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Monotonicity and Dummy Free Property for Multi-Choice Cooperative Games

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Abstract: Given a coalition of an n-person cooperative game in characteristic function form, we can associate a zero-one vector whose non-zero coordinates identify the players in the given coalition. The cooperative game with this identification is just a map on such vectors. By allowing each coordinate to take finitely many values we can define multi-choice cooperative games. In such multi-choice games we can also define Shapley value axiomatically. We show that this multi-choice Shapley value is dummy free of actions, dummy free of players, non-decreasing for non-decreasing multi-choice games, and strictly increasing for strictly increasing cooperative games. Some of these properties are closely related to some properties of independent exponentially distributed random variables. An advantage of multi-choice formulation is that it allows to model strategic behavior of players within the context of cooperation.

Introduction

Often in the modelling of cooperative transferable utility games, the strategic aspects of the players are temporarily set aside and the characteristic function is computed, based on the maximum utility sum attainable under cooperation. For example, in a market game with transferable utility [14], the characteristic function is computed by maximizing the total utility that a coalition can get by reallocating the initial bundle among the members of the coalition. In such a formulation each player in any coalition has only two options: either quit the coalition or stay in the coalition and put in the highest levels of cooperation to maximize the coalitions total utility. Any hint about how the player's relative contribution is going to be valued by the coalition through a point solution such as the Shapley value [13] or the Nucleolus [9] or any other solution restricted to the coalition might completely change the player's attitude and effort level [3]. Suppose a landlord can harvest 20 bags of rice employing one honest hardworking peasant and 24 bags employing two such peasants. As an alternative plan, the landlord can harvest only 10 bags of rice employing a leisure seeking peasant and can produce 18 bags employing two such peasants. If the honest peasants come to know that the Nucleolus will be used to measure their contribution, it is clearly to their advantage to exert less, for the Nucleolus allocates only 2 bags to each peasant in the first case inspite of the greater effort and higher output

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while it allocates 4 bags to each peasant with lower effort and lower output! Similar examples can be formulated for the Shapley value also. Thus, it may be useful to model cooperative games with the possibility that players have more than one way of acting within a coalition.

Introduction. Let $N=\{1, 2, \ldots, n\}$ be the set of players. We allow each player to have $(m+1)$ actions, say $\sigma_0, \sigma_1, \sigma_2, \ldots, \sigma_m$, where σ_0 is the action to do nothing, while σ_k is the option to work at level k, which is better than σ_{k-1}.

Let $\Gamma=\{0, 1, \ldots, m\}$. The action space of N is defined by $\Gamma^n=\{(x_1, \ldots, x_n): x_i\in\Gamma, \forall i\in N\}$. Thus $(x_1, \ldots, x_n)$ is called an action vector of N, and $x_i=k$ if and only if player i takes action σ_k.

A multi-choice cooperative game in characteristic function form is the pair (Γ^n, V) defined by: $V:\Gamma^n\to R$, such that $V(\mathbf{O})=0$, where $\mathbf{O}=(0, 0, \ldots, 0)$.

We can identify the set of all multi-choice cooperative games by: $G\simeq R^{\Gamma^n-\{\mathbf{O}\}}$. As in [5], we define the value of a multi-choice cooperative game as a matrix.

Let $\phi: G\to M_{m\times n}$ be the function such that

$$\phi(V)=\begin{pmatrix}\phi_{11}(V) & \ldots & \phi_{1n}(V)\\ \phi_{21}(V) & & \phi_{2n}(V)\\ \vdots & & \vdots\\ \phi_{m1}(V) & \ldots & \phi_{mn}(V)\end{pmatrix}=(\vec{\phi}_1(V), \ldots, \vec{\phi}_n(V))$$

and

$$\vec{\phi}_i(V)=\begin{pmatrix}\phi_{1i}(V)\\ \phi_{2i}(V)\\ \vdots\\ \phi_{mi}(V)\end{pmatrix}.$$

Here $\phi_{ji}(V)$ is the power index or the value of player i when he takes action σ_j in game V.

Let $w:\Gamma\to R_+$ be a non-negative function such that $w(0)=0$, $w(0)\le w(1)\le w(2)\le\ldots\le w(m)$. Here w is called a **weight** of Γ. In [5], we showed that there exists a unique ϕ satisfying the following four axioms.

Axiom 1. Suppose $w(0), w(1), \ldots, w(m)$ are given. If V is of the form

$$V(\mathbf{y})=\begin{cases}c>0 & \text{if } \mathbf{y}\ge\mathbf{x}\\ 0 & \text{otherwise},\end{cases}$$

then $\phi_{x_i, i}(V)$ is proportional to $w(x_i)$.

Remark 1. The a priori weights are taken to be independent of the players and depend only on their action levels. Suppose we also make the weights to depend on players. Then our Shapley value can no longer be an extension of the classical Shapley value in the following sense. The classical games correspond to taking $m=1$ and different weights for different players will correspond to valueing players asymmetrically even for the unanimity game. While the classical Shapley value is fixed using only the three axioms (dummy, symmetry and efficiency) for unanimity games it will no more be possible with such asymmetric weights depending on the players.

A vector $\mathbf{x}^* \in \Gamma^n$ is called a **carrier** of V, if $V(\mathbf{x}^* \wedge \mathbf{x}) = V(\mathbf{x})$ for all $\mathbf{x} \in \Gamma^n$.

Axiom 2. If $\mathbf{x}^*$ is a carrier of V then, for $\mathbf{m} = (m, m, \ldots, m)$ we have

$$\sum_{\substack{x_i^* \neq 0 \\ x_i^* \in \mathbf{x}^*}} \phi_{x_i^*, i}(V) = V(\mathbf{m}).$$

By $x_i^* \in \mathbf{x}^*$ we mean x_i^* is the i-th component of $\mathbf{x}^*$.

Axiom 3. $\phi(V^1 + V^2) = \phi(V^1) + \phi(V^2)$, where $(V^1 + V^2)(\mathbf{x}) = V^1(\mathbf{x}) + V^2(\mathbf{x})$.

Axiom 4. Given $\mathbf{x}^0 \in \Gamma^n$, let $V(\mathbf{x}) = 0$, whenever $\mathbf{x} \not\geq \mathbf{x}^0$. Then $\phi_{k,i}(V) = 0$, for all $k < x_i^0$ and all $i \in N$.

Definition 1. Let $(\mathbf{x} \mid x_i = k)$ denote a vector with $x_i = k$. Player i is said to be a **dummy** player if $V((\mathbf{x} \mid x_i = k)) = V((\mathbf{x} \mid x_i = 0))$ for all $\mathbf{x} \in \Gamma^n$ and for all $k \in \Gamma$.

Definition 2. Given $\mathbf{x} \in \Gamma^n$, let $S(\mathbf{x}) = \{i \mid x_i \neq 0,\ x_i \text{ is a component of } \mathbf{x}\}$. Given $S \subseteq N$, let $\mathbf{b}(S)$ be the binary vector with coordinates $b_i(S)$ satisfying

$$b_i(S) = \begin{cases} 1 & \text{if } i \in S \\ 0 & \text{otherwise.} \end{cases}$$

Let $|S|$ be the number of elements of S.

Definition 3. Given Γ^n and $w(0) = 0, w(1), \ldots, w(m)$, for any $\mathbf{x} \in \Gamma^n$, we define $\|\mathbf{x}\|_w = \sum_{r=1}^{n} w(x_r)$.

Definition 4. Given $\mathbf{x} \in \Gamma^n$, and $j \in N = \{1, 2, \ldots, n\}$ we define $M_j(\mathbf{x}) = \{i \mid x_i \neq m, i \neq j\}$.

From theorem 2 in [5] we have

$$\phi_{ij}(V) = \sum_{k=1}^{i} \sum_{\substack{x_j = k \\ \mathbf{x} \neq \mathbf{O} \\ \mathbf{x} \in \Gamma^n}} \left[\sum_{T \subseteq M_j(\mathbf{x})} (-1)^{|T|} \frac{w(x_j)}{\|\mathbf{x}\|_w + \sum_{r \in T} [w(x_r + 1) - w(x_r)]} \right] \times [V(\mathbf{x}) - V(\mathbf{x} - \mathbf{b}(\{j\}))]. \qquad (**)$$

Given $N = \{1, 2, \ldots, n\}$, $\Gamma = \{0, 1, \ldots, m\}$, and a multi-choice cooperative game (Γ^n, V), suppose $\phi(V) = (a_{ij})_{m \times n}$. Suppose we allow a dummy player, say $(n+1)$ to join the game. Then we have a new game (Γ^{n+1}, V^D) such that

$V^D((\mathbf{x}|x_{n+1}=i))=V(\mathbf{x})$, for all $\mathbf{x}\in\Gamma^n$ and all $i\in\Gamma$. (Γ^{n+1}, V^D) is called a dummy extension of (Γ^n, V).

Suppose $\phi(V^D)=(b_{ij})_{m\times(n+1)}$; it is clear that $b_{i,(n+1)}=0$, $\forall i\in\Gamma$. Now we could ask whether $a_{ij}=b_{ij}$ for all $i\in\{1, 2, \ldots, m\}$ and all $j\in N$. A solution of a multi-choice cooperative game is said to be **dummy free of players** if $a_{ij}=b_{ij}$ for all $i\in\{1, 2, \ldots, m\}$ and all $j\in N$; otherwise the solution is said to be dummy dependent of players. We will show that our extended Shapley value is dummy free of players.

Remark 2. Though in [5] we gave the explicit formula for the multi-choice Shapley value, the following representation of a multi-choice game in terms of unanimity games is useful for proving the dummy free property.

Theorem 1. A multi-choice cooperative game (Γ^n, V) can be written as:

$$V=\sum_{\substack{\mathbf{x}\in\Gamma^n\\ \mathbf{x}\neq\mathbf{O}}} a_{\mathbf{x}} V^{\mathbf{x}},$$

where $a_{\mathbf{x}}=\sum_{S\subseteq S(\mathbf{x})}(-1)^{|S|}V(\mathbf{x}-\mathbf{b}(S))$, and

$$V^{\mathbf{x}}(\mathbf{z})=\begin{cases}1 & \text{if } \mathbf{z}\geq\mathbf{x}\\ 0 & \text{otherwise}.\end{cases}$$

Proof. Given $\mathbf{x}\in\Gamma^n$, suppose $S(\mathbf{x})=\{k_1, k_2, \ldots, k_p\}$. Let

$$A=\{\mathbf{y}\,|\,\mathbf{y}\leq\mathbf{x}\}=\{\mathbf{y}\,|\,\mathbf{y}\leq\mathbf{x}-\mathbf{b}(\emptyset)\}.$$

For $S\subseteq S(\mathbf{x})$, $S\neq\emptyset$ define

$$A(S)=\{\mathbf{y}\,|\,\mathbf{y}\leq\mathbf{x},\ y_{k_i}<x_{k_i},\ \forall k_i\in S\}=\{\mathbf{y}\,|\,\mathbf{y}\leq\mathbf{x}-\mathbf{b}(S)\}.$$

Then by the inclusion-exclusion principle we have:

$$\begin{aligned}\{\mathbf{x}\}=&A-A(\{k_1\})-A(\{k_2\})-\ldots-A(\{k_p\})+A(\{k_1, k_2\})\\ &+\ldots+A(\{k_{p-1}, k_p\})-A(\{k_1, k_2, k_3\})-\ldots\\ &+(-1)^{|S|}A(S)+\ldots+(-1)^{|S(\mathbf{x})|}A(S(\mathbf{x})).\end{aligned}$$

Given the real function $a_{\mathbf{x}}$ on the finite set Γ^n one can associate a signed measure $\mu(E)=\sum_{\mathbf{x}\in E}a_{\mathbf{x}}$. The additivity of this signed measure gives

$$\begin{aligned}a_{\mathbf{x}}=&\sum_{\mathbf{y}\leq\mathbf{x}}a_y-\sum_{\mathbf{y}\leq\mathbf{x}-\mathbf{b}(\{k_1\})}a_y-\ldots-\sum_{\mathbf{y}\leq\mathbf{x}-\mathbf{b}(\{k_p\})}a_{\mathbf{y}}\\ &+\ldots+(-1)^{|S|}\sum_{\mathbf{y}\leq\mathbf{x}-\mathbf{b}(S)}a_{\mathbf{y}}+\ldots+(-1)^p\sum_{\mathbf{y}\leq\mathbf{x}-\mathbf{b}(S(\mathbf{x}))}a_{\mathbf{y}}.\end{aligned}\tag{1.1}$$

In [5] Theorem 1, we have proved that V can be written as

$$V = \sum_{\substack{\mathbf{y} \neq \mathbf{O} \\ \mathbf{y} \in \Gamma^n}} a_{\mathbf{y}} V^{\mathbf{y}}.$$

Since $V^{\mathbf{y}}\mathbf{x} = 1$ when $\mathbf{y} \leq \mathbf{x}$ it is clear that

$$V(\mathbf{x}) = \sum_{\mathbf{y} \leq \mathbf{x}} a_{\mathbf{y}}, \quad \text{for all } \mathbf{x} \in \Gamma, \mathbf{x} \neq \mathbf{O}. \tag{1.2}$$

Namely $V(\mathbf{x})$ is simply the sum of all the coefficients $a_{\mathbf{y}}$ where $\mathbf{y}$ is any vector dominated by $\mathbf{x}$. By (1.1) and (1.2) we have:

$$\begin{aligned} a_{\mathbf{x}} = {} & V(\mathbf{x}) - V(\mathbf{x} - \mathbf{b}(\{k_1\})) - \ldots - V(\mathbf{x} - \mathbf{b}(\{k_p\}) \\ & + V(\mathbf{x} - \mathbf{b}(\{k_1, k_2\})) + \ldots + (-1)^{|S|} V(\mathbf{x} - \mathbf{b}(S)) \\ & + V(-1)^p V(\mathbf{x} - \mathbf{b}(S(\mathbf{x}))). \end{aligned}$$

Therefore $a_{\mathbf{x}} = \sum_{S \subseteq S(\mathbf{x})} (-1)^{|S|} V(\mathbf{x} - \mathbf{b}(S))$. ◇

Theorem 2. The Shapley value for a multi-choice cooperative game is dummy free of players.

Proof. Given a multi-choice cooperative game (Γ^n, V) and its dummy extension (Γ^{n+1}, V^D), suppose player $(n+1)$ is the dummy player.

Now V can be written as:

$$V = \sum_{\substack{\mathbf{x} \neq \mathbf{O} \\ \mathbf{x} \in \Gamma^n}} a_{\mathbf{x}} V^{\mathbf{x}}.$$

Also, V^D can be written as:

$$V^D = \sum_{\substack{\mathbf{x} \neq \mathbf{O} \\ \mathbf{x} \in \Gamma^{n+1}}} c_{\mathbf{x}} V^{\mathbf{x}}.$$

Given $\mathbf{x} \in \Gamma^n$, define $(\mathbf{x}|k) = (x_1, \ldots, x_n, k)$. It is easy to observe that:

$$V^D = \sum_{k=0}^{m} \sum_{\substack{\mathbf{x} \neq \mathbf{O} \\ \mathbf{x} \in \Gamma^n}} c_{(\mathbf{x}|k)} V^{(\mathbf{x}|k)}.$$

Now, for any $k \neq 0$, $k \in \Gamma$, given $\mathbf{x} \in \Gamma^n$, consider $c_{(\mathbf{x}|k)}$. By Theorem 1 we have:

$$c_{(\mathbf{x}|k)} = \sum_{S \subseteq S((\mathbf{x}|k))} (-1)^{|S|} V^D((\mathbf{x}|k) - \mathbf{b}(S)).$$

Since $k \neq 0$, we have $(n+1) \in S((\mathbf{x}|k))$. Thus we can rewrite $c_{(\mathbf{x}|k)}$ as:

$$\begin{aligned} c_{(\mathbf{x}|k)} &= \sum_{\substack{(n+1) \notin S \\ S \subseteq S((\mathbf{x}|k))}} (-1)^{|S|} V^D((\mathbf{x}|k) - \mathbf{b}(S)) \\ &+ \sum_{\substack{(n+1) \in S \\ S \subseteq S((\mathbf{x}|k))}} (-1)^{|S|} V^D((\mathbf{x}|k) - \mathbf{b}(S)) \\ &= \sum_{\substack{(n+1) \notin S \\ S \subseteq S((\mathbf{x}|k))}} (-1)^{|S|} [V^D((\mathbf{x}|k) - \mathbf{b}(S)) \\ &- V^D((\mathbf{x}|k) - \mathbf{b}(S \cup \{n+1\}))]. \end{aligned}$$

But $V^D((\mathbf{x}|k) - \mathbf{b}(S)) - V^D((\mathbf{x}|k) - \mathbf{b}(S \cup \{n+1\})) = 0$, for all $\mathbf{x} \in \Gamma^n$ and all $k \neq 0$.

Therefore $c_{(\mathbf{x}|k)} = 0$, for all $k \neq 0$ and all $\mathbf{x} \in \Gamma^n$. Thus

$$V^D = \sum_{\substack{\mathbf{x} \neq \mathbf{O} \\ \mathbf{x} \in \Gamma^n}} c_{(\mathbf{x}|0)} V^{(\mathbf{x}|0)}.$$

Now it is easy to see that $c_{(\mathbf{x}|0)} = a_{\mathbf{x}}$ for all $\mathbf{x} \in \Gamma^n$, and by axiom 1, it is also clear that $\phi_{ij}(V^{(\mathbf{x}|0)}) = \phi_{ij}(V^{\mathbf{x}})$ for all $i \in \{1, 2, \ldots, m\}$ and $j \in N$.

Hence by axiom 3, we conclude that $\phi_{ij}(V) = \phi_{ij}(V^D)$ for all $i \in \{1, \ldots, m\}$ and all $j \in N$. Moreover $\phi_{i,(n+1)}(V^D) = 0 \; \forall i \in \{1, \ldots, m\}$, and the proof is completed. ◇

Remark 3. If a solution of a game is *dummy dependent of a player*, then the dummy player acts like a catalyst, and those who are not dummy will invite him to join the game if he can make their income bigger. Conversely, they will reject the dummy player joining the game if he will make their income smaller.

Definition 5. Given (Γ^n, V) and given $\mathbf{x} \in \Gamma^n - \{\mathbf{O}\}$, let $A_i(\mathbf{x}) = \{j | x_j = i\}$. We call the action σ_i a **dummy action** if $V(\mathbf{x}) = V(\mathbf{x} - \mathbf{b}(T))$, for all $T \subseteq A_i(\mathbf{x})$ and all $\mathbf{x} \in \Gamma^n$.

Given (Γ^n, V), where $\Gamma = \{0, 1, \ldots, m\}$, and given $r \in \Gamma$, allow players to have one more choice $\sigma_{r'}$, such that $\sigma_{r'}$ has a level which is in between σ_r and σ_{r+1}. When $r = m$, we assume $r' = m+1$.

Thus we have a new action space Γ_*^n, where $\Gamma_* = \{0, 1, \ldots, r, r', (r+1), \ldots, m\}$. Let (Γ_*^n, V^A) be the game such that $V^A(\mathbf{x}) = V(\mathbf{x})$ whenever $\mathbf{x} \in \Gamma_*^n \cap \Gamma^n = \Gamma^n$. If $\sigma_{r'}$ is a dummy action of V^A, then we call (Γ_*^n, V^A) a **dummy action extension** of (Γ^n, V).

After extending Γ^n to Γ_*^n, we encounter a notational difficulty with $\Gamma_* = \{0, 1, \ldots, r, r', r+1, \ldots, m\}$, where r' denotes an action whose level is no lower than r and no higher than $r+1$. Here r' is just a number but not necessarily a natural number. We decide to leave r' alone, and make the following modification: Given $\mathbf{x} \in \Gamma_*^n$ with $A_{r'} = \{j | x_j = r'\}$, for any $S \subseteq A_{r'}(\mathbf{x})$, we have $\mathbf{x} - \mathbf{b}(S) = \mathbf{y}$, where $y_i = x_i$ if $i \notin S$ and $y_i = r$ (one level lower than r') if $i \in S$. Similarly, for any $T \subseteq A_{r+1}(\mathbf{x})$, we have $\mathbf{x} - \mathbf{b}(T) = \mathbf{z}$, where $z_i = x_i$ if $i \notin T$ and $z_i = r'$ if $i \in T$.

Suppose ϕ is a solution of (Γ^n, V), and $\phi(V) = (a_{ij})_{m \times n}$, suppose after a dummy action extension,

$$\phi(V^A) = \begin{pmatrix} b_{11} & \dots & b_{1n} \\ \vdots & & \vdots \\ b_{r'1} & & b_{r'n} \\ \vdots & & \vdots \\ b_{m1} & & b_{mn} \end{pmatrix}.$$

Then the solution ϕ is said to be **dummy free of action** if $a_{ij} = b_{ij}$ for all $i \in \{1, 2, \dots, m\}$ and all $j \in N$. Otherwise, ϕ is said to be dummy dependent of action.

Theorem 3. The Shapley value for multi-choice cooperative game is dummy free of action, and the value for a player using the dummy action is same as the value using the action, one step lower.

Proof. Given (Γ^n, V) and its dummy action extension game (Γ^n_*, V^A) with the dummy action $\sigma_{r'}$, by Theorem 1.1 we have:

$$V = \sum_{\substack{\mathbf{x} \neq \mathbf{O} \\ \mathbf{x} \in \Gamma^n}} a_{\mathbf{x}} V^{\mathbf{x}}, \quad \text{where } a_{\mathbf{x}} = \sum_{S \subseteq S(\mathbf{x})} (-1)^{|S|} V(\mathbf{x} - \mathbf{b}(S))$$

and

$$V^A = \sum_{\substack{\mathbf{x} \neq \mathbf{O} \\ \mathbf{x} \in \Gamma^n_*}} d_{\mathbf{x}} V^{\mathbf{x}}, \quad \text{where } d_{\mathbf{x}} = \sum_{S \subseteq S(\mathbf{x})} (-1)^{|S|} V^A(\mathbf{x} - \mathbf{b}(S)).$$

It is easy to see that $\mathbf{x} \in \Gamma^n$ if and only if $A_{r'}(\mathbf{x}) = \emptyset$. Now given $\mathbf{x} \in \Gamma^n_*$, consider the following two cases.

Case 1. When $A_{r'}(\mathbf{x}) = \emptyset$ and $A_{r+1}(\mathbf{x}) = \emptyset$, then $V(\mathbf{x} - \mathbf{b}(S)) = V^A(\mathbf{x} - \mathbf{b}(S))$ for all $S \subseteq S(\mathbf{x})$. Hence $d_{\mathbf{x}} = a_{\mathbf{x}}$.

Case 2. When $A_{r'}(\mathbf{x}) = \emptyset$ and $A_{r+1}(\mathbf{x}) \neq \emptyset$, it is clear that $A_{r+1}(\mathbf{x}) \subseteq S(\mathbf{x})$. Since $A_{r'}(\mathbf{x}) = \emptyset$, we may consider $\mathbf{x} = \mathbf{y}$, $\mathbf{y} \in \Gamma^n$. Now, since $\sigma_{r'}$ is a dummy action we have:

$$\begin{aligned} V^D(\mathbf{x} - \mathbf{b}(S)) &= V^D(\mathbf{x} - \mathbf{b}(S) - \mathbf{b}(S \cap A_{r+1}(\mathbf{x}))) \\ &= V(\mathbf{y} - \mathbf{b}(S)) = V(\mathbf{x} - \mathbf{b}(S)), \quad \text{for all } S \subseteq S(\mathbf{x}) = S(\mathbf{y}). \end{aligned}$$

Hence $d_{\mathbf{x}} = a_{\mathbf{x}}$.

From Case 1 and Case 2 we can conclude that

$$d_{\mathbf{x}} = a_{\mathbf{x}} \quad \text{for all } \mathbf{x} \in \Gamma^n_* \cap \Gamma^n = \Gamma^n. \tag{3.1}$$

When $\mathbf{x} \in \Gamma^n_* - \Gamma^n$, then $A_{r'}(\mathbf{x}) \neq \emptyset$. Choose $j^* \in A_{r'}(\mathbf{x})$. Since $A_{r'}(\mathbf{x}) \subseteq S(\mathbf{x})$ we have:

$$\begin{aligned} d_{\mathbf{x}} &= \sum_{S \subseteq S(\mathbf{x})} (-1)^{|S|} V^A(\mathbf{x}-\mathbf{b}(S)) \\ &= \sum_{\substack{j^* \notin S \\ S \subseteq S(\mathbf{x})}} (-1)^{|S|} V^A(\mathbf{x}-\mathbf{b}(S)) \\ &\quad + \sum_{\substack{j^* \in S \\ S \subseteq S(\mathbf{x})}} (-1)^{|S|+1} V^A(\mathbf{x}-\mathbf{b}(S \cup \{j^*\})) \\ &= \sum_{\substack{j^* \notin S \\ S \subseteq S(\mathbf{x})}} (-1)^{|S|} [V^A(\mathbf{x}-\mathbf{b}(S)) - V^A(\mathbf{x}-\mathbf{b}(S \cup \{j^*\}))]. \end{aligned}$$

But $V^A(\mathbf{x}-\mathbf{b}(S)) - V^A(\mathbf{x}-\mathbf{b}(S \cup \{j^*\})) = 0$ for all $S \subseteq S(\mathbf{x})$. Therefore,

$$d_{\mathbf{x}} = 0 \quad \text{for all } \mathbf{x} \in \Gamma^n_* - \Gamma^n. \tag{3.2}$$

Given $w(0) = 0 < w(1) \leq w(2) \leq \ldots \leq w(r) \leq w(r') \leq w(r+1) \leq \ldots \leq w(m)$, by the proof of Theorem 1 in [5] we have: For $\mathbf{x} \in \Gamma^n$,

$$\vec{\phi}_j(V^{\mathbf{x}}) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \frac{w(x_j)}{\|\mathbf{x}\|_w} \\ \vdots \\ \frac{w(x_j)}{\|\mathbf{x}\|_w} \end{pmatrix} \begin{matrix} \text{1-st} \\ \\ \\ x_j\text{-th} \\ \\ m\text{-th} \end{matrix} \tag{3.3}$$

and for $\mathbf{x} \in \Gamma^n_*$ with $r' > x_j$,

$$\vec{\phi}_j(V^{\mathbf{x}}) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \frac{w(x_j)}{\|\mathbf{x}\|_w} \\ \vdots \\ \frac{w(x_j)}{\|\mathbf{x}\|_w} \\ \vdots \\ \frac{w(x_j)}{\|\mathbf{x}\|_w} \end{pmatrix} \begin{matrix} \text{1-st} \\ \\ x_j\text{-th} \\ \\ \\ r'\text{-th} \\ \\ m\text{-th} \end{matrix} \tag{3.4}$$

By (3.1), (3.2), (3.3), (3.4) and axiom 3, we conclude that $\phi_{ij}(V) = \phi_{ij}(V^A)$, for all $i \neq r'$ and all $j \in N$. Moreover, $\phi_{r',j}(V) = \phi_{r,j}(V)$, for all $j \in N$. ◇

Remark 4. When players are playing a multi-choice cooperative game, until and unless the solution is dummy free of action some players could claim to be playing the game (Γ^n, V), while some other players could say we are playing (Γ^n_*, V) and as a result the Shapley value for a specific action can be changed by introducing *spurious* actions.

Remark 5. Since the extended Shapley value is dummy free of action, in many cases we can insert some dummy actions to make the calculations of the Shapley value easier. For example, if $w(0)=0$, $w(1)=1$, $w(2)=3$, $\Gamma=\{0, 1, 2\}$, then we may insert a dummy action σ_{1^1} such that $w(0)=0$, $w(1)=1$, $w(1^1)=2$, $w(2)=3$, and $\Gamma_*=\{0, 1, 1^1, 2\}$. Then Γ_* has the property that when the level of action increases, the weights also increase at a fixed rate, and this property *can* make the calculation easier. See formula (4.1) below.

Remark 6. Given a game (Γ^n, V) if we can make $w(i)=i$ without inserting too many dummy actions, then using formula (4.1) is easier than using formula (**) for the following reason. If we use a computer program to calculate the Shapley value using formula (**), since $w(r+1)-w(r)$ is not a constant for all $r\in T$ in (**), we must write a computer program that writes out all the elements in T and all the T's$\in M_j(\mathbf{x})$. If we use formula (4.1), all we have to do is to consider the number of elements in T.

Lemma 1. Let $f(z) = \sum_{k=0}^{m} (-1)^k \binom{m}{k} \cdot \frac{1}{z+k}$ then

$$f(z) = \frac{m!}{\prod_{k=0}^{m} (z+k)}.$$

Proof.

$$\begin{aligned} f(z) &= \sum_{k=0}^{m} (-1)^k \binom{m}{k} \cdot \int_0^1 x^{z+k-1} dx \\ &= \int_0^1 \sum_{k=0}^{m} (-1)^k \binom{m}{k} \cdot x^{z+k-1} dx \\ &= \int_0^1 x^{z-1} \sum_{k=0}^{m} (-1)^k \binom{m}{k} \cdot x^k dx \\ &= \int_0^1 x^{z-1} (1-x)^{m+1-1} dx = \frac{m!}{\prod_{k=0}^{m} (z+k)}. \end{aligned}$$

◇

Theorem 4. Given (Γ^n, V) if $w(0)=0$, $w(1)=1$, $w(2)=2$, $\ldots$, $w(m)=m$, then the Shapley value is:

$$\phi_{ij}(V) = \sum_{k=1}^{i} \sum_{\substack{x_j=k \\ \mathbf{x}\neq \mathbf{O} \\ \mathbf{x}\in\Gamma^n}} \frac{k\cdot |M_j(\mathbf{x})|!}{\prod_{t=0}^{|M_j(\mathbf{x})|} \left[\sum_{r=1}^{n} x_r + t\right]} [V(\mathbf{x}) - V(\mathbf{x}-\mathbf{b}(\{j\}))]. \tag{4.1}$$

Proof. When $w(i)=i$, $\forall i\in\Gamma$, we have $\|\mathbf{x}\|_w = \sum_{r=1}^{n} x_r$. Since $w(i+1)-w(i)=1$, $\forall i\in\{0, 1, \ldots, m-1\}$ $t = \sum_{r\in T} [w(x_r+1)-w(x_r)] = |T|$. By Lemma 1 applied to the formula (**) we get (4.1).

Remark 7. Given a game (Γ^n, V) if we can make $w(i)=i$ without inserting too many dummy actions, the formula (4.1) looks somewhat simpler to calculate the Shapley value.

Definition 6. A multi-choice cooperative game (Γ^n, V) is said to be non-decreasing, if $x\geq y\Rightarrow V(\mathbf{x})\geq V(\mathbf{y})$.

Given a non-decreasing multi-choice cooperative game (Γ^n, V), and its Shapley value $\phi_{i,j}(V)$, naturally we expect that

$$0=\phi_{0,j}(V)\leq\phi_{1,j}(V)\leq\phi_{2,j}(V)\leq\ldots\leq\phi_{m,j}(V), \quad \text{for all } j\in N.$$

Before proving this expected result, we would prove the following inequality, due to shapley [12].

Theorem 5. Given a set $M=\{1, 2, \ldots, m\}$, and scalars $x>0$, $y_i\geq 0$, $i=1, \ldots, m$ let

$$f(x, y_1, \ldots, y_m) = \sum_{S\subseteq M} (-1)^{|S|} \frac{1}{x+\sum_{i\in S} y_i}.$$

Then $f(x, y_1, \ldots, y_m)\geq 0$.

We would like to present the following elegant proof which replaces shapley's inductive proof [12]. Also see [6].

Proof. (R. B. Bapat) We will temporarily assume that the y_j's are positive. Let $y(S)$ denote the sum $\sum_{i\in S} y_i$. Consider the integral

$$\int_0^1 p^{x-1} \prod_{i=1}^{m} (1-p^{y_i})dp\geq 0.$$

This reduces to

$$\int_0^1 p^{x-1} \sum_{S\subseteq M} (-1)^{|S|} p^{y(S)}dp$$

which when integrated gives the required inequality. Since $x>0$, for each S, the function $g_S(y_1, y_2, \ldots, y_n) = \frac{1}{x+\sum_{i\in S} y_i}$ is continuous at all y_i's nonnegative. Thus the general case follows by continuity.

Remark 8. Consider independent exponential random variables X, Y_i, $i=1, 2, \ldots, m$ with parameters x, y_i, $i=1, 2, \ldots, m$ respectively. We will assume that all the parameters are positive. Then the above inequality is precisely the same as the expectation inequality[3]

$$\mathrm{E}\{\max(X, Y_1, Y_2, \ldots, Y_m)-\max(Y_1, Y_2, \ldots, Y_m)\}\geq 0.$$

The same inequality can also be proved[4] by expanding the inequality $P(\cup A_i)\leq 1$ via the inclusion-exclusion principle. Here the events $A_i=\{X\leq Y_i\}$, $i=1, 2\ldots$, and the events $A_iA_j=\{X\leq \min\{Y_i, Y_j\}\}$ and so on. Since the minimum of two independent exponential random variables is exponential with its parameter as the sum of their parameters [1], the inequality follows from

$$1\geq P(\cup A_i)=\sum_i \frac{x}{x+y_i}-\sum_{i\neq j}\frac{x}{x+y_i+y_j}+\ldots$$

Definition 7. Let $\mathbf{x}$, $\mathbf{y}\in\Gamma^n$, we say that $\mathbf{x}$ is strictly greater than $\mathbf{y}$, denoted by $\mathbf{x}>\mathbf{y}$, if there exists a $j\in N$ such that $x_j<y_j$. A multi-choice cooperative game (Γ^n, V) is said to be strictly increasing if $V(\mathbf{x})>V(\mathbf{y})$ whenever $\mathbf{x}>\mathbf{y}$.

Theorem 6. Given a non-decreasing multi-choice cooperative game (Γ^n, V) and its Shapley value $\phi_{ij}(V)$, we have

$$0=\phi_{0,j}(V)\leq\phi_{1,j}(V)\leq\phi_{2,j}(V)\leq\ldots\leq\phi_{m,j}(V),$$

for all $j\in N$. In case the multi-choice cooperative game is strictly increasing, then we have

$$0=\phi_{0j}(V)<\phi_{1j}(V)<\phi_{2j}(V)<\ldots<\phi_{mj}(V),\quad \text{for all } j\in N.$$

Proof. Since V is non-decreasing, $[V(\mathbf{x})-V(\mathbf{x}-\mathbf{b}(\{j\}))]\geq 0$ for all $\mathbf{x}\in\Gamma^n$ in equation (**). Moreover, by Theorem 5 we have

$$\sum_{T\subseteq M_j(\mathbf{x})}(-1)^{|T|}\frac{w(x_j)}{\|\mathbf{x}\|_w+\sum_{r\in T}[w(x_r+1)-w(x_r)]}\geq 0,\quad \forall\mathbf{x}\in\Gamma^n.$$

Therefore, the conclusion follows. The second case follows easily the same way. ◇

Remark 9. Suppose we know that a multi-choice cooperative game is strictly increasing. Then a desirable property for a solution is to have increased payments to a

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[3] This was pointed out by Ravindra Bapat at the First Game Theory and Economics Conference, 1990 at I.S.I., New Delhi.

[4] This was pointed out by El-Neweihi.

player for higher levels of his action. Theorem 7 shows that the Shapley value for multi-choice cooperative games has this property.

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