

# The Convergence in $L^1$ of Singular Integrals in Harmonic Analysis and Ergodic Theory

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Communicated by Aline Bonami

**ABSTRACT.** We study the behavior of the ergodic singular integral  $\mathcal{T}$  associated to a nonsingular measurable flow  $\{\tau_t : t \in \mathbb{R}\}$  on a finite measure space and a Calderón–Zygmund kernel with support in  $(0, \infty)$ . We show that if the flow preserves the measure or, with more generality, if the flow is such that the semiflow  $\{\tau_t : t \geq 0\}$  is Cesàro-bounded,  $f$  and  $\mathcal{T}f$  are integrable functions, then the truncations of the singular integral converge to  $\mathcal{T}f$  not only in the a.e. sense but also in the  $L^1$ -norm. To obtain this result we study the problem for the singular integrals in the real line and in the setting of the weighted  $L^1$ -spaces.

## 1. Introduction

A function  $K \in L^1_{\text{loc}}(\mathbb{R} - \{0\})$  is a Calderón–Zygmund kernel if it satisfies the following properties:

- (1.1) There exists a constant  $B_1$  such that  $\left| \int_{\varepsilon < |x| < N} K(x) dx \right| \leq B_1$ , for all  $\varepsilon$  and  $N$  with  $0 < \varepsilon < N$ , and there exists the limit  $\lim_{\varepsilon \rightarrow 0} \int_{\varepsilon < |x| < 1} K(x) dx$ .
- (1.2) There exists a constant  $B_2$  such that  $|K(x)| \leq \frac{B_2}{|x|}$ , for all  $x \neq 0$ .
- (1.3) There exists a constant  $B_3$  such that  $|K(x-y) - K(x)| \leq B_3|y||x|^{-2}$ , for all  $x$  and  $y$  with  $|x| > 2|y| > 0$ .

Associated to  $K$  and for all  $\varepsilon > 0$  we define the truncated

$$T_\varepsilon f(x) = \int_{|y| > \varepsilon} K(y) f(x-y) dy$$

and the singular integral  $Tf(x) = \lim_{\varepsilon \rightarrow 0} T_\varepsilon f(x)$ .

*Math Subject Classifications.* primary 28D05; secondary 42B20.

*Keywords and Phrases.* one-sided singular integrals, Cesàro-bounded flows, weights.

*Acknowledgements and Notes.* This research has been partially supported by a D.G.I.C.Y.T. grant (PB94-1496), a D.G.E.S. grant (PB97-1097) and Junta de Andalucía.

It is known [7] that if  $\omega$  is a nonnegative function that satisfies the Muckenhoupt  $A_p$  condition,  $1 < p < \infty$  [12], then the singular integral  $Tf$  is defined as a pointwise limit of  $T_\varepsilon f$  for all  $f \in L^p(\omega)$  and  $T$  is bounded from  $L^p(\omega)$  into  $L^p(\omega)$ . For  $p = 1$  we have that if  $\omega$  satisfies  $A_1$ , then  $Tf$  is defined in the same way for all  $f \in L^1(\omega)$  and  $T$  is of weak type (1,1) with respect to  $\omega$ . These results together with the corresponding ones for the maximal operator  $T^*f(x) = \sup_{\varepsilon>0} |T_\varepsilon f(x)|$  give easily that if  $\omega$  satisfies  $A_p$ ,  $1 < p < \infty$ , then  $T_\varepsilon f$  converges to  $Tf$  in the  $L^p(\omega)$ , for all  $f \in L^p(\omega)$ . In the same way, if  $\omega \in A_1$ , the weak type inequality (1,1) implies that  $T_\varepsilon f$  converges to  $Tf$  in measure for all  $f \in L^1(\omega)$ . Then a natural question arises: if  $f \in L^1(\omega)$  and  $Tf \in L^1(\omega)$ , when does  $T_\varepsilon f$  converge to  $Tf$  in the  $L^1(\omega)$ -norm? The answer to this question was put forth by Calderón and Capri [4] in 1984 for  $\omega = 1$  and generalized to general weights by Capri and Segovia [5] in 1986. In [5] it is shown that if  $\omega$  satisfies  $A_1$ ,  $f \in L^1(\omega)$ , and  $Tf \in L^1(\omega)$ , then  $T_\varepsilon f$  converges to  $Tf$  in the  $L^1(\omega)$ -norm.

In 1991 Asmar et al. [2] studied the same problem but in the setting of the ergodic theory and, more precisely, for the ergodic Hilbert transform. Given a finite measure space  $(X, \mathfrak{M}, \mu)$ , it is said that  $\{\tau_t : t \in \mathbb{R}\}$  is a flow defined on  $X$  if for all  $t \in \mathbb{R}$ ,  $\tau_t$  is a measurable map from  $X$  to  $X$  such that

- (1)  $\tau_0$  is the identity on  $X$ ,
- (2)  $\tau_{t+s} = \tau_t \circ \tau_s$ ,
- (3) the map  $(x, t) \rightarrow \tau_t x$  from  $X \times \mathbb{R}$  into  $X$  is  $\tilde{\mathfrak{M}}\text{-}\mathfrak{M}$  measurable, where  $\tilde{\mathfrak{M}}$  is the completion of the product- $\sigma$ -algebra  $\mathfrak{M} \otimes \mathcal{B}$  of  $\mathfrak{M}$  with the Borel sets, and the completion is taken with respect to the product measure  $\mu$  on  $\mathfrak{M}$  and the Lebesgue measure on  $\mathcal{B}$ .

We shall say that the flow is measure preserving if

- (4)  $\mu(\tau_t E) = \mu(E)$  for all  $t \in \mathbb{R}$  and  $E \in \mathfrak{M}$ .

Associated to one of these flows and to a Calderón–Zygmund kernel we can define an ergodic singular integral by

$$\mathcal{T}f(x) = \lim_{\varepsilon \rightarrow 0} \mathcal{T}_\varepsilon f(x) = \lim_{\varepsilon \rightarrow 0} \int_{\varepsilon < |t| < 1/\varepsilon} K(t) f(\tau_t x) dt$$

where the limit will be understood in the a.e. sense. In particular if  $K(t) = \frac{1}{t}$  we obtain the ergodic Hilbert transform

$$\mathcal{H}f(x) = \lim_{\varepsilon \rightarrow 0} \mathcal{H}_\varepsilon f(x) = \lim_{\varepsilon \rightarrow 0} \int_{\varepsilon < |t| < 1/\varepsilon} \frac{f(\tau_t x)}{t} dt.$$

It is known (see, [8], [13] for these results) that if  $\{\tau_t : t \in \mathbb{R}\}$  is a measure preserving flow, then  $\mathcal{H}f$  is well defined as an almost everywhere limit for all  $f \in L^p(d\mu)$ ,  $1 \leq p < \infty$ ,  $\mathcal{H}$  is of strong type  $(p, p)$  for  $p > 1$  and of weak type (1,1). Furthermore,  $\mathcal{H}_\varepsilon f$  converges to  $\mathcal{H}f$  in the  $L^p(d\mu)$ -norm if  $1 < p < \infty$  for all  $f \in L^p(d\mu)$  and  $\mathcal{H}_\varepsilon f$  converges to  $\mathcal{H}f$  in measure for all  $f \in L^1(d\mu)$ . As in the case of the singular integrals in  $\mathbb{R}$ , this leaves open the following question: if  $f \in L^1(d\mu)$  and  $\mathcal{H}f \in L^1(d\mu)$ , does  $\mathcal{H}_\varepsilon f$  converge to  $\mathcal{H}f$  in the  $L^1(d\mu)$ -norm? This question was answered in the affirmative in [2] in a more general setting. In 1994 Martín–Reyes and the author [11] studied the behavior of the ergodic Hilbert transform associated to a nonsingular measurable flow which need not preserve the measure  $\mu$  but which is a Cesàro-bounded flow, which means the following:

- (a) if  $E \in \mathfrak{M}$  and  $\mu(E) = 0$ , then  $\mu(\tau_t E) = 0$  for all  $t \in \mathbb{R}$  and
- (b)  $\sup_{\varepsilon>0} \|\mathcal{A}_\varepsilon f\|_{L^1(d\mu)} \leq C \|f\|_{L^1(d\mu)}$  where

$$\mathcal{A}_\varepsilon f(x) = \frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} f(\tau_t x) dt.$$

They proved that if  $\{\tau_t : t \in \mathbb{R}\}$  is a Cesàro-bounded flow,  $f \in L^1(d\mu)$ , and  $\mathcal{H}f \in L^1(d\mu)$ , then  $\mathcal{H}_\varepsilon f$  converges to  $\mathcal{H}f$  in the  $L^1(d\mu)$ -norm as  $\varepsilon$  goes to 0. Analogous results could be obtained for general ergodic singular integrals  $\mathcal{T}f$  defined as above.

The final aim of the paper is to study the same results for ergodic singular integrals associated to a Calderón–Zygmund kernel with support in  $(0, \infty)$ , assuming that the semiflow  $\{\tau_t : t \geq 0\}$  is Cesàro-bounded. In order to prove our result in ergodic theory, the idea is to first check the problem in the real line and then to transfer the results to the ergodic theory. Notice that we are not transferring only maximal inequalities.

Therefore, let us go back now to the setting of the singular integrals in  $\mathbb{R}$ . Recently, Aimar et al. [1] have studied singular integrals associated to a Calderón–Zygmund kernel  $K$  with support in  $(0, \infty)$  or  $(-\infty, 0)$ . These are the one-sided singular integrals. They proved that the good weights for these operators are the one-sided  $A_p$  weights (see [14]). More precisely, they proved that if the support of  $K$  is contained in  $(0, \infty)$ ,  $1 < p < \infty$ ,  $\frac{1}{p} + \frac{1}{p'} = 1$ ,  $\omega$  is a nonnegative function and there exists a constant  $C > 0$  such that

$$(A_p^-) \quad \sup_{a < b < c} \int_b^c \omega \left( \int_a^b \omega^{1-p'} \right)^{p-1} \leq C(c-a)^p$$

then  $T$  and  $T^*$  are bounded from  $L^p(\omega)$  into  $L^p(\omega)$  and, thus,  $T_\varepsilon f$  converges to  $Tf$  in the  $L^p(\omega)$ -norm for all  $f \in L^p(\omega)$ . With the same kind of kernels and for  $p = 1$  they proved that if  $\omega$  satisfies

$$(A_1^-) \quad M^+ \omega(x) = \sup_{\varepsilon > 0} \frac{1}{\varepsilon} \int_x^{x+\varepsilon} \omega(t) dt \leq C\omega(x) \quad \text{a.e.}$$

for some constant  $C > 0$ , then  $T$  and  $T^*$  are of weak type  $(1,1)$  with respect to  $\omega$  and, consequently,  $T_\varepsilon f$  converges to  $Tf$  in measure for all  $f \in L^1(\omega)$ . This again leaves open the convergence of  $T_\varepsilon f$  to  $Tf$  in the  $L^1(\omega)$ -norm. In Section 2 we study this kind of singular integrals and we obtain the one-sided version of the result of Capri and Segovia. This is established in the following theorem:

**Theorem 1.**

Let  $K$  be a Calderón–Zygmund kernel with support in  $(0, \infty)$  and let  $T$  be the singular integral operator associated to  $K$ . Let  $\omega \in A_1^-$ . If  $f$  and  $Tf$  belong to  $L^1(\omega)$ , then  $T_\varepsilon f$  converges to  $Tf$  in the  $L^1(\omega)$ -norm as  $\varepsilon$  goes to 0.

We shall follow the steps of the proof of Capri and Segovia [5]. However, we point out that the proof is more difficult in our case because the hypothesis on the weight is weaker: this is the reason why we must work with functions with support contained in  $[0, \infty)$  (see Theorem 5 for instance).

We also define the class of  $A_1^+$  weights, which we shall use later: we say that  $\omega \in A_1^+$  if there exists a constant  $C > 0$  such that

$$M^- \omega(x) = \sup_{\varepsilon > 0} \frac{1}{\varepsilon} \int_{x-\varepsilon}^x \omega(t) dt \leq C\omega(x) \quad \text{a.e.}$$

As we said above, the final result is about ergodic singular integrals defined by a Calderón–Zygmund kernel with support in  $(0, \infty)$  and to a flow  $\{\tau_t : t \in \mathbb{R}\}$ , i.e.,

$$\mathcal{T}f(x) = \lim_{\varepsilon \rightarrow 0} \mathcal{T}_\varepsilon f(x) = \lim_{\varepsilon \rightarrow 0} \int_\varepsilon^{\frac{1}{\varepsilon}} K(t) f(\tau_t x) dt .$$

Under an additional assumption on  $K$ , we obtain the following theorem which is the one-sided version of the result in [11].

**Theorem 2.**

Let  $(X, \mathfrak{M}, \nu)$  be a finite measure space and let  $\{\tau_t : t \in \mathbb{R}\}$  be a nonsingular measurable flow such that

$$\sup_{\varepsilon > 0} \| \mathcal{A}_\varepsilon^+ f \|_{L^1(d\nu)} \leq C \| f \|_{L^1(d\nu)} ,$$

for all  $f \in L^1(d\nu)$ , where  $\mathcal{A}_\varepsilon^+ f(x) = \frac{1}{\varepsilon} \int_0^\varepsilon f(\tau_t x) dt$ . Let  $\mathcal{T}$  be a one-sided singular integral associated to the flow and to a Calderón–Zygmund kernel  $K$  with support in  $(0, \infty)$  for which there exists the limit  $\lim_{\varepsilon \rightarrow 0} \int_1^{\frac{1}{\varepsilon}} K(t) dt$ . If  $f \in L^1(d\nu)$  is such that  $\mathcal{T}f \in L^1(d\nu)$ , then  $\mathcal{T}_\varepsilon f$  converges to  $\mathcal{T}f$  in the  $L^1(d\nu)$ -norm as  $\varepsilon$  goes to 0.

We ask for the existence of the limit  $\lim_{\varepsilon \rightarrow 0} \int_1^{\frac{1}{\varepsilon}} K(t) dt$  to assure the existence of  $\mathcal{T}f$ , at least for  $f \in L^1(d\nu)$  which is constant on each orbit, i.e., such that  $f(\tau_t x) = f(x)$  for all  $t \in \mathbb{R}$  and  $x \in X$ .

The result is similar to Theorem 1 and, in fact, it can be considered as Theorem 1 transferred to the ergodic setting. Furthermore, our theorem extends the result in [2] to one-sided singular integrals assuming that the flow is Cesàro-bounded. Three main ideas are in the proof of this theorem: the first one is to show that the functions of  $L^1(d\nu)$  can be approximated by a suitable class of functions (Theorem 7); the second one is to express the action of the ergodic singular integrals in terms of the action of singular integrals on the real line and then to use the results of Section 2; the third one is to reduce the problem to the case in which the measure is preserved by the flow.

Throughout this paper, the letter  $C$  will always mean a positive constant not necessarily the same at each occurrence and if  $1 < p < \infty$ , then  $p'$  will denote its conjugate exponent, i.e., the number  $p'$  such that  $p + p' = pp'$ . Furthermore, the maximal operators associated to the averages  $\mathcal{A}_\varepsilon^+$ ,  $\mathcal{A}_\varepsilon^-$  and to the ergodic singular integral  $\mathcal{T}$  will be denoted by  $\mathcal{M}^+$ ,  $\mathcal{M}^-$ , and  $\mathcal{T}^*$ , respectively. Therefore,

$$\mathcal{M}^+ f(x) = \sup_{\varepsilon > 0} \mathcal{A}_\varepsilon^+ |f|(x), \quad \mathcal{M}^- f(x) = \sup_{\varepsilon > 0} \mathcal{A}_\varepsilon^- |f|(x) \quad \text{and} \quad \mathcal{T}^* f(x) = \sup_{\varepsilon > 0} |\mathcal{T}_\varepsilon f(x)|.$$

## 2. One-Sided Singular Integrals on the Real Line: Proof of Theorem 1

In the proof of Theorem 1 we can suppose that  $\omega > 0$  a.e. We shall need some results that we state in the following theorems:

### Theorem 3.

Let  $\varphi \geq 0$  be an integrable function with support in  $[0, \infty)$  and nonincreasing in  $[0, \infty)$ . For each  $\varepsilon > 0$ , let  $\varphi_\varepsilon(x) = \frac{1}{\varepsilon} \varphi\left(\frac{x}{\varepsilon}\right)$ . Then the following hold:

- (1) For all nonnegative  $f$  and for almost all  $x \in \mathbb{R}$ ,  $|f * \varphi_\varepsilon(x)| \leq M^- f(x) \int_0^\infty \varphi$ , where  $M^-$  is the one-sided Hardy–Littlewood maximal function  $M^- f(x) = \sup_{\varepsilon > 0} \frac{1}{\varepsilon} \int_{x-\varepsilon}^x |f(t)| dt$ .
- (2) If  $1 \leq p < \infty$ ,  $\omega \in A_p^-$  and  $f \in L^p(\omega)$ , then  $\|f * \varphi_\varepsilon\|_{L^p(\omega)} \leq C \|f\|_{L^p(\omega)} \int_0^\infty \varphi$ , where the constant  $C$  is independent of  $f$  and  $\varepsilon$ .
- (3) If  $1 \leq p < \infty$ ,  $\omega \in A_p^-$  and  $f \in L^p(\omega)$ , then, for almost every  $x \in \mathbb{R}$  (in particular for all  $x$  belonging to the Lebesgue set of  $f$ ),  $\lim_{\varepsilon \rightarrow 0} f * \varphi_\varepsilon(x) = f(x) \int_0^\infty \varphi(y) dy$ .
- (4) If  $1 \leq p < \infty$ ,  $\omega \in A_p^-$  and  $f \in L^p(\omega)$ , then  $\lim_{\varepsilon \rightarrow 0} \|f * \varphi_\varepsilon - f \int_0^\infty \varphi\|_{L^p(\omega)} = 0$ .

### Theorem 4.

Let  $g \geq 0$  be a bounded function with compact support contained in  $[0, \infty)$ . If  $f \in L^1(\omega)$  and  $\omega \in A_1^-$ , then  $f * g \in L^1(\omega)$ .

The following results show the behavior of the one-sided singular integrals acting over convolutions.

**Theorem 5.**

Let  $g \geq 0$  be as before and let  $T$  be a singular integral operator associated to a Calderón–Zygmund kernel  $K$  such that the support of  $K$  is contained in  $(0, \infty)$ . Let  $\omega \in A_1^-$  and  $f \in L^1(\omega)$ . If  $Tf \in L^1(\omega)$ , then

$$T(f * g)(x) = Tf * g(x) \quad \text{a.e.}$$

**Theorem 6.**

Let  $g \geq 0$  be an integrable function with support in  $[0, \infty)$ , nonincreasing in  $[0, \infty)$  and suppose that  $g \in L^{p_0}(\mathbb{R})$  for some  $p_0, 1 < p_0 \leq \infty$ . Let  $T$  be as in Theorem 5,  $\omega \in A_1^-$ , and  $f \in L^1(\omega)$ . Then

$$T(f * g)(x) = f * Tg(x) \quad \text{a.e.}$$

**Proof of Theorem 3.** Let  $f \geq 0$ . For each  $s > 0$  let  $h(s) = \sup\{t > 0 : \varphi(t) \geq s\}$  and  $\bar{s} = \sup\{s : h(s) > 0\}$ . Then, changing variables and using Fubini’s theorem, we have the following:

$$\begin{aligned} f * \varphi_\varepsilon(x) &= \int_0^\infty f(x-y) \frac{1}{\varepsilon} \varphi\left(\frac{y}{\varepsilon}\right) dy = \int_0^\infty \varphi(t) f(x-\varepsilon t) dt \\ &= \int_0^\infty f(x-\varepsilon t) \int_0^{\varphi(t)} 1 ds dt = \int_0^\infty \int_{\{t>0:\varphi(t)\geq s\}} f(x-\varepsilon t) dt ds \\ &= \int_0^{\bar{s}} \int_0^{h(s)} f(x-\varepsilon t) dt ds = \int_0^{\bar{s}} \frac{1}{\varepsilon} \int_0^{\varepsilon h(s)} f(x-u) du ds \\ &= \int_0^{\bar{s}} \frac{1}{\varepsilon} \int_{x-\varepsilon h(s)}^x f(y) dy ds = \int_0^{\bar{s}} h(s) \frac{1}{\varepsilon h(s)} \int_{x-\varepsilon h(s)}^x f(y) dy ds \\ &\leq \int_0^{\bar{s}} h(s) M^- f(x) ds \leq M^- f(x) \int_0^\infty |\{t > 0 : \varphi(t) \geq s\}| ds \\ &= M^- f(x) \int_0^\infty \int_{\{t>0:\varphi(t)\geq s\}} 1 dt ds = M^- f(x) \int_0^\infty \int_0^{\varphi(t)} 1 ds dt \\ &= M^- f(x) \int_0^\infty \varphi(t) dt . \end{aligned}$$

This proves (1).

To prove (2), we first suppose that  $p = 1$ . Then, changing variables, we have

$$\begin{aligned} \|f * \varphi_\varepsilon\|_{L^1(\omega)} &\leq \int_0^\infty \varphi_\varepsilon(y) \int_{\mathbb{R}} |f(x-y)| \omega(x) dx dy \\ &= \int_{\mathbb{R}} |f(x)| \int_0^\infty \varphi_\varepsilon(y) \omega(x+y) dy dx = \int_{\mathbb{R}} |f(x)| \varphi_\varepsilon * \bar{\omega}(-x) dx , \end{aligned} \quad (2.1)$$

where  $\bar{\omega}(x) = \omega(-x)$ . Taking into account (1) and the fact that  $\omega \in A_1^-$  we get

$$\begin{aligned} \int_{\mathbb{R}} |f(x)| \varphi_\varepsilon * \bar{\omega}(-x) dx &\leq \int_{\mathbb{R}} |f(x)| M^- \bar{\omega}(-x) dx \int_0^\infty \varphi = \int_{\mathbb{R}} |f(x)| M^+ \omega(x) dx \int_0^\infty \varphi \\ &\leq C \int_{\mathbb{R}} |f(x)| \omega(x) dx \int_0^\infty \varphi = C \|f\|_{L^1(\omega)} \int_0^\infty \varphi . \end{aligned} \quad (2.2)$$

Inequalities (2.1) and (2.2) immediately give (2) for  $p = 1$ .

Now let  $p > 1$ . Then Theorem 1 of [14] and (1) imply the following:

$$\|f * \varphi_\varepsilon\|_{L^p(\omega)} \leq \int_0^\infty \varphi(y) dy \left( \int_{\mathbb{R}} (M^- f(x))^p \omega(x) dx \right)^{1/p} \leq C \int_0^\infty \varphi(y) dy \|f\|_{L^p(\omega)} .$$

This proves (2) for  $p > 1$ .

Next, we are going to prove (3). Changing variables we obtain

$$\begin{aligned}
 \left| f * \varphi_\varepsilon(x) - f(x) \int_0^\infty \varphi(y) dy \right| &= \left| \int_0^\infty f(x-y) \frac{1}{\varepsilon} \varphi\left(\frac{y}{\varepsilon}\right) dy - f(x) \int_0^\infty \varphi(y) dy \right| \\
 &= \left| \int_0^\infty f(x-\varepsilon y) \varphi(y) dy - \int_0^\infty f(x) \varphi(y) dy \right| \leq \int_0^\infty |f(x-\varepsilon y) - f(x)| \varphi(y) dy \\
 &= \int_0^\infty |f(x-\varepsilon y) - f(x)| \int_0^{\varphi(y)} 1 ds dy \\
 &= \int_0^\infty \int_{\{y>0:\varphi(y)\geq s\}} |f(x-\varepsilon y) - f(x)| dy ds. \tag{2.3}
 \end{aligned}$$

Let  $h(s)$  and  $\bar{s}$  as before. Then

$$\begin{aligned}
 \int_0^\infty \int_{\{y>0:\varphi(y)\geq s\}} |f(x-\varepsilon y) - f(x)| dy ds &= \int_0^{\bar{s}} \int_0^{h(s)} |f(x-\varepsilon y) - f(x)| dy ds \\
 &= \int_0^{\bar{s}} \frac{1}{\varepsilon} \int_0^{\varepsilon h(s)} |f(x-y) - f(x)| dy ds \\
 &= \int_0^{\bar{s}} h(s) \frac{1}{\varepsilon h(s)} \int_0^{\varepsilon h(s)} |f(x-y) - f(x)| dy ds. \tag{2.4}
 \end{aligned}$$

The function

$$g(s) = h(s) \chi_{(0,\bar{s})}(s) \frac{1}{\varepsilon h(s)} \int_0^{\varepsilon h(s)} |f(x-y) - f(x)| dy$$

is bounded for a function of  $L^1(\mathbb{R})$ :

$$\begin{aligned}
 |g(s)| &= g(s) \leq h(s) \left( \frac{1}{\varepsilon h(s)} \int_0^{\varepsilon h(s)} |f(x-y)| dy + |f(x)| \right) \chi_{(0,\bar{s})}(s) \\
 &\leq h(s) (M^- f(x) + |f(x)|) \chi_{(0,\bar{s})}(s).
 \end{aligned}$$

Then, the dominated convergence theorem, the differentiation Lebesgue theorem, (2.3) and (2.4) give

$$\lim_{\varepsilon \rightarrow 0} \left| f * \varphi_\varepsilon(x) - f(x) \int_0^\infty \varphi \right| \leq \int_0^{\bar{s}} h(s) \lim_{\varepsilon \rightarrow 0} \left( \frac{1}{\varepsilon h(s)} \int_0^{\varepsilon h(s)} |f(x-y) - f(x)| dy \right) ds = 0,$$

for almost all  $x \in \mathbb{R}$ .

To prove (4) suppose first that  $p = 1$ . Then

$$\begin{aligned}
 \int_{\mathbb{R}} \left| f * \varphi_\varepsilon(x) - f(x) \int_0^\infty \varphi \right| \omega(x) dx &= \int_{\mathbb{R}} \left| f * \varphi_\varepsilon(x) - f(x) \int_0^\infty \varphi_\varepsilon \right| \omega(x) dx \\
 &\leq \int_{\mathbb{R}} \left( \int_0^\infty |f(x-y) - f(x)| \varphi_\varepsilon(y) dy \right) \omega(x) dx \\
 &= \int_0^\infty g(y) \varphi_\varepsilon(y) dy = \bar{g} * \varphi_\varepsilon(0), \tag{2.5}
 \end{aligned}$$

where  $g(y) = \int_{\mathbb{R}} |f(x-y) - f(x)| \omega(x) dx$ . Since  $g$  is a continuous function, then part (3) gives

$$\lim_{\varepsilon \rightarrow 0} \int_0^\infty g(y) \varphi_\varepsilon(y) dy = g(0) \int_0^\infty \varphi(y) dy = 0. \tag{2.6}$$

From (2.5) and (2.6) we get (4) for  $p = 1$ .

If  $p > 1$ , we apply (1) and obtain

$$\left| f * \varphi_\varepsilon(x) - f(x) \int_0^\infty \varphi \right|^p \leq C \left( \int_0^\infty \varphi \right)^p \left( (M^- f(x))^p + |f(x)|^p \right).$$

The function on the right-hand side belongs to  $L^1(\omega)$  by Theorem 1 of [14]. Then the dominated convergence theorem, together with (3), gives (4) for  $p > 1$ .  $\square$

**Proof of Theorem 4.** Suppose that the support of  $g$  is contained in  $[0, L]$ . Then, using the facts that  $\omega \in A_1^-$  and  $f \in L^1(\omega)$  we get

$$\begin{aligned} \int_{\mathbb{R}} |f * g(x)| \omega(x) dx &= \int_{\mathbb{R}} \left| \int_0^L f(x-t)g(t)dt \right| \omega(x) dx \leq \int_0^L |g(t)| \int_{\mathbb{R}} |f(x-t)| \omega(x) dx dt \\ &= \int_0^L |g(t)| \int_{\mathbb{R}} |f(x)| \omega(x+t) dx dt = \int_{\mathbb{R}} |f(x)| \int_0^L |g(t)| \omega(x+t) dt dx \\ &\leq L \|g\|_\infty \int_{\mathbb{R}} |f(x)| \frac{1}{L} \int_x^{x+L} \omega(t) dt dx \leq L \|g\|_\infty \int_{\mathbb{R}} |f(x)| M^+ \omega(x) dx \\ &\leq CL \|g\|_\infty \int_{\mathbb{R}} |f(x)| \omega(x) dx < \infty. \end{aligned} \quad \square$$

To prove Theorem 5 we shall use the following results:

**Lemma 1.**

Let  $T$  be as in Theorem 1. If  $\omega \in A_1^-$ , then  $T$  is of weak type  $(1,1)$  with respect to  $\omega$ , i.e., there exists a constant  $C$  such that for all  $f \in L^1(\omega)$  and all  $\lambda > 0$ ,

$$\omega(\{x : |Tf(x)| > \lambda\}) \leq \frac{C}{\lambda} \|f\|_{L^1(\omega)}.$$

If  $\omega \in A_p^-$ , then  $T$  is of strong type  $(p,p)$  with respect to  $\omega$ , i.e., there exists a constant  $C$  such that for all  $f \in L^p(\omega)$ ,

$$\|Tf\|_{L^p(\omega)} \leq C \|f\|_{L^p(\omega)}.$$

The same holds for  $T^*$ .

**Lemma 2.**

Let  $\omega \in A_1^-$  and  $R > 0$ . Then, for almost all  $x \in [-R, R]$ , the following inequality holds:

$$\frac{1}{\omega(x)} \leq C \frac{3R}{\int_R^{2R} \omega(y) dy}$$

where  $C$  is a positive constant independent on  $x$  and  $R$ .

**Proof of Lemma 2.** Using that  $\omega \in A_1^-$ , it is very easy to see that for almost all  $x \in [-R, R]$  we have

$$\frac{1}{3R} \int_R^{2R} \omega(y) dy \leq \frac{1}{2R-x} \int_x^{2R} \omega(y) dy \leq C \omega(x),$$

which gives the desired inequality.  $\square$

**Proof of Theorem 5.** By Theorem 4,  $f * g \in L^1(\omega)$ . Therefore,  $T(f * g)$  makes sense. On the other hand, since  $f \in L^1(\omega)$ ,  $Tf \in L^1(\omega)$ , and  $\omega \in A_1^-$ , then Theorem 4 implies that  $Tf * g$  makes sense and belongs to  $L^1(\omega)$ . Suppose that  $\text{supp } g \subset [0, L]$ . Let  $\{P_j\}_{j=1}^\infty$ ,  $P_j = \{t_{j,0}, t_{j,1}, \dots, t_{j,N_j}\}$ ,

be a sequence of partitions of the interval  $[0, L]$  such that  $P_j \subset P_{j+1}$  and the diameters decrease to 0 as  $j \rightarrow \infty$ . Define

$$S_j f(x) = \sum_{k=1}^{N_j} f(x - t_{j,k}) \int_{t_{j,k-1}}^{t_{j,k}} g(t) dt. \quad (2.7)$$

**Claim:** We claim that for each  $R > 0$  fixed we have

$$\lim_{j \rightarrow \infty} \int_{|x| \leq R} |S_j f(x) - f * g(x)| dx = 0. \quad (2.8)$$

First we are going to see that it suffices to prove the claim for  $f \in L^1(\mathbb{R})$ . As a consequence of Lemma 2, if  $f \in L^1(\omega)$  and  $m > 0$ , then  $f \chi_{[-m, m]} \in L^1(\mathbb{R})$  because

$$\int_{-m}^m |f(x)| dx = \int_{-m}^m |f(x)| \omega(x) \frac{1}{\omega(x)} dx \leq C \frac{3m}{\int_m^{2m} \omega(y) dy} \int_{-m}^m |f(x)| \omega(x) dx < \infty.$$

Now we observe that for  $|x| \leq R$ ,

$$S_j f(x) - f * g(x) = S_j (f \chi_{[-R-L, R+L]})(x) - (f \chi_{[-R-L, R+L]}) * g(x)$$

and since  $f \chi_{[-R-L, R+L]} \in L^1(\mathbb{R})$  we have seen that it is enough to prove the claim for  $f \in L^1(\mathbb{R})$ . Assuming that  $f \in L^1(\mathbb{R})$  we have

$$\begin{aligned} & \int_{|x| \leq R} |S_j f(x) - f * g(x)| dx \\ &= \int_{|x| \leq R} \left| \sum_{k=1}^{N_j} f(x - t_{j,k}) \int_{t_{j,k-1}}^{t_{j,k}} g(t) dt - \sum_{k=1}^{N_j} \int_{t_{j,k-1}}^{t_{j,k}} f(x - t) g(t) dt \right| dx \\ &\leq \sum_{k=1}^{N_j} \int_{|x| \leq R} \int_{t_{j,k-1}}^{t_{j,k}} |f(x - t_{j,k}) - f(x - t)| |g(t)| dt dx \\ &= \sum_{k=1}^{N_j} \int_{t_{j,k-1}}^{t_{j,k}} |g(t)| \int_{|x| \leq R} |f(x - t_{j,k}) - f(x - t)| dx dt. \end{aligned} \quad (2.9)$$

Given  $\varepsilon > 0$ , since  $f \in L^1(\mathbb{R})$ , by the translations lemma there exists  $j_0 \in \mathbb{N}$  such that for all  $j \geq j_0$

$$\int_{|x| \leq R} |f(x - t_{j,k}) - f(x - t)| dx < \frac{\varepsilon}{\|g\|_{L^1(\mathbb{R})}},$$

for all  $k \in \{1, \dots, N_j\}$ . This inequality and (2.9) give (2.8).

In order to prove that  $T(f * g)(x) = Tf * g(x)$  for almost every  $x \in \mathbb{R}$ , we are going to see that for all fixed  $P > 0$  we have that  $T(f * g)(x) = Tf * g(x)$  for almost every  $x \in [-P, P]$ . For fixed positive  $P$  and  $\varepsilon$  we shall estimate the measure of the set

$$\{x \in [-P, P] : |T(f * g)(x) - Tf * g(x)| > \varepsilon\}.$$

Since  $f \in L^1(\omega)$  and  $f * g \in L^1(\omega)$  by Theorem 4, we can choose  $R > 3L + P$  such that

$$\int_{|y| \geq R-L} |f(y)| \omega(y) dy < \varepsilon^2 \quad (2.10)$$



and

$$\int_{|y|>R} |f * g(y)| \omega(y) dy < \varepsilon^2. \quad (2.11)$$

Now, by (2.8), we can choose  $j \in \mathbb{N}$  such that

$$\int_{|y|\leq R} |S_j f(y) - f * g(y)| dy < \varepsilon^2 \quad (2.12)$$

and

$$\int_{|y|\leq R} |S_j (Tf)(y) - Tf * g(y)| dy < \varepsilon^2. \quad (2.13)$$

Then we have the following:

$$\begin{aligned} & |\{x \in [-P, P] : |T(f * g)(x) - Tf * g(x)| > \varepsilon\}| \\ & \leq \left| \left\{ x \in [-P, P] : |T(f * g)(x) - T((f * g)\chi_{[-R, R]})(x)| > \frac{\varepsilon}{4} \right\} \right| \\ & \quad + \left| \left\{ x \in [-P, P] : |T((f * g)\chi_{[-R, R]})(x) - T((S_j f)\chi_{[-R, R]})(x)| > \frac{\varepsilon}{4} \right\} \right| \\ & \quad + \left| \left\{ x \in [-P, P] : |T((S_j f)\chi_{[-R, R]})(x) - S_j(Tf)(x)| > \frac{\varepsilon}{4} \right\} \right| \\ & \quad + \left| \left\{ x \in [-P, P] : |S_j(Tf)(x) - Tf * g(x)| > \frac{\varepsilon}{4} \right\} \right| = \text{I} + \text{II} + \text{III} + \text{IV}. \quad (2.14) \end{aligned}$$

Let us estimate I. Let

$$E = \left\{ x \in [-P, P] : |T(f * g)(x) - T((f * g)\chi_{[-R, R]})(x)| > \frac{\varepsilon}{4} \right\}.$$

Now, Lemmas 2 and 1 and inequality (2.11) give

$$\begin{aligned} \text{I} &= \int_E 1 dx \leq \frac{C3P}{\int_P^{2P} \omega(y) dy} \int_E \omega(x) dx \\ &= C\omega \left( \left\{ x \in [-P, P] : |T(f * g - (f * g)\chi_{[-R, R]})(x)| > \frac{\varepsilon}{4} \right\} \right) \\ &\leq C \frac{4}{\varepsilon} \|(f * g)(1 - \chi_{[-R, R]})\|_{L^1(\omega)} \leq C\varepsilon. \quad (2.15) \end{aligned}$$

Observe that the constant  $C$  depends on  $P$  and  $L$ , but both numbers are fixed.

The estimation of II is very easy. We have to use only the weak-type inequality (1,1) of  $T$  with respect to the Lebesgue's measure and (2.12) to obtain

$$\text{II} \leq \frac{C}{\varepsilon} \int_{|x|\leq R} |S_j f(x) - f * g(x)| dx < C\varepsilon. \quad (2.16)$$

The estimation of III is more difficult. First of all, by the definition of  $T$  and  $S_j$  we have

$$\begin{aligned} & T((S_j f)\chi_{[-R, R]})(x) - S_j(Tf)(x) \\ &= \sum_{k=1}^{N_j} \lim_{\eta \rightarrow 0} \int_{-\infty}^{x-t_{j,k}-\eta} K(x-s-t_{j,k}) f(s) (\chi_{[-R, R]}(s+t_{j,k}) - 1) ds \int_{t_{j,k-1}}^{t_{j,k}} g(t) dt. \quad (2.17) \end{aligned}$$

Observe that if  $|s+t_{j,k}| \leq R$ , then  $\chi_{[-R, R]}(s+t_{j,k}) - 1 = 0$  and if  $|s+t_{j,k}| > R$ , then  $|x - (s+t_{j,k})| \geq |s+t_{j,k}| - |x| \geq R - P > 3L$ , hence,  $|x - (s+t_{j,k})|$  is far away from 0. On the other hand, if

$L - R \leq s < x$ , then  $\chi_{[-R,R]}(s + t_{j,k}) - 1 = 0$ . Therefore, we obtain

$$\begin{aligned} & |T((S_j f) \chi_{[-R,R]})(x) - S_j(Tf)(x)| \\ & \leq \sum_{k=1}^{N_j} \int_{-\infty}^{L-R} |K(x-s-t_{j,k}) - K(x-s)| |f(s)| ds \int_{t_{j,k-1}}^{t_{j,k}} g(t) dt \\ & \quad + \left| \sum_{k=1}^{N_j} \int_{-\infty}^x K(x-s) (\chi_{[-R,R]}(s+t_{j,k}) - 1) f(s) ds \int_{t_{j,k-1}}^{t_{j,k}} g(t) dt \right| \\ & = I(x) + |Th(x)|, \end{aligned} \quad (2.18)$$

where

$$h(s) = f(s) \sum_{k=1}^{N_j} (\chi_{[-R,R]}(s+t_{j,k}) - 1) \int_{t_{j,k-1}}^{t_{j,k}} g(t) dt.$$

Thus,

$$\text{III} \leq \left| \left\{ x \in [-P, P] : I(x) > \frac{\varepsilon}{8} \right\} \right| + \left| \left\{ x \in [-P, P] : |Th(x)| > \frac{\varepsilon}{8} \right\} \right|. \quad (2.19)$$

Acting as in the estimation of I we have that

$$\begin{aligned} \left| \left\{ x \in [-P, P] : I(x) > \frac{\varepsilon}{8} \right\} \right| & \leq \frac{C3P}{\int_P^{2P} \omega} \omega \left( \left\{ x \in [-P, P] : I(x) > \frac{\varepsilon}{8} \right\} \right) \\ & \leq C \int_{\{x \in [-P, P] : I(x) > \frac{\varepsilon}{8}\}} \omega(x) dx \\ & \leq \frac{C}{\varepsilon} \int_{-P}^P I(x) \omega(x) dx. \end{aligned} \quad (2.20)$$

Observe that  $L - R < 0$ . Then, if  $s \in (-\infty, L - R)$ , we have  $|s| > |L - R| = R - L$  and  $|x - s| \geq |s| - |x| > R - L - P > 2L \geq 2|t_{j,k}|$ . Now we use condition (1.3) of the kernel  $K$  and we obtain

$$\begin{aligned} \int_{-P}^P I(x) \omega(x) dx & \leq C \sum_{k=1}^{N_j} \int_{t_{j,k-1}}^{t_{j,k}} |g(t)| dt \int_{-P}^P \left( \int_{-\infty}^{L-R} |f(s)| \frac{|t_{j,k}|}{|x-s|^2} \right) \omega(x) dx \\ & \leq CL \sum_{k=1}^{N_j} \int_{t_{j,k-1}}^{t_{j,k}} |g(t)| dt \int_{-\infty}^{L-R} |f(s)| \int_{-P}^P \frac{\omega(x)}{(x-s)^2} dx ds. \end{aligned} \quad (2.21)$$

Taking into account that  $-\infty < s < L - R$  implies  $2L \leq -P - s$ , we get

$$\begin{aligned} \int_{-P}^P \frac{\omega(x)}{(x-s)^2} dx & = \int_{-P-s}^{P-s} \omega(x+s) \frac{1}{x^2} dx = \int_{-P-s}^{P-s} \bar{\omega}(-s-x) \frac{1}{x^2} dx \\ & \leq \int_{2L}^{\infty} \bar{\omega}(-s-x) \frac{1}{x^2} dx \leq \int_0^{\infty} \bar{\omega}(-s-x) \psi(x) dx, \end{aligned} \quad (2.22)$$

where  $\psi$  is defined by  $\psi(x) = \frac{1}{(2L)^2}$  if  $0 \leq x \leq 2L$  and  $\psi(x) = \frac{1}{x^2}$  if  $2L \leq x$ . Notice that  $\psi$  has support in  $[0, \infty)$ , is nonincreasing in  $[0, \infty)$ , and is integrable. Thus, part (1) in Theorem 3 and the fact that  $\omega \in A_1^-$  give

$$\int_0^{\infty} \bar{\omega}(-s-x) \psi(x) dx \leq M^- \bar{\omega}(-s) \int_0^{\infty} \psi(x) dx = CM^+ \omega(s) \leq C\omega(s) \quad (2.23)$$

for almost every  $s \in (-\infty, L - R)$ . If we put together inequalities (2.20), (2.21), (2.22), and (2.23) and we use (2.10), we get

$$\begin{aligned} \left| \left\{ x \in [-P, P] : I(x) > \frac{\varepsilon}{8} \right\} \right| &\leq \frac{C}{\varepsilon} \|g\|_{L^1(\mathbb{R})} \int_{-\infty}^{L-R} |f(s)|\omega(s)ds \\ &\leq \frac{C}{\varepsilon} \|g\|_{L^1(\mathbb{R})} \int_{|s| \geq R-L} |f(s)|\omega(s)ds \\ &\leq C\varepsilon \|g\|_{L^1(\mathbb{R})} . \end{aligned} \tag{2.24}$$

On the other hand, using Lemmas 1 and 2, we obtain

$$\begin{aligned} \left| \left\{ x \in [-P, P] : |Th(x)| > \frac{\varepsilon}{8} \right\} \right| &\leq \frac{C3P}{\int_P^{2P} \omega} \omega \left( \left\{ x \in [-P, P] : |Th(x)| > \frac{\varepsilon}{8} \right\} \right) \\ &\leq \frac{C}{\varepsilon} \int_{\mathbb{R}} |h(x)|\omega(x)dx \\ &\leq \frac{C}{\varepsilon} \sum_{k=1}^{N_j} \int_{t_{j,k-1}}^{t_{j,k}} |g(t)|dt \int_{\mathbb{R}} |f(x)| |\chi_{[-R,R]}(x + t_{j,k}) - 1| \omega(x)dx \\ &\leq \frac{C}{\varepsilon} \|g\|_{L^1(\mathbb{R})} \int_{|x| > R-L} |f(x)|\omega(x)dx \leq C\varepsilon \|g\|_{L^1(\mathbb{R})} . \end{aligned} \tag{2.25}$$

Finally, let us estimate IV. Using (2.13) and the fact that if  $|x| \leq P$  then  $|x| \leq R$  we obtain

$$\begin{aligned} \left| \left\{ x \in [-P, P] : |S_j(Tf)(x) - Tf * g(x)| > \frac{\varepsilon}{4} \right\} \right| \\ \leq \frac{4}{\varepsilon} \int_{|x| \leq R} |S_j(Tf)(x) - Tf * g(x)| dx \leq 4\varepsilon . \end{aligned} \tag{2.26}$$

Putting together inequalities (2.14), (2.15), (2.16), (2.18), (2.19), (2.24), (2.25), and (2.26)) we get that

$$\lim_{\varepsilon \rightarrow 0} |\{x \in [-P, P] : |T(f * g)(x) - Tf * g(x)| > \varepsilon\}| = 0 ,$$

which finishes the proof of Theorem 5.  $\square$

**Proof of Theorem 6.** We have that  $\omega \in A_1^-$ . Then, by the proof of the Remark (C) in [14], there exists  $\delta > 1$  such that  $\omega^p \in A_1^-$  for all  $p, 1 < p \leq \delta$ . Let us choose  $p$  such that  $1 < p \leq \min\{\delta, p_0\}$ . It is clear that  $\omega^p$  is a locally integrable function and  $g \in L^p(\mathbb{R})$ . Then, by part (1) of Theorem 3 and the fact that  $\omega^p \in A_1^-$ , we obtain that for almost all  $y \in \mathbb{R}$

$$\begin{aligned} \int_{\mathbb{R}} |g(x - y)|^p \omega^p(x)dx &= \int_0^\infty |g(x)|^p \omega^p(y + x)dx = \int_0^\infty |g(x)|^p \bar{\omega}^p(-y - x)dx \\ &\leq M^- \bar{\omega}^p(-y) \int_0^\infty |g(x)|^p dx = M^+ \omega^p(y) \|g\|_{L^p(\mathbb{R})}^p \\ &\leq C\omega^p(y) \|g\|_{L^p(\mathbb{R})}^p . \end{aligned} \tag{2.27}$$

As a consequence, by Minkowsky's integral inequality and (2.27) we obtain

$$\begin{aligned} \|f * g\|_{L^p(\omega^p)} &\leq \int_{\mathbb{R}} \left( \int_{\mathbb{R}} |g(x - y)|^p |f(y)|^p \omega^p(x)dx \right)^{1/p} dy \\ &\leq C \|g\|_{L^p(\mathbb{R})} \int_{\mathbb{R}} |f(y)|\omega(y)dy = C \|g\|_{L^p(\mathbb{R})} \|f\|_{L^1(\omega)} . \end{aligned} \tag{2.28}$$

On the other hand, since  $\omega^p \in A_1^- \subset A_p^-$ , by Lemma 1, we have

$$\int_{\mathbb{R}} |Tg(x-y)|^p \omega^p(x) dx \leq C \int_{\mathbb{R}} |g(x-y)|^p \omega^p(x) dx. \quad (2.29)$$

Consequently,

$$\|f * Tg\|_{L^p(\omega^p)} \leq C \|g\|_{L^p(\mathbb{R})} \|f\|_{L^1(\omega)}. \quad (2.30)$$

For each  $\varepsilon > 0$  let  $K_\varepsilon = K \chi_{(\varepsilon, \infty)}$ . We have that  $f * g \in L^p(\omega^p)$  by (2.28) and since  $\omega^p \in A_p^-$ , we have that  $(f * g) * K_\varepsilon$  converges to  $T(f * g)$  in the  $L^p(\omega^p)$ -norm as  $\varepsilon$  goes to 0, by the result in [1].

To prove that  $T(f * g) = f * Tg$  a.e. it suffices to prove the following:

$$\lim_{\varepsilon \rightarrow 0} \|f * (g * K_\varepsilon) - f * Tg\|_{L^p(\omega^p)} = 0.$$

Observe that since  $\omega < \infty$  a.e., inequality (2.27) gives that the function  $g_y(x) = g(x-y)$  belongs to  $L^p(\omega^p)$  for almost all  $y \in \mathbb{R}$ . Then, by Lemma 1, we get

$$\lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}} |g_y * K_\varepsilon(x) - Tg_y(x)|^p \omega^p(x) dx = 0. \quad (2.31)$$

On the other hand, inequality (2.27) and Lemma 1 give

$$\begin{aligned} \int_{\mathbb{R}} |g_y * K_\varepsilon(x) - Tg_y(x)|^p \omega^p(x) dx &= \int_{\mathbb{R}} |g * K_\varepsilon(x-y) - Tg(x-y)|^p \omega^p(x) dx \\ &\leq 2^p \int_{\mathbb{R}} |T^*g(x-y)|^p \omega^p(x) dx \\ &\leq C 2^p \int_{\mathbb{R}} |g(x-y)|^p \omega^p(x) dx \leq C \omega^p(y) \|g\|_{L^p(\mathbb{R})}^p. \end{aligned} \quad (2.32)$$

Then, by Minkowsky's integral inequality

$$\begin{aligned} \|f * (g * K_\varepsilon) - f * Tg\|_{L^p(\omega^p)} &= \|f * (g * K_\varepsilon - Tg)\|_{L^p(\omega^p)} \\ &= \left( \int_{\mathbb{R}} \left| \int_{\mathbb{R}} (g * K_\varepsilon(x-y) - Tg(x-y)) f(y) dy \right|^p \omega^p(x) dx \right)^{1/p} \\ &\leq \int_{\mathbb{R}} \left( \int_{\mathbb{R}} |g * K_\varepsilon(x-y) - Tg(x-y)|^p |f(y)|^p \omega^p(x) dx \right)^{1/p} dy \\ &= \int_{\mathbb{R}} |f(y)| \left( \int_{\mathbb{R}} |g * K_\varepsilon(x-y) - Tg(x-y)|^p \omega^p(x) dx \right)^{1/p} dy. \end{aligned}$$

Thus,

$$\lim_{\varepsilon \rightarrow 0} \|f * (g * K_\varepsilon) - f * Tg\|_{L^p(\omega^p)} = 0, \quad (2.33)$$

by the dominated convergence theorem. This finishes the proof of Theorem 6.  $\square$

**Proof of Theorem 1.** Let  $\varphi \geq 0$  be a bounded function with compact support contained in  $[0, \infty)$ , nonincreasing in  $[0, \infty)$  and such that  $\text{supp } \varphi \subset [0, 1]$  and  $\int_0^1 \varphi(x) dx = 1$ . Let  $\varphi_\varepsilon(x) = \frac{1}{\varepsilon} \varphi\left(\frac{x}{\varepsilon}\right)$ . Define

$$\delta_\varepsilon(x) = T\varphi_\varepsilon(x) - K_\varepsilon(x).$$

If  $g$  is a bounded function with compact support in  $\mathbb{R}$ , then its least decreasing radial majorant belongs to  $L^1(\mathbb{R}) \cap L^{p_0}(\mathbb{R})$  for  $p_0 > 1$ . Then, since  $\varphi_\varepsilon \in L^1(\mathbb{R})$ , applying Lemma 6 in [5] in the

case  $\omega = 1$ , we have that  $T(g * \varphi_\varepsilon)(x) = Tg * \varphi_\varepsilon(x)$  a.e. On the other hand, since  $\varphi_\varepsilon \in L^{p_0}(\mathbb{R})$  for all  $p_0 > 1$  and  $g \in L^1(\omega)$ , Theorem 6 gives that  $T(g * \varphi_\varepsilon)(x) = g * T\varphi_\varepsilon(x)$  a.e. Then,

$$g * \delta_\varepsilon(x) = g * T\varphi_\varepsilon(x) - g * K_\varepsilon(x) = T(g * \varphi_\varepsilon)(x) - g * K_\varepsilon(x) = Tg * \varphi_\varepsilon(x) - g * K_\varepsilon(x), \quad \text{a.e.}$$

We claim that

$$\lim_{\varepsilon \rightarrow 0} \|g * \delta_\varepsilon\|_{L^1(\omega)} = 0. \quad (2.34)$$

We first prove that  $\lim_{\varepsilon \rightarrow 0} \|g * \delta_\varepsilon\|_{L^2(\omega)} = 0$ , following the ideas in [5]. The fact that  $\omega \in A_1^- \subset A_2^-$  gives that  $T$  is bounded in  $L^2(\omega)$  [1]. Then, both  $Tg$  and  $g * K_\varepsilon$  belong to  $L^2(\omega)$ , since  $g \in L^2(\omega)$ , and  $\lim_{\varepsilon \rightarrow 0} \|Tg - g * K_\varepsilon\|_{L^2(\omega)} = 0$ . This, together with part (4) of Theorem 3, gives

$$\lim_{\varepsilon \rightarrow 0} \|g * \delta_\varepsilon\|_{L^2(\omega)} \leq \lim_{\varepsilon \rightarrow 0} \|Tg * \varphi_\varepsilon - Tg\|_{L^2(\omega)} + \lim_{\varepsilon \rightarrow 0} \|Tg - g * K_\varepsilon\|_{L^2(\omega)} = 0. \quad (2.35)$$

To prove (2.34), we suppose that  $\text{supp } g \subset [-N, N]$  and we take  $\varepsilon$  with  $0 < \varepsilon < N$ . Then, for  $x \geq 4N$  we have

$$T(g * \varphi_\varepsilon)(x) = \int_{-\infty}^x K(x-y)(g * \varphi_\varepsilon)(y)dy = \int_{|y| \leq 2N} K(x-y)(g * \varphi_\varepsilon)(y)dy$$

and

$$g * K_\varepsilon(x) = \int_{-\infty}^{x-\varepsilon} K(x-y)g(y)dy = \int_{|y| \leq N} K(x-y)g(y)dy = \int_{|y| \leq 2N} K(x-y)g(y)dy.$$

For  $x \leq -4N$ , one has that  $T(g * \varphi_\varepsilon)(x) = 0$  and  $g * K_\varepsilon(x) = 0$ . As a consequence, if  $|x| \geq 4N$  we have

$$g * \delta_\varepsilon(x) = \int_{|y| \leq 2N} K(x-y)(g * \varphi_\varepsilon(y) - g(y))dy. \quad (2.36)$$

We are going to see that  $\int_{|y| \leq 2N} (g * \varphi_\varepsilon(y) - g(y))dy = 0$ . By Fubini's theorem we have

$$\begin{aligned} \int_{|y| \leq 2N} g * \varphi_\varepsilon(y)dy &= \int_{|y| \leq 2N} \int_{|y-z| \leq \varepsilon; |z| \leq N} g(z)\varphi_\varepsilon(y-z)dzdy \\ &= \int_{|z| \leq N} g(z) \int_{|y-z| \leq \varepsilon; |y| \leq 2N} \varphi_\varepsilon(y-z)dydz \\ &= \int_{|z| \leq N} g(z)dz = \int_{|z| \leq 2N} g(z)dz. \end{aligned}$$

Then, by (2.36), for all  $x$  with  $|x| \geq 4N$  we have

$$g * \delta_\varepsilon(x) = \int_{|y| \leq 2N} (K(x-y) - K(x))(g * \varphi_\varepsilon(y) - g(y))dy.$$

Therefore, Fubini's theorem and condition (1.3) of the kernel give

$$\begin{aligned} \int_{|x| \geq 4N} |g * \delta_\varepsilon(x)| \omega(x)dx &= \int_{x \geq 4N} |g * \delta_\varepsilon(x)| \omega(x)dx \\ &\leq \int_{|y| \leq 2N} \left( \int_{x \geq 2|y|} |K(x-y) - K(x)| \omega(x)dx \right) |g * \varphi_\varepsilon(y) - g(y)|dy \\ &\leq C \int_{|y| \leq 2N} \left( \int_{x \geq 2|y|} \frac{|y|}{|x-y|^2} \omega(x)dx \right) |g * \varphi_\varepsilon(y) - g(y)|dy \\ &\leq C \int_{|y| \leq 2N} |g * \varphi_\varepsilon(y) - g(y)| \int_{x \geq 4N} \frac{\omega(x)}{(x-y)^2} dx dy. \end{aligned} \quad (2.37)$$

The same argument used in (2.22) gives that

$$\int_{x \geq 4N} \frac{\omega(x)}{(x-y)^2} dx \leq C\omega(y), \quad \text{a.e. } y \in [-2N, 2N].$$

This inequality, together with (2.37), gives

$$\int_{|x| \geq 4N} |g * \delta_\varepsilon(x)| \omega(x) dx \leq C \int_{|y| \leq 2N} |g * \varphi_\varepsilon(y) - g(y)| \omega(y) dy, \quad (2.38)$$

which converges to 0 as  $\varepsilon$  goes to 0 by part (4) in Theorem 3.

Now, using Hölder's inequality, we get

$$\begin{aligned} \int_{|x| \leq 4N} |g * \delta_\varepsilon(x)| \omega(x) dx &\leq \left( \int_{|x| \leq 4N} \omega(x) dx \right)^{1/2} \left( \int_{|x| \leq 4N} |g * \delta_\varepsilon(x)|^2 \omega(x) dx \right)^{1/2} \\ &\leq C \|g * \delta_\varepsilon\|_{L^2(\omega)}, \end{aligned} \quad (2.39)$$

which tends to 0 as  $\varepsilon$  goes to 0 by (2.35). This proves (2.34) for all functions  $g$ , bounded with compact support.

The function  $\varphi$  is nonnegative, with support in  $[0, \infty)$ , nonincreasing in  $[0, \infty)$ , and it is integrable. If we define  $\tilde{\varphi}(x) = \varphi(x)$  for  $x \geq 0$  and  $\tilde{\varphi}(x) = \varphi(-x)$  for  $x < 0$ , we get that  $\tilde{\varphi}$  has the same properties as those in the proof of Theorem 1 in [5]. Then, by what it is proved in pages 28 and 29 in [5], there exists a nonincreasing, radial, and integrable function  $\tilde{\Delta}$  such that

$$|T\tilde{\varphi}_\varepsilon(x) - K_\varepsilon(x)| \leq \tilde{\Delta}_\varepsilon(x) = \frac{1}{\varepsilon} \tilde{\Delta}\left(\frac{x}{\varepsilon}\right).$$

Then, the function  $\Delta = \tilde{\Delta}\chi_{[0, \infty)}$  has support in  $[0, \infty)$ , it is nonincreasing in  $[0, \infty)$ , it is integrable, and  $|\delta_\varepsilon(x)| \leq \Delta_\varepsilon(x)$  for all  $x$ .

Observe now that, by Theorems 5 and 6,  $\|f * K_\varepsilon\|_{L^1(\omega)} \leq \|Tf * \varphi_\varepsilon\|_{L^1(\omega)} + \|f * \delta_\varepsilon\|_{L^1(\omega)}$ . Since  $f \in L^1(\omega)$ ,  $Tf \in L^1(\omega)$ , and  $\omega \in A_1^-$ , part (2) in Theorem 3 gives  $\|Tf * \varphi_\varepsilon\|_{L^1(\omega)} \leq C\|Tf\|_{L^1(\omega)}$  and  $\|f * \delta_\varepsilon\|_{L^1(\omega)} \leq \|f\| * \Delta_\varepsilon\|_{L^1(\omega)} \leq C\|f\|_{L^1(\omega)} \int_0^\infty \Delta = C\|f\|_{L^1(\omega)}$ . In particular, we have obtained that for all  $\varepsilon > 0$ ,  $f * K_\varepsilon = T_\varepsilon f \in L^1(\omega)$ .

Given  $\eta > 0$ , let us choose  $g$  bounded with compact support such that  $\|f - g\|_{L^1(\omega)} < \eta$ . Then

$$\|f * K_\varepsilon - Tf\|_{L^1(\omega)} \leq \|Tf * \varphi_\varepsilon - Tf\|_{L^1(\omega)} + \|(f - g) * \delta_\varepsilon\|_{L^1(\omega)} + \|g * \delta_\varepsilon\|_{L^1(\omega)}. \quad (2.40)$$

By Theorem 3 and condition (2.34) we have

$$\lim_{\varepsilon \rightarrow 0} \|Tf * \varphi_\varepsilon - Tf\|_{L^1(\omega)} + \|g * \delta_\varepsilon\|_{L^1(\omega)} = 0.$$

On the other hand,

$$\|(f - g) * \delta_\varepsilon\|_{L^1(\omega)} \leq \|f - g\| * \Delta_\varepsilon\|_{L^1(\omega)} \leq C\|f - g\|_{L^1(\omega)} \leq C\eta.$$

Since this inequality is valid for all  $\eta > 0$  we have

$$\lim_{\varepsilon \rightarrow 0} \|f * K_\varepsilon - Tf\|_{L^1(\omega)} = \lim_{\varepsilon \rightarrow 0} \|T_\varepsilon f - Tf\|_{L^1(\omega)} = 0.$$

This finishes the proof of Theorem 1.  $\square$

### 3. One-Sided Singular Integrals Associated to a Flow Cesàro-Bounded to the Right: Proof of Theorem 2

We begin by stating the boundedness of the maximal operator.

**Theorem 7.**

Let  $(X, \mathfrak{M}, \nu)$ ,  $\{\tau_t : t \in \mathbb{R}\}$ ,  $K$  and  $\mathcal{T}$  be as in Theorem 2. Let  $\mathcal{T}^*$  and  $\mathcal{M}^+$  be the maximal operators defined as in Section 1. Let  $1 \leq p < \infty$ . Then  $\mathcal{M}^+$  and  $\mathcal{T}^*$  are bounded from  $L^p(d\nu)$  into  $L^p(d\nu)$  if  $p > 1$  and they are of weak type  $(1,1)$  in  $L^1(d\nu)$ .

As a consequence of this theorem we obtain the following corollary.

**Corollary.**

Under the same assumptions as that in Theorem 7 we have the following:

- (1) If  $p = 1$  and  $f \in L^1(d\nu)$ , then  $\mathcal{A}_\varepsilon^+ f$  converges in the  $L^1(d\nu)$ -norm as  $\varepsilon$  goes to infinity and  $\mathcal{T}_\varepsilon f$  converges in measure as  $\varepsilon$  goes to 0.
- (2) If  $1 < p < \infty$  and  $f \in L^p(d\nu)$ , then  $\mathcal{A}_\varepsilon^+ f$  converges in the  $L^p(d\nu)$ -norm as  $\varepsilon$  goes to infinity and  $\mathcal{T}_\varepsilon f$  converges in the  $L^p(d\nu)$ -norm as  $\varepsilon$  goes to 0.

We omit the proof of Theorem 7 because it is almost the same as the one in [11] (the proof follows standard arguments as the truncation of the maximal operators and transference methods (see [3] and [6]) to use the valid results in  $\mathbb{R}$ ; in the case of  $\mathcal{M}^+$  we use the results in  $\mathbb{R}$  for the one-sided Hardy–Littlewood maximal function  $M^+$ ; in the case of  $\mathcal{T}^*$  we use the results of Section 2). The difference yields in the fact that when we define the measure  $\mu$  equivalent to  $\nu$  such that the flow preserves the measure  $\mu$ , and  $d\nu = \omega d\mu$  then we obtain that the functions  $\omega^x : \mathbb{R} \rightarrow \mathbb{R}$ ,  $\omega^x(t) = \omega(\tau_t x)$ , satisfy  $A_p^+$  for almost every  $x \in X$  with the same constant. The proof of the corollary follows from the weak type inequalities of  $\mathcal{M}^+$  and  $\mathcal{T}^*$  and the fact that the convergence holds for functions  $f \in L^1(d\mu) \cap L^p(d\nu)$  which is dense in  $L^p(d\nu)$ .

For  $f \in L^1(d\nu)$  and  $\varphi \in L^1(\mathbb{R})$  we define the convolution of  $f$  and  $\varphi$  by  $f * \varphi(x) = \int_{\mathbb{R}} f(\tau_t x) \varphi(t) dt$  (it makes sense by the same reason as in [11]). The following result is about the behavior of the convolution.

**Theorem 8.**

Under the same assumptions as that in Theorem 2, if  $f \in L^1(d\nu)$  and  $g \geq 0$  is a function with support in  $[0, \infty)$ , nonincreasing in  $[0, \infty)$  and  $g \in L^1(\mathbb{R})$ , then

- (1)  $|f * g(x)| \leq \|g\|_{L^1(\mathbb{R})} \mathcal{M}^+ f(x)$  a.e.
- (2)  $\|f * g\|_{L^1(d\nu)} \leq C \|g\|_{L^1(\mathbb{R})} \|f\|_{L^1(d\nu)}$ .

As in [2] and [11] the proof of Theorem 2 requires to approximate the functions in  $L^1(d\nu)$  by suitable functions. This is the next result.

**Theorem 9.**

Assume that we are under the same hypothesis as that in Theorem 2. Let

$$A = \left\{ f \in L^1(d\nu) : f(x) = f(\tau_t x) \text{ for all } t \in \mathbb{R} \text{ and a.e. } x \in X \right\},$$

$$B_1 = \left\{ f \in L^1(d\nu) : f = f_0 * \varphi : f_0 \in L^\infty(d\nu), \varphi \in C_c^1[0, \infty) \text{ and } \int_0^\infty \varphi = 0 \right\},$$

where  $C_c^1[0, \infty)$  is the set of functions of class 1 and compact support in  $[0, \infty)$  and

$$B_2 = \left\{ f \in L^1(d\nu) : f = f_0 * \varphi : f_0 \in L^\infty(d\nu) \text{ and } \varphi \in L_0^1[0, \infty) \right\},$$

where  $L_0^1[0, \infty)$  is the set of functions with support in  $[0, \infty)$ , with  $\int_0^\infty |\varphi| < \infty$  and  $\int_0^\infty \varphi = 0$ . Denote by  $\overline{B_1}$  and  $\overline{B_2}$  the  $L^1(d\nu)$ -closure of the linear manifold generated by  $B_1$  and  $B_2$ , respectively. Then  $A \oplus \overline{B_1} = A \oplus \overline{B_2} = L^1(d\nu)$ .

We shall need a result similar to Theorem 6 in [11]:

**Theorem 10.**

Under the same hypothesis as that in Theorem 9, if  $f \in \overline{B_1}$  is such that  $\mathcal{T}f \in L^1(d\nu)$ , then  $\mathcal{T}f \in \overline{B_1}$

**Proof of Theorem 8.** Denote by  $f^x$  the function  $f^x(t) = f(\tau_t x)$  and by  $\bar{g}$  the function  $\bar{g}(t) = g(-t)$ . Then, by well-known results in  $\mathbb{R}$ ,

$$\begin{aligned} |f * g(x)| &\leq \int_{\mathbb{R}} |f(\tau_t x)| |g(t)| dt = \int_{\mathbb{R}} |\overline{f^x}(-t)| |g(t)| dt \\ &= \int_{\mathbb{R}} |\overline{f^x} \chi_{(-\infty, 0]}(-t)| |g(t)| dt = |\overline{f^x} \chi_{(-\infty, 0]}| * g(0) \\ &\leq \|g\|_{L^1(\mathbb{R})} \mathcal{M}^+ f(x), \end{aligned}$$

as in the proof of (1) in Theorem 3. This proves (1).

To prove (2) observe that if  $g$  has support in  $[0, \infty)$  and it is nonincreasing in  $[0, \infty)$ , then  $\bar{g}$  has support in  $(-\infty, 0]$  and it is nondecreasing in  $(-\infty, 0]$ . As in [11], there exists a measure  $\mu$  equivalent to  $\nu$  such that the flow preserves the measure  $\mu$ . If we define  $\omega = \frac{d\nu}{d\mu}$ , then  $\omega \in A_1^+$ . Therefore,

$$\begin{aligned} \|f * g\|_{L^1(d\nu)} &= \int_X \left| \int_0^\infty f(\tau_t x) g(t) dt \right| \omega(x) d\mu(x) \leq \int_0^\infty g(t) \int_X |f(x)| \omega(\tau_{-t} x) d\mu(x) \\ &= \int_X |f(x)| \int_{-\infty}^0 \omega(\tau_t x) \bar{g}(t) dt d\mu(x) \leq \|\bar{g}\|_{L^1(\mathbb{R})} \int_X |f(x)| \mathcal{M}^- \omega(x) d\mu(x) \\ &\leq C \|g\|_{L^1(\mathbb{R})} \int_X |f(x)| \omega(x) d\mu(x) = C \|g\|_{L^1(\mathbb{R})} \|f\|_{L^1(d\nu)}. \end{aligned}$$

This completes the proof of Theorem 8. □

**Proof of Theorem 9.** It is clear that  $B_1 \subset B_2$  and then  $\overline{B_1} \subset \overline{B_2}$ . Observe that it suffices to prove that  $A \oplus \overline{B_2} = L^1(d\nu)$  by (2) in Theorem 8 and the fact that  $C_c^1[0, \infty) \cap L_0^1[0, \infty)$  is dense in  $L_0^1[0, \infty)$  implies that  $\overline{B_1} = \overline{B_2}$  in  $L^1(d\nu)$ .

Let us see that  $A \cap B_2 = \{0\}$ . It is clear that  $\lim_{\varepsilon \rightarrow \infty} \mathcal{A}_\varepsilon^+ f(x) = f(x)$  for  $f \in A$ . If  $f \in B_2$ , then  $f = f_0 * \varphi$ , where  $f_0 \in L^\infty(d\nu)$  and  $\varphi \in L_0^1[0, \infty)$ . Observe that since  $\int_0^\infty \varphi = 0$  we have

$$\mathcal{A}_\varepsilon^+ f(x) = \frac{1}{\varepsilon} \int_0^\varepsilon \int_0^\infty \varphi(s) f_0(\tau_{t+s} x) ds dt = \int_0^\infty \varphi(s) \frac{1}{\varepsilon} \int_0^\varepsilon (f_0(\tau_{t+s} x) - f_0(\tau_t x)) dt ds.$$

Now, the dominated convergence theorem and the fact that  $\lim_{\varepsilon \rightarrow \infty} \mathcal{A}_\varepsilon^+ f_0(x) = \lim_{\varepsilon \rightarrow \infty} \mathcal{A}_\varepsilon^+ f_0(\tau_t x)$  give that  $\lim_{\varepsilon \rightarrow \infty} \mathcal{A}_\varepsilon^+ f(x) = 0$  and thus  $A \cap B_2 = \{0\}$ . The inequality  $\|\mathcal{A}_\varepsilon^+ f\|_{L^1(d\nu)} \leq C \|f\|_{L^1(d\nu)}$  and the fact that this limit is a limit in the  $L^1(d\nu)$ -norm give that  $A \cap \overline{B_2} = \{0\}$ .

Let us now prove that  $A \oplus \overline{B_2} = L^1(d\nu)$ . Let  $f \in L^1(d\nu)$  and denote  $F = \lim_{\varepsilon \rightarrow \infty} \mathcal{A}_\varepsilon^+ f$ . It is obvious that  $F \in A$ . Then it suffices to show that  $f - F \in \overline{B_2}$ . Suppose that  $h \in L^\infty(d\nu)$  and  $\int_X h G d\nu = 0$  for all functions  $G \in \overline{B_2}$ . Then

$$\int_X h(x) \int_0^\infty g(\tau_s x) \varphi(s) ds d\nu(x) = 0 \tag{3.1}$$



for all functions  $g \in L^\infty(d\nu)$  and all  $\varphi \in L^1_0[0, \infty)$ . Let now  $g \in L^1(d\nu)$  and  $\varphi \in L^1_0[0, \infty)$ . Then, given  $\varepsilon > 0$  there exists  $g_0 \in L^\infty(d\nu)$  such that  $\|g - g_0\|_{L^1(d\nu)} < \varepsilon$ . Therefore, by (3.1) and part (2) of Theorem 8,

$$\begin{aligned} & \left| \int_X h(x) \int_0^\infty g(\tau_s x) \varphi(s) ds d\nu(x) \right| \\ & \leq \left| \int_X h(x) \int_0^\infty (g(\tau_s x) - g_0(\tau_s x)) \varphi(s) ds d\nu(x) \right| \\ & \quad + \left| \int_X h(x) \int_0^\infty g_0(\tau_s x) \varphi(s) ds d\nu(x) \right| \\ & \leq \|h(g - g_0) * \varphi\|_{L^1(d\nu)} \\ & \leq \|h\|_{L^\infty(d\nu)} \|g - g_0\|_{L^1(d\nu)} \|\varphi\|_{L^1(\mathbb{R})} < C\varepsilon, \end{aligned}$$

where  $\psi$  is the least majorant nonincreasing in  $[0, \infty)$  of  $\varphi$ . Then, equality (3.1) is valid for  $g \in L^1(d\nu)$  and  $\varphi \in L^1_0[0, \infty)$ . Applying this equality to  $g = f$  and  $\varphi = (1/\varepsilon)\chi_{(0,\varepsilon)} - (1/\eta)\chi_{(0,\eta)}$  we get

$$\int_X h(x) \frac{1}{\varepsilon} \int_0^\varepsilon f(\tau_s x) ds d\nu(x) = \int_X h(x) \frac{1}{\eta} \int_0^\eta f(\tau_s x) ds d\nu(x).$$

Since  $(1/\eta) \int_0^\eta f(\tau_s x) ds$  converges to  $F$  in the  $L^1(d\nu)$ -norm as  $\eta$  goes to  $\infty$ , by the corollary and  $(1/\varepsilon) \int_0^\varepsilon f(\tau_s x) ds$  converges to  $f$  in the  $L^1(d\nu)$ -norm as  $\varepsilon$  goes to 0, by Wiener's theorem and the fact that  $\|\mathcal{A}_\varepsilon^+ f\|_{L^1(d\nu)} \leq C\|f\|_{L^1(d\nu)}$ , we obtain  $\int_X (f - F)h d\nu = 0$  for the functions  $h \in L^\infty(d\nu)$  as above. This means that  $f - F \in \overline{B_2}$ .  $\square$

**Proof of Theorem 10.** We start by proving this theorem, assuming that the flow preserves the measure  $\nu$ , and we call the measure  $\mu$  instead of  $\nu$  in this case. We shall need the following result, which gives the existence of a bounded approximation of the identity in  $L^1_0[0, \infty)$ . The proof is given at the end of the section.

**Lemma 3.**

There exists a sequence  $\{\psi_n\}_{n=1}^\infty \subset L^1[0, \infty)$  such that

- (i)  $\int_0^\infty \psi_n = 0$ , for all  $n \in \mathbb{N}$ .
- (ii)  $\|\psi_n\|_{L^1[0,\infty)} \leq 2$ , for all  $n \in \mathbb{N}$ .
- (iii)  $\lim_{n \rightarrow \infty} \|f * \psi_n - f\|_{L^1[0,\infty)} = 0$ , for all  $f \in L^1_0[0, \infty)$ .

Let  $f \in \overline{B_1}$  be such that  $\mathcal{T}f \in L^1(d\mu)$ . Then, by Theorem 9, there exists  $F_1 \in A$  and  $F_2 \in \overline{B_1}$  such that  $\mathcal{T}f = F_1 + F_2$ . Let  $\{\psi_n\}_{n=1}^\infty$  as in Lemma 3. Let us see that

$$\lim_{n \rightarrow \infty} \|\psi_n * F_2 - F_2\|_{L^1(d\mu)} = 0. \tag{3.2}$$

Assume first that  $F_2 \in B_1$ , i.e.,  $F_2 = F * \phi$ , for some  $F \in L^\infty(d\mu)$  and  $\phi \in C^1_c[0, \infty)$  with  $\int_0^\infty \phi = 0$ . Therefore,

$$\|\psi_n * F_2 - F_2\|_{L^1(d\mu)} = \|(\psi_n * \phi) * F - \phi * F\|_{L^1(d\mu)} \leq \|\psi_n * \phi - \phi\|_{L^1(\mathbb{R})} \|F\|_{L^1(d\mu)},$$

which converges to 0 as  $n$  tends to infinity by Lemma 3. It is clear that (3.2) holds for all  $F_2$  in the linear manifold generated by  $B_1$ . Now let  $F_2 \in \overline{B_1}$ . Then, fixed  $\varepsilon > 0$ , there exists  $G$  in the linear manifold generated by  $B_1$  such that  $\|F_2 - G\|_{L^1(d\mu)} < \varepsilon$ . Therefore,

$$\begin{aligned} \|\psi_n * F_2 - F_2\|_{L^1(d\mu)} & \leq \|\psi_n\|_{L^1(\mathbb{R})} \|F_2 - G\|_{L^1(d\mu)} + \|\psi_n * G - G\|_{L^1(d\mu)} \\ & \quad + \|F_2 - G\|_{L^1(d\mu)} \leq 2\varepsilon + \|\psi_n * G - G\|_{L^1(d\mu)} + \varepsilon. \end{aligned}$$

Since  $G$  is in the linear manifold generated by  $B_1$ , we can use (3.2). Consequently,  $\{\psi_n * F_2\}$  converges to  $F_2$  in the  $L^1(d\mu)$ -norm as  $n$  tends to infinity. On the other hand,  $\psi_n * F_1(x) = \int_{\mathbb{R}} \psi_n(t) F_1(\tau_t x) dt = F_1(x) \int_{\mathbb{R}} \psi_n(t) dt = 0$ . Then  $\psi_n * \mathcal{T}f = \psi_n * F_1 + \psi_n * F_2 = \psi_n * F_2$ . Therefore,  $\{\psi_n * \mathcal{T}f\}$  converges to  $F_2$  in the  $L^1(d\mu)$ -norm. By Corollary 3.15 in [2] with  $\mathcal{T}$  instead of the ergodic Hilbert transform, we have that  $\psi_n * \mathcal{T}f(x) = \mathcal{T}(\psi_n * f)(x)$  a.e. and therefore  $\{\mathcal{T}(\psi_n * f)\}$  converges to  $F_2$  in the  $L^1(d\mu)$ -norm. On the other hand, since  $f \in \overline{B_1}$ , applying (3.2), we have that  $\{\psi_n * f\}$  converges to  $f$  in the  $L^1(d\mu)$ -norm, which implies that  $\{\mathcal{T}(\psi_n * f)\}$  converges to  $\mathcal{T}f$  in measure. Then  $F_2 = \mathcal{T}f$  a.e. and thus  $\mathcal{T}f \in \overline{B_1}$ .

In order to prove Theorem 10 for general measures, we need the following result:

**Lemma 4.**

Assume that we are under the same hypothesis as that in Theorem 9, but with a measure  $\mu$  which is preserved by the flow. Let  $\mathcal{T}$  be a one-sided singular integral associated to the flow and to a Calderón-Zygmund kernel  $K$  with support in  $(0, \infty)$  for which there exists the limit  $\lim_{\varepsilon \rightarrow 0} \int_{\varepsilon}^{\frac{1}{\varepsilon}} K(t) dt$ . If  $f \in \overline{B_2}$  is such that  $\mathcal{T}f \in L^1(d\mu)$ , then there exist  $g_0 \in L^1_0[0, \infty)$  and  $f_0 \in \overline{B_2}$  such that  $f = g_0 * f_0$ .

**Proof.** Let

$$C = \left\{ f \in \overline{B_2} : \mathcal{T}f \in L^1(d\mu) \right\} .$$

We claim that if  $g \in L^1_0[0, \infty)$  and  $f \in C$ , then  $g * f \in C$ .

We first suppose that  $f \in B_2$ , i.e.,  $f = h * f_0$  for some  $f_0 \in L^\infty(d\mu)$  and  $h \in L^1_0[0, \infty)$ . Then  $g * f = (g * h) * f_0$  and  $g * h \in L^1_0[0, \infty)$ . Consequently,  $g * f \in B_2$  and, since  $\mathcal{T}f \in L^1(d\mu)$ , we obtain by Corollary 3.15 in [2] with  $\mathcal{T}$  instead of the ergodic Hilbert transform, that  $\mathcal{T}(g * f) = g * \mathcal{T}f \in L^1(d\mu)$ . Thus,  $g * f \in C$ . We obtain the same for  $f$  in the linear manifold generated by  $B_2$ . If  $f \in \overline{B_2}$ , then there exists a sequence  $\{f_n\}_{n=1}^\infty$  in the linear manifold generated by  $B_2$  such that  $f = \lim_{n \rightarrow \infty} f_n$  in the  $L^1(d\mu)$ -norm and, since  $g * f_n$  belongs to the linear manifold generated by  $B_2$ , for all  $n \in \mathbb{N}$ , it follows that  $g * f$  belongs to  $\overline{B_2}$ , since  $g * f = \lim_{n \rightarrow \infty} g * f_n$  in  $L^1(d\mu)$ . Furthermore,  $\mathcal{T}(g * f) = g * \mathcal{T}f \in L^1(d\mu)$ .

If we define

$$\|f\|_C = \|f\|_{L^1(d\mu)} + \|\mathcal{T}f\|_{L^1(d\mu)} ,$$

then  $\|\cdot\|_C$  is a norm in  $C$  and  $(C, \|\cdot\|_C)$  is a Banach space. Now, by Theorem 32.22 of [9] we have that  $L^1_0[0, \infty) * C$  is a linear subspace closed in  $C$ . If we prove that it is dense in  $C$ , we will get that  $C = L^1_0[0, \infty) * C$  and this will complete the proof of Lemma 4.

Let  $f \in C$  and let  $\{\psi_n\}_{n=1}^\infty$  be the approximation of the identity given by Lemma 3. Let us consider  $\{\psi_n * f\}_{n=1}^\infty \subset L^1_0[0, \infty) * C$ . By what we have proved in the case that the flow preserves the measure, we have that if  $G \in \overline{B_1} = \overline{B_2}$ , then  $\psi_n * G$  converges to  $G$  in the  $L^1(d\mu)$ -norm and  $\mathcal{T}(\psi_n * f)$  converges to  $\mathcal{T}f$  in the  $L^1(d\mu)$ -norm as  $n$  tends to infinity. As a consequence,

$$\lim_{n \rightarrow \infty} \|\psi_n * f - f\|_C = \lim_{n \rightarrow \infty} \|\psi_n * f - f\|_{L^1(d\mu)} + \lim_{n \rightarrow \infty} \|\mathcal{T}(\psi_n * f) - \mathcal{T}f\|_{L^1(d\mu)} = 0 .$$

Therefore,  $\psi_n * f$  converges to  $f$  in the  $C$ -norm, which implies that  $C = L^1_0[0, \infty) * C$ . This finishes the proof of Lemma 4.  $\square$

Consider as in [11] the sets

$$X_n = \left\{ x \in X : 2^n \leq \sup_{t \in \mathbb{R}} \omega^{-1}(\tau_t x) < 2^{n+1} \right\} ,$$

where  $\omega = \frac{d\nu}{d\mu}$  is as above. Since  $L^1(X_n, d\nu) \subset L^1(X_n, d\mu)$ , we have that  $f \in \overline{B_1} = \overline{B_2}$  (working in  $X_n$ ) and  $\mathcal{T}f \in L^1(X_n, d\mu)$ . Then  $f \in C$ , defined as in the proof of Lemma 4 (in  $X_n$ ). Therefore,

for each  $n \in \mathbb{N}$ , there exist  $F_n \in \mathcal{C}$  and  $\phi_n \in L^1_0[0, \infty)$  such that  $f = F_n * \phi_n$  in  $X_n$ . Since  $F_n \in \mathcal{C}$  we have that  $\mathcal{T}F_n \in L^1(X_n, d\mu)$ . Then we can apply the theorem in the case that the flow preserves the measure to obtain that  $\mathcal{T}f = \mathcal{T}F_n * \phi_n$  in  $X_n$ . On the other hand, it follows from Theorem 9 that  $\mathcal{T}f = f_1 + f_2$ , where  $f_1 \in A$  and  $f_2 \in \overline{B_1}$ . Then,

$$\lim_{\varepsilon \rightarrow \infty} \mathcal{A}_\varepsilon^+(\mathcal{T}f) = \lim_{\varepsilon \rightarrow \infty} (\mathcal{A}_\varepsilon^+ f_1 + \mathcal{A}_\varepsilon^+ f_2) = \lim_{\varepsilon \rightarrow \infty} \mathcal{A}_\varepsilon^+ f_1 = f_1 .$$

Since the limits of the averages are constant on each orbit and  $\int_0^\infty \phi_n = 0$ , we have that

$$\lim_{\varepsilon \rightarrow \infty} \mathcal{A}_\varepsilon^+(\mathcal{T}f) = \lim_{\varepsilon \rightarrow \infty} \mathcal{A}_\varepsilon^+(\mathcal{T}F_n * \phi_n) = 0$$

in  $X_n$ , which implies that  $f_1 = 0$  in  $X_n$  and therefore  $f_1 = 0$  in  $X$ . Thus,  $\mathcal{T}f = f_2 \in \overline{B_1}$ . □

**Proof of Theorem 2.** Let  $f \in L^1(d\nu)$  such that  $\mathcal{T}f \in L^1(d\nu)$ , then, by Theorem 9, there exist  $f_1 \in A$  and  $f_2 \in \overline{B_1}$  such that  $f = f_1 + f_2$ . We know that  $\mathcal{T}_\varepsilon f_1$  converges to  $\mathcal{T}f_1$  a.e. and in the  $L^1(d\nu)$ -norm. Thus, we can suppose that  $f \in \overline{B_1}$ . Then, let  $f \in \overline{B_1}$  such that  $\mathcal{T}f \in L^1(d\nu)$ . By Theorem 10,  $\mathcal{T}f \in \overline{B_1}$ . Let us consider a function  $\varphi \geq 0$ ,  $\varphi \in C^1_c[0, \infty)$ , nonincreasing in  $[0, \infty)$  and such that  $\int_0^\infty \varphi = 1$ . For each  $\varepsilon > 0$  let  $\varphi_\varepsilon(t) = \frac{1}{\varepsilon} \varphi(\frac{t}{\varepsilon})$  and define

$$\delta_\varepsilon(x) = T\varphi_\varepsilon(x) - K\chi_{(\varepsilon, \infty)}(x) ,$$

where  $T$  is the singular integral associated to  $K$  in  $\mathbb{R}$ . Let  $K_\varepsilon = K\chi_{(\varepsilon, \frac{1}{\varepsilon})}$ . Then

$$K_\varepsilon = T(\varphi_\varepsilon - \varphi_{1/\varepsilon}) + \delta_{1/\varepsilon} - \delta_\varepsilon . \tag{3.3}$$

Let  $\Delta_\varepsilon = \delta_{1/\varepsilon} - \delta_\varepsilon$ . Then  $\Delta_\varepsilon \in L^1(\mathbb{R})$  for all  $\varepsilon > 0$  and  $\|\Delta_\varepsilon\|_{L^1(\mathbb{R})} \leq C$ , where  $C$  does not depend on  $\varepsilon$ . Since  $K_\varepsilon \in L^1(\mathbb{R})$  it follows from (3.3) that  $T(\varphi_\varepsilon - \varphi_{1/\varepsilon}) \in L^1(\mathbb{R})$ . Observe that  $\delta_\varepsilon$  and  $\Delta_\varepsilon$  have both support in  $[0, \infty)$  and that the least majorant nonincreasing in  $[0, \infty)$  of  $\Delta_\varepsilon$  belongs to  $L^1(\mathbb{R})$ . Then Theorem 8 gives that

$$\|\Delta_\varepsilon * F\|_{L^1(d\nu)} \leq C \|F\|_{L^1(d\nu)} ,$$

for all  $F \in L^1(d\nu)$ .

Let  $\gamma > 0$ . Choose  $g$  in the linear manifold generated by  $B_1$  such that  $\|f - g\|_{L^1(d\nu)} < \gamma$ . Notice that Lemma 4 holds for  $f \in L^1(d\nu)$ , since it suffices to restrict to  $X_n$ . Therefore, using this fact and Corollary 3.15 of [2] with  $\mathcal{T}$  instead of the ergodic Hilbert transform (again restricting the things to  $X_n$ ) we obtain

$$\begin{aligned} \|\mathcal{T}_\varepsilon f - \mathcal{T}f\|_{L^1(d\nu)} &= \|K_\varepsilon * f - \mathcal{T}f\|_{L^1(d\nu)} \\ &= \|\Delta_\varepsilon * f + T(\varphi_\varepsilon - \varphi_{1/\varepsilon}) * f - \mathcal{T}f\|_{L^1(d\nu)} \\ &= \|\Delta_\varepsilon * (f - g) + \Delta_\varepsilon * g + \varphi_\varepsilon * \mathcal{T}f - \varphi_{1/\varepsilon} * \mathcal{T}f - \mathcal{T}f\|_{L^1(d\nu)} \\ &\leq \|\Delta_\varepsilon * (f - g)\|_{L^1(d\nu)} + \|\Delta_\varepsilon * g\|_{L^1(d\nu)} \\ &\quad + \|\varphi_\varepsilon * \mathcal{T}f - \mathcal{T}f\|_{L^1(d\nu)} + \|\varphi_{1/\varepsilon} * \mathcal{T}f\|_{L^1(d\nu)} \\ &= \text{I} + \text{II} + \text{III} + \text{IV} . \end{aligned} \tag{3.4}$$

For the first term we have

$$\text{I} \leq \|\Delta_\varepsilon\|_{L^1(\mathbb{R})} \|f - g\|_{L^1(d\nu)} \leq C\gamma . \tag{3.5}$$

Notice that  $B_1 \subset L^2(d\nu)$ , then  $g \in L^2(d\nu)$  which implies that  $\mathcal{T}g \in L^2(d\nu) \subset L^1(d\nu)$ . We have the following:

$$\begin{aligned}
 \text{II} &= \|\Delta_\varepsilon * g\|_{L^1(d\nu)} \\
 &\leq \|\varphi_{1/\varepsilon} * \mathcal{T}g\|_{L^1(d\nu)} + \|\mathcal{T}_\varepsilon g - \mathcal{T}g\|_{L^1(d\nu)} + \|\varphi_\varepsilon * \mathcal{T}g - \mathcal{T}g\|_{L^1(d\nu)} \\
 &\leq \|\varphi_{1/\varepsilon} * \mathcal{T}g\|_{L^1(d\nu)} + C \|\mathcal{T}_\varepsilon g - \mathcal{T}g\|_{L^2(d\nu)} + \|\varphi_\varepsilon * \mathcal{T}g - \mathcal{T}g\|_{L^1(d\nu)} \\
 &= \text{V} + \text{VI} + \text{VII}.
 \end{aligned} \tag{3.6}$$

The corollary of Theorem 7 gives

$$\lim_{\varepsilon \rightarrow 0} \|\mathcal{T}_\varepsilon g - \mathcal{T}g\|_{L^2(d\nu)} = 0. \tag{3.7}$$

Finally, it is clear that the proof will be complete if we show that

$$\lim_{\varepsilon \rightarrow 0} (\text{III} + \text{IV} + \text{V} + \text{VII}) = 0. \tag{3.8}$$

This is a consequence of the following result and the fact that, by Theorem 10, both,  $\mathcal{T}f$  and  $\mathcal{T}g$  belong to  $\overline{B_1}$ .

**Lemma 5.**

Suppose that we are under the same hypothesis as that in Theorem 9. Let  $\varphi \geq 0$ ,  $\varphi \in C_c^1[0, \infty)$ , nonincreasing in  $[0, \infty)$  and such that  $\int_0^\infty \varphi = 1$ . For each  $\varepsilon > 0$  let  $\varphi_\varepsilon(t) = \frac{1}{\varepsilon} \varphi\left(\frac{t}{\varepsilon}\right)$ .

- (a) If  $f \in L^1(d\nu)$ , then  $f * \varphi_\varepsilon$  converges to  $f$  in the  $L^1(d\nu)$ -norm as  $\varepsilon$  goes to 0.
- (b) If  $f \in \overline{B_1}$ , then  $f * \varphi_{1/\varepsilon}$  converges to 0 in the  $L^1(d\nu)$ -norm as  $\varepsilon$  goes to 0.

The proof of this theorem follows the same pattern as the proof of Theorem 5 in [11]. Therefore, we omit it.  $\square$

**Proof of Lemma 3.** Consider a continuous function  $\varphi$  with support contained in  $[0, 1]$ ,  $\varphi \geq 0$  and such that  $\int_0^1 \varphi = 1$ . For each  $n \in \mathbb{N}$  let

$$\psi_n(t) = \frac{1}{n} \varphi\left(\frac{t}{n}\right) - \frac{1}{n} \varphi\left(\frac{t}{n} - 1\right).$$

It is very easy to see that the sequence  $\{\psi_n\}_{n=1}^\infty$  satisfies (i) and (ii). To prove (iii) it is sufficient to show that the second convolution tends to 0, i.e.,  $\lim_{n \rightarrow \infty} \|f * \phi_n\|_{L^1[0, \infty)} = 0$ , where  $\phi_n(t) = \frac{1}{n} \varphi\left(\frac{t}{n} - 1\right)$ , for  $f \in L_0^1[0, \infty)$ . If  $f = 0$  a.e., then there is nothing to prove. Suppose that  $\|f\|_{L^1[0, \infty)} > 0$ . Let  $\varepsilon > 0$ , then there exists  $k > 0$  such that  $\int_k^\infty |f| < \frac{\varepsilon}{4}$ . Therefore, using the fact that  $\int_0^\infty f = 0$  and Fubini's theorem we obtain

$$\begin{aligned}
 \|f * \phi_n\|_{L^1[0, \infty)} &\leq \int_0^\infty \int_0^\infty |f(t)| |\phi_n(x-t) - \phi_n(x)| dt dx \\
 &= \int_0^k |f(t)| \int_0^\infty |\phi_n(x-t) - \phi_n(x)| dx dt \\
 &\quad + \int_k^\infty |f(t)| \int_0^\infty |\phi_n(x-t) - \phi_n(x)| dx dt = A + B.
 \end{aligned}$$

It is obvious that

$$B \leq 2 \|\phi_n\|_{L^1[0, \infty)} \int_k^\infty |f| < \frac{\varepsilon}{2}.$$

To estimate  $A$ , observe that for all  $t \in [0, k]$  and for all natural  $n \geq k$  we have that

$$\begin{aligned} \int_0^\infty |\phi_n(x-t) - \phi_n(x)| dx &= \int_n^{2n+t} \frac{1}{n} \left| \varphi\left(\frac{x-t}{n} - 1\right) - \varphi\left(\frac{x}{n} - 1\right) \right| dx \\ &= \int_{-\frac{t}{n}}^1 \left| \varphi(x) - \varphi\left(x + \frac{t}{n}\right) \right| dx = \int_{-1}^1 \left| \varphi(x) - \varphi\left(x + \frac{t}{n}\right) \right| dx. \end{aligned}$$

Now using that  $\varphi$  is uniformly continuous, we have that there exists  $\delta > 0$  such that  $|u - y| < \delta$  implies that  $|\varphi(u) - \varphi(y)| < \frac{\varepsilon}{4\|f\|_{L^1[0,\infty]}}$ . Choose  $N \in \mathbb{N}$  such that  $N \geq k$  and  $\frac{k}{N} < \delta$ . Then, for all  $n \geq N$  we have that

$$\int_{-1}^1 \left| \varphi(x) - \varphi\left(x + \frac{t}{n}\right) \right| dx \leq \frac{\varepsilon}{2\|f\|_{L^1[0,\infty]}}.$$

This gives that

$$A \leq \int_0^k |f(t)| \frac{\varepsilon}{2\|f\|_{L^1[0,\infty]}} dt < \frac{\varepsilon}{2}.$$

Thus,  $\|f * \phi_n\|_{L^1[0,\infty]} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ , for all  $n \geq N$ . This completes the proof of Lemma 3.  $\square$

## Acknowledgment

We want to thank the referees for their helpful comments and indications on how to improve the paper. In particular, one of the referees pointed out a mistake in the proof of Lemma 3 and the way to overcome it.

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Received July 20, 1998

Revision received April 2, 1999

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