H_n v_n=\varphi(H_n u_n)+2\nabla\varphi\cdot\mathbf{D}_n u_n+(\Delta\varphi)\,u_n.$$

Hence

$$\begin{aligned}\|H_n v_n\| &\leqq \|\varphi\|_\infty(\|(H_n+1)\,u_n\|+\|u_n\|)+2\|\nabla\varphi\|_\infty\|\mathbf{D}_n u_n\|+\|\Delta\varphi\|_\infty\|u_n\|\\ &\leqq\|(H+1)\,u\|\,[2\|\varphi\|_\infty+2\|\nabla\varphi\|_\infty+\|\Delta\varphi\|_\infty]=:a.\end{aligned}$$

By Lemma 4 we have $\|v_n\|_\infty\leqq\|\varphi\|_\infty\|u_n\|_\infty\leqq\|\varphi\|_\infty\|(H+1)\,u\|_\infty$ and since $v_n\in\mathscr{D}(H_n)\cap L^\infty$, $\operatorname{supp} v_n$ compact, we conclude $v_n\in W^1(\mathbb{R}^m)$. In view of $\operatorname{div}(\mathbf{a}_n v_n)=(\operatorname{div}\mathbf{a}_n)\,v_n+\mathbf{a}_n\cdot\nabla v_n$, (3.12), (3.13), (3.14) we obtain

$$H_n v_n=-\Delta v_n+2i\mathbf{a}_n\cdot\nabla v_n+(i\cdot\operatorname{div}\mathbf{a}_n+\mathbf{a}_n^2+q)\,v_n. \tag{4.5}$$

Since $\mathbf{a}_n\in C^\infty$, (4.5) shows $\Delta v_n\in L^2(\mathbb{R}^m)$, i.e. $v_n\in\mathscr{D}(-\Delta)$.

Now let Ω be a bounded subset of $\mathbb{R}^m$, such that $\operatorname{supp} v_n\subset\operatorname{supp}\varphi\subset\Omega$ and let $c>0$ be a constant satisfying

$$\|\operatorname{div}\mathbf{a}_n\|_{L^2(\Omega)}+\|\mathbf{a}_n^2\|_{L^2(\Omega)}\leqq c.$$

Due to Lemma 8, (4.3) we obtain

$$\begin{aligned}\|\Delta v_n\|&\leqq 2\,\|H_n v_n\|+d\,\|v_n\|_\infty\\ &\leqq 2a+d\,\|\varphi\|_\infty\,\|(H+1)u\|_\infty<\infty.\end{aligned} \tag{4.6}$$

By (4.6), using the weak compactness of the unit ball, we may extract a weakly convergent subsequence of (Δv_n). Since $v_n=\varphi u_n\to\varphi u$ in $L^2(\mathbb{R}^m)$ we conclude

$\varphi u \in \mathcal{D}(-\Delta)$, thus bearing in mind Lemma 4 and Lemma 7, we have shown (4.4). □

Theorem 2. *Assume* (C.2) *and let H be the selfadjoint operator in Lemma* 1. *Then $C_0^\infty(\mathbb{R}^m)$ is an operator core of H, i.e. $\mathcal{H} = -(\nabla - i\mathbf{a})^2 + q$ is essentially selfadjoint on $C_0^\infty(\mathbb{R}^m)$.*

Proof. In view of (3.16), (4.4) the set $\mathcal{C}$ is an operator core of H satisfying $\mathcal{C} \subset \mathcal{D}(-\Delta) \cap L^\infty(\mathbb{R}^m) \cap W^{1,4}(\mathbb{R}^m)$. Thus, using (3.12), (3.13) and $\operatorname{div}(\mathbf{a}u) = (\operatorname{div}\mathbf{a})u + \mathbf{a}\cdot\nabla u$, we have for each $u \in \mathcal{C}$

$$Hu = -\Delta u + 2i\mathbf{a}\cdot\nabla u + (i\cdot\operatorname{div}\mathbf{a} + \mathbf{a}^2 + q)u. \tag{4.7}$$

Define $u_\varepsilon = J_\varepsilon u$, where J_ε is the Friedrichs mollifier and $\varepsilon < 1$, such that $u_\varepsilon \in C_0^\infty(\mathbb{R}^m)$ with a common support in $\Omega = \operatorname{supp} u + \{x \mid |x| \leqq 1\}$, $\|u_\varepsilon\|_\infty \leqq \|u\|_\infty$ and $u_\varepsilon \to u$ in $W^2(\mathbb{R}^m) = \mathcal{D}(-\Delta)$. We have also $u_\varepsilon \to u$ a.e. pointwise along some subsequence $\varepsilon = \varepsilon_n \to 0$. By (4.2) we get

$$\begin{aligned}\|\mathbf{a}\cdot\nabla u_\varepsilon - \mathbf{a}\cdot\nabla u\| &\leqq \|\mathbf{a}\|_{L^4(\Omega)^m} \|\nabla(u_\varepsilon - u)\|_{L^4(\Omega)^m} \\ &\leqq c\,\|\mathbf{a}\|_{L^4(\Omega)^m} \|u\|_\infty \|\Delta u_\varepsilon - \Delta u\|.\end{aligned}$$

Thus, considering each term in (4.7), we conclude $Hu_\varepsilon \to Hu$ in $L^2(\mathbb{R}^m)$ as $\varepsilon \to 0$. Since $\mathcal{C}$ is a core of H, $C_0^\infty(\mathbb{R}^m)$ turns out to be a core of H, too. □

To prove the essential selfadjointness of $\mathcal{H} = -(\nabla - i\mathbf{a})^2 + q$ on $C_0^\infty(\mathbb{R}^m)$ including also negative parts of q, we need Lemma 6. But since (C.2) is now (in this section) assumed, a nice proof of $|(H+\lambda)^{-1} f| \leqq (-\Delta + \lambda)^{-1}|f|$ can be given using Theorem 2.

Lemma 10. *Let $\lambda > 0$, $f \in L^2(\mathbb{R}^m)$ and assume* (C.2). *Let H be the selfadjoint operator of Lemma* 1. *Then*

$$|(H+\lambda)^{-1} f| \leqq (-\Delta + \lambda)^{-1} |f|. \tag{4.8}$$

Remark 2. Iterating (4.8) and using the well known formula

$$e^{-tA} = \lim_{n\to\infty} \left(1 + \frac{tA}{n}\right)^{-n}$$

one easily obtains from (4.8)

$$|e^{-tH} f| \leqq e^{t\Delta} |f| \qquad (t > 0). \tag{4.9}$$

Proof of Lemma 10. Choose $\mathbf{a}_n \in C^\infty$ such that $\mathbf{a}_n \to \mathbf{a}$, $\operatorname{div}\mathbf{a}_n \to \operatorname{div}\mathbf{a}$ in L^2_{loc}. Denote by H_n the selfadjoint operator $H(\mathbf{a}_n)$ of Lemma 1. Let $\varphi, \Phi \in C_0^\infty(\mathbb{R}^m)$, $0 \leqq \Phi$, $\Phi = 1$ near $\operatorname{supp}\varphi$, and let $\varphi_\varepsilon = [|\varphi|^2 + \varepsilon^2]^{\frac{1}{2}}$ with $\varepsilon > 0$. Then an elementary calculation (see also [7, 13, (X.47)]) shows

$$\begin{aligned}(-\Delta + \lambda)\varphi_\varepsilon &\leqq \operatorname{Re}\left(\frac{\bar\varphi}{\varphi_\varepsilon}(H_n + \lambda)\varphi\right) + \lambda\left(\varphi_\varepsilon - \frac{|\varphi|^2}{\varphi_\varepsilon}\right) \\ &\leqq |(H_n + \lambda)\varphi| + \lambda\left(\varphi_\varepsilon - \frac{|\varphi|^2}{\varphi_\varepsilon}\right).\end{aligned}$$

Thus

$$\begin{aligned}(-\Delta+\lambda)(\Phi\varphi_\varepsilon)&=\Phi(-\Delta+\lambda)\varphi_\varepsilon-2\nabla\Phi\cdot\nabla\varphi_\varepsilon-(\Delta\Phi)\varphi_\varepsilon\\&=\Phi(-\Delta+\lambda)\varphi_\varepsilon-(\Delta\Phi)\varphi_\varepsilon\\&\leqq|(H_n+\lambda)\varphi|+\lambda\Phi\left(\varphi_\varepsilon-\frac{|\varphi|^2}{\varphi_\varepsilon}\right)-(\Delta\Phi)\varphi_\varepsilon.\end{aligned}$$

Define $u^\varepsilon=\lambda\Phi\left(\varphi_\varepsilon-\frac{|\varphi|^2}{\varphi_\varepsilon}\right)-(\Delta\Phi)\varphi_\varepsilon$, then using the fact that $(-\Delta+\lambda)^{-1}$ is positivity preserving (see Remark 1), we obtain

$$\Phi\varphi_\varepsilon\leqq(-\Delta+\lambda)^{-1}|(H_n+\lambda)\varphi|+(-\Delta+\lambda)^{-1}u^\varepsilon.$$

Since $\Phi\varphi_\varepsilon\to|\varphi|$, $u^\varepsilon\to 0$ in $L^2(\mathbb{R}^m)$ as $\varepsilon\to 0$ we have

$$|\varphi|\leqq(-\Delta+\lambda)^{-1}|(H_n+\lambda)\varphi|$$

Since $H_n\varphi\to H\varphi$, this yields

$$|\varphi|\leqq(-\Delta+\lambda)^{-1}|(H+\lambda)\varphi|. \tag{4.10}$$

By Theorem 2, $C_0^\infty(\mathbb{R}^m)$ is a core of H, thus (4.10) is valid for $\varphi\in\mathscr{D}(H)$, too. Now let $\varphi=(H+\lambda)^{-1}f\in\mathscr{D}(H)$, then $|(H+\lambda)^{-1}f|\leqq(-\Delta+\lambda)^{-1}|f|$. □

Theorem 3. *Suppose* $\mathbf{a}\in L^4_{\mathrm{loc}}(\mathbb{R}^m)^m$, $\operatorname{div}\mathbf{a}\in L^2_{\mathrm{loc}}(\mathbb{R}^m)$ *and* $q\in L^2_{\mathrm{loc}}(\mathbb{R}^m)$. *Assume that* $q_-=\min(q,0)$ *is* Δ*-bounded with relative bound* $a<1$.

Then $\mathscr{H}=-(\nabla-i\mathbf{a})^2+q$ *is essentially selfadjoint on* $C_0^\infty(\mathbb{R}^m)$ *and semibounded from below.*

Proof. Let $\lambda>0$ and $f\in L^2(\mathbb{R}^m)$. Let $q_+=\max(q,0)$ and let H_0 be the selfadjoint operator of Lemma 1 associated with $\mathscr{H}_0=-(\nabla-i\mathbf{a})^2+q_+$. In view of (4.8) we have

$$|(H_0+\lambda)^{-1}f|\leqq(-\Delta+\lambda)^{-1}|f|.$$

Thus

$$|q_-(H_0+\lambda)^{-1}f|\leqq|q_-|(-\Delta+\lambda)^{-1}|f|$$

and therefore

$$\|q_-(H_0+\lambda)^{-1}f\|\leqq\|q_-(-\Delta+\lambda)^{-1}|f|\|.$$

Since q_- is Δ-bounded with relative bound $a<1$, there exist $a<a^*<1$, $\lambda^*>0$ such that for each $g\in L^2(\mathbb{R}^m)$

$$\|q_-(-\Delta+\lambda^*)g\|\leqq a^*\|g\|$$

Let $\varphi\in C_0^\infty(\mathbb{R}^m)$ and define $g=(H_0+\lambda^*)\varphi$, then

$$\|q_-\varphi\|\leqq a^*\|(H_0+\lambda^*)\varphi\|\leqq a^*\|H_0\varphi\|+a^*\lambda^*\|\varphi\|. \tag{4.11}$$

Since $C_0^\infty(\mathbb{R}^m)$ is a core of H_0 (Theorem 2), we conclude by Rellich-Kato [13, Theorem X.12] that $\mathscr{H}=-(\nabla-i\mathbf{a})^2+q$ is essentially selfadjoint on $C_0^\infty(\mathbb{R}^m)$, semibounded from below and in addition $\overline{\mathscr{H}|C_0^\infty(\mathbb{R}^m)}=H_0+q_-$. □

Theorem 4. *Suppose* $\mathbf{a} \in L^4_{\text{loc}}(\mathbb{R}^m)^m$, $\operatorname{div} \mathbf{a} \in L^2_{\text{loc}}(\mathbb{R}^m)$ *and assume* $q = q_1 + q_2$ *with* q_1, $q_2 \in L^2_{\text{loc}}(\mathbb{R}^m)$, $q_2 \leqq 0$. *Let* q_2 *be* Δ*-bounded with relative bound* $a < 1$ *and suppose* q_1 *satisfies* $q_1(x) \geqq -c|x|^2$ *with a constant* $c \geqq 0$. *Then* $\mathscr{H} = -(\nabla - i\mathbf{a})^2 + q$ *is essentially selfadjoint on* $C_0^\infty(\mathbb{R}^m)$.

To prove Theorem 4 we first derive an a-priori estimate of $\|\mathbf{D}\varphi\|$ in terms of $\|\varphi\|$ and $\|H_n\varphi\|$, where H_n is the operator

$$(4.12) \qquad \begin{aligned} \mathscr{D}(H_n) &= C_0^\infty(\mathbb{R}^m) \\ H_n &= -\mathbf{D}^2 + \max(q_1, -cn^2) + q_2 \\ &= -\Delta + 2i\mathbf{a}\cdot\nabla + i \operatorname{div} \mathbf{a} + \mathbf{a}^2 + \max(q_1, -cn^2) + q_2. \end{aligned}$$

Lemma 10. *Assume the conditions of Theorem* 4 *and let* H_n *be the operator defined in* (4.12). *Then there exists a constant* $d > 0$, *such that*

$$(4.13) \qquad \|\mathbf{D}\varphi\|^2 \leqq d \cdot \|\varphi\| [n^2 \|\varphi\| + \|H_n\varphi\|] \quad \textit{for all } \varphi \in C_0^\infty(\mathbb{R}^m).$$

Proof of Lemma 10. Let $V_n = \max(q_1, -cn^2)$, thus $V_n + cn^2 \geqq 0$. Let $\mathscr{D}(S_n) = \mathscr{D}(V_n) = \mathscr{D}(q_2) = C_0^\infty \mathbb{R}^m)$ and $S_n = -\mathbf{D}^2 + V_n$, so that $H_n = S_n + q_2$. Since q_2 is Δ-bounded with relative bound $a < 1$ we know due to (4.11) for each $\varphi \in C_0^\infty(\mathbb{R}^m)$

$$(4.14) \qquad \|q_2\varphi\| \leqq a^* \|(S_n + cn^2)\varphi\| + b\|\varphi\|$$

with constants $a^* < 1$, $b = a^* \lambda^*$ independent of n. This yields

$$\|S_n\varphi\| \leqq \|H_n\varphi\| + \|q_2\varphi\| \leqq \|H_n\varphi\| + a^*\|S_n\varphi\| + (b + a^* cn^2)\|\varphi\|$$

and then

$$(4.15) \qquad \|S_n\varphi\| \leqq (1-a^*)^{-1}[\|H_n\varphi\| + (b + a^* cn^2)\|\varphi\|].$$

Since

$$(S_n\varphi, \varphi) = \|\mathbf{D}\varphi\|^2 + (V_n\varphi, \varphi) \geqq \|\mathbf{D}\varphi\|^2 - cn^2\|\varphi\|^2$$

we get

$$\|\mathbf{D}\varphi\|^2 \leqq cn^2\|\varphi\|^2 + \|\varphi\|\,\|S_n\varphi\|$$

combining last inequality with (4.15) we obtain

$$\begin{aligned} \|\mathbf{D}\varphi\|^2 &\leqq cn^2\|\varphi\|^2 + (1-a^*)^{-1}\|\varphi\|\,\|H_n\varphi\| + \frac{b + a^* cn^2}{1-a^*}\|\varphi\|^2 \\ &\leqq (1-a^*)^{-1}[(b + cn^2)\|\varphi\|^2 + \|\varphi\|\,\|H_n\varphi\|] \end{aligned}$$

which gives estimate (4.13). □

Proof of Theorem 4. Let $\lambda \in \{+i, -i\}$ and let $\mathscr{D}(H) = C_0^\infty(\mathbb{R}^m)$,

$$H = -(\nabla - i\mathbf{a})^2 + q = -\Delta + 2i\mathbf{a}\cdot\nabla + i\operatorname{div}\mathbf{a} + \mathbf{a}^2 + q.$$

Then we have to show, that $R(H+\lambda)$ is dense in $L^2(\mathbb{R}^m)$. Suppose $f \in R(H+\lambda)^\perp$, that is $(f, (H+\lambda)\varphi) = 0$ for each $\varphi \in C_0^\infty(\mathbb{R}^m)$.

In view of Theorem 3 there exist $\varphi_n \in C_0^\infty(\mathbb{R}^m)$ such that (remind (4.12))

(4.16) $$\|(H_n+\lambda)\varphi_n - f\| \leqq \frac{1}{n}.$$

Now choose $\Phi \in C_0^\infty(\mathbb{R}^m)$ satisfying $0 \leqq \Phi \leqq 1$,

(4.17) $$\Phi(x) = \begin{cases} 1, & |x| \leqq \frac{1}{2} \\ 0, & |x| \geqq 1 \end{cases}$$

and put $\Phi_n = \Phi\left(\frac{\cdot}{n}\right)$. Then for any $\varphi \in C_0^\infty(\mathbb{R}^m)$

(4.18) $$\Phi_n H \varphi = \Phi_n H_n \varphi.$$

We have

$$\begin{aligned}
\|f\|^2 &= \lim_{n\to\infty} (f\,\Phi_n, (H_n+\lambda)\varphi_n) \qquad ((4.16), (4.17)) \\
&= \lim_{n\to\infty} (f, \Phi_n(H+\lambda)\varphi_n) \qquad ((4.18)) \\
&= \lim_{n\to\infty} (f, (H+\lambda)(\Phi_n\varphi_n) + 2\nabla\Phi_n \cdot \mathbf{D}\,\varphi_n + (\Delta\Phi_n)\varphi_n) \qquad ((3.14)) \\
&= \lim_{n\to\infty} (f, 2\nabla\Phi_n \cdot \mathbf{D}\,\varphi_n + (\Delta\Phi_n)\varphi_n), \qquad \text{since } f \in R(H+\lambda)^\perp.
\end{aligned}$$

Due to (4.16) and Lemma 10, (4.13) we get the estimates

$$\|(\Delta\Phi_n)\varphi_n\| \leqq \frac{1}{n^2}\|\Delta\Phi\|_\infty \|\varphi_n\| \leqq \frac{1}{n^2}\|\Delta\Phi\|_\infty(\|f\|+1)$$

$$\begin{aligned}
\|\nabla\Phi_n \cdot \mathbf{D}\,\varphi_n\|^2 &\leqq \frac{1}{n^2}\|\nabla\Phi\|_\infty^2 \cdot d\,\|\varphi_n\|\,[n^2\|\varphi_n\| + \|H_n\varphi_n\|] \\
&\leqq \frac{1}{n^2}\|\nabla\Phi\|_\infty^2\, d(\|f\|+1)^2(2+n^2)
\end{aligned}$$

thus $\|(\Delta\Phi_n)\varphi_n + 2\nabla\Phi_n \cdot \mathbf{D}\,\varphi_n\| \leqq c^* < \infty$.

The last estimate yields

$$\|f\|^2 \leqq \lim_{n\to\infty} \|f \cdot \chi_{\left\{x \mid |x| > \frac{n}{2}\right\}}\| \cdot c^* = 0,$$

thus H is essentially selfadjoint on $C_0^\infty(\mathbb{R}^m)$. □

Appendix

The purpose of this appendix is to give the proof (Theorem A) of the facts, concerning truncation method, being decisively used in Lemmas 2–6. We like to mention that the assumptions in Theorem A are not the weakest possible ones. For related results with the sharpest assumptions we refer the reader to [9, 10] (see also [2, I.7.4]). We have chosen the assumptions in Theorem A such that simple proofs can be given. However, all relevant applications are included.

First we introduce some notations. By $\mathscr{Lip}(\mathbb{R}^k;\mathbb{R})$ we denote the space of uniformly Lipschitz continuous functions $f\colon \mathbb{R}^k\to\mathbb{R}$ endowed with the norm

$$[f]=\sup_{\substack{\mathbf{x},\mathbf{y}\in\mathbb{R}^k\\ \mathbf{x}\neq\mathbf{y}}}\frac{|f(\mathbf{x})-f(\mathbf{y})|}{|\mathbf{x}-\mathbf{y}|}.$$

Let Ω be an open subset of $\mathbb{R}^m$ and let $\mathscr{M}(\Omega)$ be the space of real measurable functions defined in Ω. Given a function $f\in\mathscr{Lip}(\mathbb{R}^k;\mathbb{R})$ we define a mapping T_f: $\mathscr{M}(\Omega)^k\to\mathscr{M}(\Omega)$ by

$$T_f\mathbf{u}=f\circ\mathbf{u},\qquad \mathbf{u}=(u_1,\dots,u_k)\in\mathscr{M}(\Omega)^k.$$

By $W^{1,p}(\Omega;\mathbb{R})$ we shall mean the set of realvalued functions in $W^{1,p}(\Omega)$.

Theorem A. *Let k, $m\in\mathbb{N}$, $1\leqq p<\infty$, $\Omega\subset\mathbb{R}^m$ open and assume $f\in\mathscr{Lip}(\mathbb{R}^k;\mathbb{R})$ satisfies $f(\mathbf{0})=0$. In addition we assume $f\in C^1(\mathbb{R}^k\backslash\Gamma)$, where Γ is any closed countable subset of $\mathbb{R}^k$.*

Then T_f maps $W^{1,p}(\Omega;\mathbb{R})^k$ continuously into $W^{1,p}(\Omega;\mathbb{R})$. Moreover the chain rule

$$\partial_j(f\circ\mathbf{u})=(\mathbf{g}\circ\mathbf{u})\,\partial_j\mathbf{u},\qquad \mathbf{u}\in W^{1,p}(\Omega;\mathbb{R}^k),\quad 1\leqq j\leqq m$$

holds true for any Borel function $\mathbf{g}\colon \mathbb{R}^k\to\mathbb{R}^k$ satisfying $\mathbf{g}=\nabla\mathbf{f}$ on $\mathbb{R}^k\backslash\Gamma$.

Proof. We devide the proof in three steps.

Step 1. In Step 1 we assume $\Gamma=\varnothing$. Let $\mathbf{u}\in W^{1,p}(\Omega;\mathbb{R})^k$, then there exists a sequence $(\mathbf{u}_n)\subset W^{1,p}(\Omega;\mathbb{R})^k\cap C^1(\Omega)^k$ such that $\mathbf{u}_n\to\mathbf{u}$ in $W^{1,p}(\Omega)^k$. In addition we may assume

$$\mathbf{u}_n\to\mathbf{u},\quad \partial_j\mathbf{u}_n\to\partial_j\mathbf{u},\quad |\mathbf{u}_n|+|\partial_j\mathbf{u}_n|\leqq w\quad \text{a.e. in }\Omega$$

with a suitable function $0\leqq w\in L^p(\Omega)$. Since

$$|f\circ\mathbf{u}_n|\leqq[f]\,|\mathbf{u}_n|\leqq[f]\,w$$

$$|\partial_j(f\circ\mathbf{u}_n)|=|(\nabla f\circ\mathbf{u}_n)\cdot\partial_j\mathbf{u}_n|\leqq|\nabla f|\circ\mathbf{u}_n\,|\partial_j\mathbf{u}_n|\leqq[f]\,w$$

and $f\circ\mathbf{u}_n\to f\circ\mathbf{u}$, $\partial_j(f\circ\mathbf{u}_n)\to(\nabla f\circ\mathbf{u})\cdot\partial_j\mathbf{u}$ a.e. in Ω as $n\to\infty$, we conclude by Lebesgue's dominated convergence theorem that $f\circ\mathbf{u}_n\to f\circ\mathbf{u}$ in $W^{1,p}(\Omega)$ and thus $\partial_j(f\circ\mathbf{u})=(\nabla f\circ\mathbf{u})\cdot\partial_j\mathbf{u}$.

Step 2. In Step 2 we show $\partial_j\mathbf{u}=\mathbf{0}$ on $\mathbf{u}^{-1}(N)$ for $1\leqq j\leqq m$, when N is any closed countable subset of $\mathbb{R}^k$.

Since $\mathbf{u}^{-1}(N)\subset\bigcap_{\nu=1}^{k}u_\nu^{-1}(pr_\nu N)$, we may assume $k=1$. Furthermore (by intersecting with compact intervalls) we may assume N to be compact. Choose $\varphi_n\in C_0^0(\mathbb{R})$ satisfying $\varphi_n=1$ on N, $0\leqq\varphi_n\leqq 1$ and $\varphi_n\to\chi_N$ pointwise as $n\to\infty$. Set $\psi_n(t)=\int_0^t\varphi_n$, then $\psi_n\to 0$ pointwise and (using Step 1)

$$|\psi_n\circ u|\leqq|u|,\qquad |\nabla(\psi_n\circ u)|=|(\varphi_n\circ u)\nabla u|\leqq|\nabla u|.$$

Thus, by Lebesgue's dominated convergence theorem $\psi_n \circ u \to 0$ in $W^{1,p}(\Omega)$ and $\mathbf{0} = \nabla 0 = (\chi_N \circ u) \nabla u$, which means $\nabla u = \mathbf{0}$ a.e. on $u^{-1}(N)$.

Step 3. Finally, in Step 3 we complete the proof of Theorem A. Consider the function $f_\varepsilon = J_\varepsilon f - (J_\varepsilon f)(\mathbf{0})$, where J_ε is the Friedrichs mollifier, then $f_\varepsilon \in C^\infty(\mathbb{R}^k) \cap \mathscr{Lip}(\mathbb{R}^k; \mathbb{R})$ satisfies

$$[f_\varepsilon] \leqq [f], \quad f_\varepsilon(\mathbf{0}) = 0$$

$$\nabla f_\varepsilon(\mathbf{x}) = J_\varepsilon(\nabla f)(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^k \backslash \Gamma, \quad \varepsilon < \operatorname{dist}(\mathbf{x}, \Gamma).$$

Moreover $f_\varepsilon \to f$ pointwise on $\mathbb{R}^k$ and $\nabla f_\varepsilon \to \nabla f$ pointwise on $\mathbb{R}^k \backslash \Gamma$. By Step 1 we know $f_\varepsilon \circ \mathbf{u} \in W^{1,p}(\Omega)$ and

$$|f_\varepsilon \circ \mathbf{u}| \leqq [f] \, |\mathbf{u}|$$

$$|\partial_j(f_\varepsilon \circ \mathbf{u})| = |(\nabla f_\varepsilon \circ \mathbf{u}) \cdot \partial_j \mathbf{u}| \leqq |\nabla f_\varepsilon| \circ \mathbf{u} \, |\partial_j \mathbf{u}| \leqq [f] \, |\partial_j \mathbf{u}|.$$

As $\mathbf{g} = \nabla f$ on $\mathbb{R}^k \backslash \Gamma$ and since $\partial_j \mathbf{u} = \mathbf{0}$ a.e. on $\mathbf{u}^{-1}(\Gamma)$ by Step 2, we have $f_\varepsilon \circ \mathbf{u} \to f \circ \mathbf{u}$ as well as

$$\partial_j(f_\varepsilon \circ \mathbf{u}) = (\nabla f_\varepsilon \circ \mathbf{u}) \, \partial_j \mathbf{u} \to (\mathbf{g} \circ \mathbf{u}) \cdot \partial_j \mathbf{u} \quad \text{a.e. in } \Omega \text{ as } \varepsilon \to 0.$$

Thus by Lebesgue's dominated convergence theorem we get $f_\varepsilon \circ \mathbf{u} \to f \circ \mathbf{u}$ in $W^{1,p}(\Omega)$ and $\partial_j(f \circ \mathbf{u}) = (\mathbf{g} \circ \mathbf{u}) \, \partial_j \mathbf{u}$ for any $1 \leqq j \leqq m$. To show the continuity of T_f, we consider a sequence $(\mathbf{u}_n) \subset W^{1,p}(\Omega; \mathbb{R})^k$ such that $\mathbf{u}_n \to \mathbf{u}$ in $W^{1,p}(\Omega)^k$. By looking at a suitable subsequence (denoted hereafter again by $(\mathbf{u}_n)$) we may assume in addition $\mathbf{u}_n \to \mathbf{u}$, $\partial_j \mathbf{u}_n \to \partial_j \mathbf{u}$, $|\mathbf{u}_n| + |\partial_j \mathbf{u}_n| \leqq w$ a.e. in Ω with suitable $w \in L^p(\Omega)$. Now we have

$$f \circ \mathbf{u}_n \to f \circ \mathbf{u}, \quad |f \circ \mathbf{u}_n| \leqq [f] \, |\mathbf{u}_n| \leqq [f] \, w \quad \text{a.e. in } \Omega$$

and similarly (looking at $\Omega \backslash u^{-1}(\Gamma)$ resp. $u^{-1}(\Gamma)$)

$$\partial_j(f \circ \mathbf{u}_n) = (\mathbf{g} \circ \mathbf{u}_n) \cdot \partial_j \mathbf{u}_n \to (\mathbf{g} \circ \mathbf{u}) \cdot \partial_j \mathbf{u} = \partial_j(f \circ \mathbf{u})$$

$$|\partial_j(f \circ \mathbf{u}_n)| = |(\mathbf{g} \circ \mathbf{u}_n) \cdot \partial_j \mathbf{u}_n| \leqq [f] \, |\partial_j \mathbf{u}_n| \leqq [f] \, w \quad \text{a.e. in } \Omega$$

where $\mathbf{g}\colon \mathbb{R}^k \to \mathbb{R}^k$ satisfies $\mathbf{g} = \nabla f$ on $\mathbb{R}^k \backslash \Gamma$ and $\mathbf{g} = \mathbf{0}$ on Γ. Thus we conclude by Lebesgue's dominated convergence theorem, that $T_f \mathbf{u}_n \to T_f \mathbf{u}$ in $W^{1,p}(\Omega)$. □

Let us now give two applications of Theorem A, often used in this note.

Corollary 1. *Let* $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$. *Then* $|u| \in W^{1,1}_{\mathrm{loc}}(\Omega; \mathbb{R})$ *and*

$$\partial_j |u| = \operatorname{Re}\left(\frac{\bar{u}}{|u|} \partial_j u\right) \quad (1 \leqq j \leqq m).$$

Proof. Considering a bounded open subset Ω' with $\Omega' \subset \Omega$ we may assume $u \in W^{1,1}(\Omega)$. Clearly $u_1 = \operatorname{Re} u$ and $u_2 = \operatorname{Im} u$ belong to $W^{1,1}(\Omega; \mathbb{R})$. Thus, considering the function $f \in \mathscr{Lip}(\mathbb{R}^2; \mathbb{R}) \cap C^1(\mathbb{R} \backslash \{0\})$, given by $f(x) = |x|$, we conclude by Theorem A, that

$$|u| = f \circ (u_1, u_2) \in W^{1,1}(\Omega)$$

and

$$\partial_j |u| = \begin{cases} \dfrac{u_1 \partial_j u_1 + u_2 \partial_j u_2}{|u|}, & u \neq 0 \\ 0, & u = 0 \end{cases}$$

since $\mathrm{Re}(\bar{u}\partial_j u) = u_1 \partial_j u_1 + u_2 \partial_j u_2$, we have shown Corollary 1. □

Corollary 2. *Let* $u \in W^{1,p}(\Omega; \mathbb{R})$. *Then* $u_+ = \max(u, 0)$ resp. $u_- = \min(u, 0)$ *belong to* $W^{1,p}(\Omega; \mathbb{R})$ *and*

$$\nabla u_+ = \begin{cases} \nabla u, & u > 0 \\ \mathbf{0}, & u \leqq 0 \end{cases} \quad \text{resp.} \quad \nabla u_- = \begin{cases} \mathbf{0}, & u \geqq 0 \\ \nabla u, & u < 0. \end{cases}$$

Proof. Consider the functions $f_+, f_- \in \mathscr{Lip}(\mathbb{R}; \mathbb{R}) \cap C^1(\mathbb{R} \setminus \{0\})$ given by $f_+(x) = \max(x, 0)$, $f_-(x) = \min(x, 0)$. In view of Theorem A we have $u_{(\pm)} = f_{(\pm)} \circ u \in W^{1,p}(\Omega)$ and

$$\nabla(f_{(\pm)} \circ u) = \begin{cases} (f'_{(\pm)} \circ u) \nabla u, & u \neq 0 \\ \mathbf{0}, & u = 0. \end{cases} \quad \square$$

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