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Estimates, Decay Properties, and Computation of the Dual Function for Gabor Frames

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ABSTRACT. *We present a simple proof of Ron and Shen's frame bounds estimates for Gabor frames. The proof is based on the Heil and Walnut's representation of the frame operator and shows that it can be decomposed into a continuous family of infinite matrices. The estimates then follow from a simple application of Gershgorin's theorem to each matrix. Next, we show that, if the window function has exponential decay, also the dual function has some exponential decay. Then, we describe a numerical method to compute the dual function and give an estimate of the error. Finally, we consider the spline of order 2; we investigate numerically the region of the time-frequency plane where it generates a frame and we compute the dual function for some values of the parameters.*

1. Introduction

In this paper we study expansions of square integrable functions into families of translates and modulates of a given function g. If g is a function in $L^2(\mathbf{R})$ and t_o, ω_o are two positive parameters, the family

$$g_{m\omega_o,nt_o}(t) = g(t - nt_o)\, e^{im\omega_o t}$$

$m, n \in \mathbf{Z}$, is called *a Gabor frame* in $L^2(\mathbf{R})$ if there exist two constants $0 < A \le B < \infty$ such that

$$A\|f\|^2 \le \sum_{m,n} \left|\langle f, g_{m\omega_o,nt_o}\rangle\right|^2 \le B\|f\|^2 \tag{1.1}$$

for all functions $f \in L^2(\mathbf{R})$. If $A = B$, the frame is called a *tight frame*. Denoting by F the operator $F : L^2(\mathbf{R}) \to l^2(\mathbf{Z}^2)$ $(Ff)_{m,n} = \langle f, g_{m\omega_o,nt_o}\rangle$, and by F^* its adjoint, we may rewrite (1.1) as follows:

$$AI \le F^*F \le BI\,. \tag{1.2}$$

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Hence, the spectral bounds of the operator F^*F are the optimal constants in (1.1). The operator F^*F is called *the frame operator;* note that it depends on g, ω_o, and t_o. It is well known that if $\{g_{m\omega_o,nt_o}\}$ is a frame, then the parameters t_o, ω_o must satisfy the condition $t_o\omega_o \leq 2\pi$. Throughout the paper we shall assume that this condition holds.

In general frames, even tight frames, are not a basis of $L^2(\mathbf{R})$, because some of their vectors may be in the closed linear span of the others. This redundancy implies that the operator F is not surjective. However, since by (1.2) its range is closed, F has a generalized inverse, which is a bounded operator from the range of F to $L^2(\mathbf{R})$. The inversion formula is

$$f = \sum_{m,n} \left\langle f, \tilde{g}_{m\omega_o,nt_o}\right\rangle g_{m\omega_o,nt_o}\ , \tag{1.3}$$

where $\tilde{g} = (F^*F)^{-1}g$ and $\tilde{g}_{m\omega_o,nt_o}(t) = \tilde{g}(t - nt_o)e^{im\omega_o t}$. The function $\tilde{g}$ is called the *frame dual function* and the family $\{\tilde{g}_{mt_o,n\omega_o}, m, n \in \mathbf{Z}, \}$ is a frame, the *dual frame.* The coefficients in the expansion (1.3) are characterized by the property of having minimal norm among all the sequences $(c_{m,n})$ in $l^2(\mathbf{Z})$ such that $f = \sum c_{m,n}\, g_{m\omega_o,nt_o}$.

In applications, once the window function g has been chosen, the basic questions to investigate for a Gabor analysis are: find the values of the time-frequency parameters such that $\{g_{m\omega_o,nt_o}\}$ is a frame and compute good estimates of the frame bounds. For reconstruction, it is also necessary to compute the dual function $\tilde{g}$ and to determine its time-frequency localization properties.

In [1], Daubechies established estimates of the frame bounds for a window function decaying sufficiently fast at infinity [see formula (2.6)]. These estimates, obtained by means of Schwarz inequality, are not sharp, so in general they do not provide the best bounds. Sharper estimates have been obtained by Ron and Shen as a consequence of their Gramian analysis of Weyl–Heisenberg frames for multiwindow frames in $L^2(\mathbf{R}^n)$ [15]. In this paper we present a simple proof of these estimates for a single window in one dimension. Starting from a representation of the frame operator due to Heil and Walnut, we show that the frame operator can be decomposed into a continuous family of infinite matrices $\{M(t), t \in \mathbf{R}\}$ [7]. The estimates then follow from a simple application of Gershgorin's theorem to each matrix.

Properties of the dual function as consequence of analogous assumptions on the function g generating a frame have been investigated by Janssen; in [8] it is proved that if ρ_o in rational, $\tilde{g}$ satisfies the Tolimieri–Orr condition if g does. In [10] the same author shows that if g is in the Schwartz space $\mathcal{S}$, then $\tilde{g}$ is in $\mathcal{S}$. We prove that if the function g has exponential decay, then the dual function $\tilde{g}$ also has exponential decay.

To compute the dual function, first Daubechies solved the equation

$$F^*F\tilde{g} = g \tag{1.4}$$

by using an iterative algorithm based on the Neumann series $\tilde{g} = \frac{2}{A+B}\sum_{k=0}^{\infty}\left(I - \frac{2}{A+B}F^*F\right)^k g$ [1]. Daubechies et al. proposed a simpler and faster recursive formula, based on the Wexler–Raz identity [4]. These formulas depend on the computed frame bounds. However, in general the true values of A and B are difficult to estimate. Alternative methods have been introduced by various authors. Qiu and Feichtinger obtain the dual function by exploiting the band and block structure and the sparsity of the Gabor matrices [14]. Their paper contains an outline of some of the practical approaches proposed by various authors. By using the Zak transform, if $\rho_o = 2\pi/t_o\omega_o$ is a rational number, Zibulski and Zeevi reduce the problem of solving equation (1.4) to the problem of solving a finite system of linear equations whose coefficients are periodic functions defined on $\mathbf{R}^2$ [19]. Lastly, we mention Janssen's technique, which consists in writing the Wexler–Raz biorthogonal conditions in the time domain as a collection of decoupled linear systems involving samples of g as knowns and samples of $\tilde{g}$ as unknowns [11]. The method we propose to compute the dual function is based on the decomposition of the frame operator into the family of infinite matrices $\{M(t), t \in \mathbf{R}\}$.

The paper is organized as follows. In Section 2 we describe how Ron and Shen's fibration and the new estimates of the frame bounds follow from Heil and Walnut's identities via Gerschgorin's theorem. In Section 3 we show that if the function g generating a frame has exponential decay, then the dual $\tilde{g}$ also has some exponential decay. The proof is obtained by combining some of the ideas in [10] with a representation of the dual function given in [9]. The key step of the proof consists in showing that if the elements of a positive definite matrix U decay exponentially away from the main diagonal, then the elements of U^{-1} also have some exponential decay. In Section 4 we present the numerical technique to compute the dual function and, by using the results of the previous section, we give an estimate of the error. In Section 5 we present several numerical results on frames generated by cardinal splines. For the spline of order 2, by using the frame bounds estimates given in Section 2, we find, for several values of t_o, the values of the frequency parameter ω_o such that $\{g_{m\omega_o,nt_o}\}$ is a frame. Finally we apply the numerical technique described in Section 4 to find the dual function of the spline of order 2 for some values of the time-frequency parameters.

2. The Structure of the Frame Operator and the Frame Bounds

In this section we collect some results on the frame operator to be used later. We begin by introducing some notation. The Fourier transform of a function f in $L^1(\mathbf{R})$ is

$$\mathcal{F}f(\xi) = \hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int f(t) e^{-it\xi} dt \,.$$

With this definition $\mathcal{F}$ is an isometry on $L^2(\mathbf{R})$. The convolution of two functions f and g is

$$f * g(x) = \frac{1}{\sqrt{2\pi}} \int f(x-y) g(y)\, dy \,,$$

so that $\mathcal{F}(f * g) = \mathcal{F}f\, \mathcal{F}g$.

Let $\hat{F}$ denote the operator $\hat{F} : L^2(\mathbf{R}) \to l^2(\mathbf{Z}^2)$ defined by $(\hat{F} f)_{m,n} = \langle f, \hat{g}_{mt_o,n\omega_o}\rangle$. From the identity

$$\left(g_{m\omega_o,nt_o}\right)^{\wedge} = \left(\hat{g}\right)_{-nt_o,m\omega_o} e^{imnt_o\omega_o}$$

it follows that

$$\hat{F}^* \hat{F} = \mathcal{F} F^* F \mathcal{F}^{-1} \,. \tag{2.1}$$

Therefore, $\{g_{m\omega_o,nt_o}\}$ is a frame if and only if $\{\hat{g}_{mt_o,n\omega_o}\}$ is a frame and the frame bounds are the same.

In [7] Heil and Walnut proved that the frame operators F^*F and $\hat{F}^*\hat{F}$ can be represented as linear combinations of translation operators with coefficients which are periodic functions (see Proposition 1 below). Let τ_a denote the translation by $a \in \mathbf{R}$, i.e., the operator mapping the function $f(t)$ into the function $f(t+a)$. To simplify the notation we shall denote by T_k, $k \in \mathbf{Z}$, the translation by $\frac{2\pi k}{\omega_o}$ and by $\hat{T_k}$ the translation by $\frac{2\pi k}{t_o}$. Let S_k and $\hat{S_k}$ be the functions

$$S_k(t) = \frac{2\pi}{\omega_o} \sum_{n\in\mathbf{Z}} g\,(t - nt_o)\, \bar{g}\left(t - nt_o + k\frac{2\pi}{\omega_o}\right) \tag{2.2}$$

$$\hat{S_k}(t) = \frac{2\pi}{t_o} \sum_{n\in\mathbf{Z}} \hat{g}\,(t - n\omega_o)\, \bar{\hat{g}}\left(t - n\omega_o + k\frac{2\pi}{t_o}\right) . \tag{2.3}$$

Denote by $W(L^\infty, L^1)$ the space of all functions such that for some positive number a

$$\|g\|_{W,a} = \sum_n \left\|\tau_{na} g \cdot \chi_{[0,a)}\right\|_\infty < \infty \,.$$

Proposition 1.

If g or $\hat{g} \in W(L^\infty, L^1)$, then the operators F and $\hat{F}$ are bounded. Moreover,

$$F^*F = \sum_{k\in\mathbf{Z}} S_k T_k , \tag{2.4}$$

$$\hat{F}^*\hat{F} = \sum_{k\in\mathbf{Z}} \hat{S_k} \hat{T_k} , \tag{2.5}$$

where the right-hand sides converge in the operator norm.

From Proposition 1 it is straightforward to derive Daubechies' estimates for the frame bounds [1].

Corollary 1.

In the hypothesis of Proposition 1, the spectral bounds A and B of F^*F satisfy the estimates

$$\begin{aligned} A &\geq \max\left\{\text{ess inf}\left(S_0 - 2\sum_{k=1}^{\infty}\|S_k\|_\infty\right), \text{ ess inf}\left(\hat{S_0} - 2\sum_{k=1}^{\infty}\|\hat{S_k}\|_\infty\right)\right\} \\ B &\leq \min\left\{\text{ess sup}\left(S_0 + 2\sum_{k=1}^{\infty}\|S_k\|_\infty\right), \text{ ess sup}\left(\hat{S_0} + 2\sum_{k=1}^{\infty}\|\hat{S_k}\|_\infty\right)\right\} . \end{aligned} \tag{2.6}$$

Moreover if ess inf $S_0 > 0$ *there exists a $\omega_o{}^c > 0$ such that for each $\omega \in]0, \omega_o{}^c]$, the family $\{g_{m\omega_o,nt_o}\}$ is a frame.*

Remark 1. Conditions ess inf $S_0 > 0$, ess sup $S_0 < \infty$ are also necessary (see [7] p. 649). Therefore, if g has support $[-a, a]$, frames are possible only if $t_o \leq 2a$. In this case the expression of F^*F becomes

$$F^*F = \sum_{|k|=0}^{N_o} S_k T_k ,$$

where $N_o = \left[\frac{2a}{h}\right]$, $h = \frac{2\pi}{\omega_o}$ (here $[x]$ denotes the greatest integer less than x). In particular, if $\omega_o \leq \pi/a$, then F^*F is the operator of multiplication by S_0, the frame bounds are $A =$ ess inf S_0 and $B =$ ess sup S_0, and the dual function is $\tilde{g} = S_0{}^{-1}g$.

Remark 2. From Proposition 1 it follows that if $g \in W(L^\infty, L^1)$, then $\{g_{m\omega_o,nt_o}\}$ is a tight frame if and only if the sums S_k, $|k| \neq 0$, are zero and S_0 is a constant. If $\|g\| = 1$, then $S_0 = \frac{2\pi}{t_o\omega_o}$.

Estimates (2.6), obtained by means of the Schwarz inequality, are not sharp, so in general they do not provide the best bounds. Sharper estimates have been obtained by Ron and Shen as a consequence of their Gramian analysis of Weyl–Heisenberg frames [15]. They prove that the frame operator F^*F is unitarily equivalent to a collection of "fiber" operators. We describe briefly how this fibration follows from Heil and Walnut's identities (2.4) and (2.5). To simplify the notation, we shall denote by h the ratio $\frac{2\pi}{\omega_o}$. By (2.4), for every $f \in L^2(\mathbf{R})$,

$$F^*Ff(t - jh) = \sum_{k\in\mathbf{Z}} S_k(t - jh)\, f(t - (j - k)h) .$$

From (2.2) it follows that $S_{j-l}(t - jh) = \overline{S_{l-j}}(t - lh)$. Therefore,

$$F^*Ff(t - jh) = \sum_{l\in\mathbf{Z}} m_{jl}(t)\, f(t - lh) \tag{2.7}$$

where

$$m_{jl}(t) = \overline{S_{l-j}}(t - lh) . \tag{2.8}$$

We shall denote by $M(t)$ the infinite Hermitian matrix whose entries are $m_{jl}(t)$, $j, l \in \mathbf{Z}$. We shall identify infinite matrices with the densely defined operators on $l^2(\mathbf{Z})$ associated to them via the canonical basis. So, the norm of a matrix is the norm of the corresponding operator. Identify isometrically $L^2(\mathbf{R})$ with the Hilbert space $L^2\Big([0, h]; l^2(\mathbf{Z})\Big)$ of the $l^2(\mathbf{Z})$-valued functions on $[0, h]$ by means of the isomorphism mapping a function f in $L^2(\mathbf{R})$ into the $l^2(\mathbf{Z})$-valued function

$$t \mapsto (f(t - nh))_{n\in\mathbf{Z}} ,$$

$0 \le t \le h$. Then by (2.7), F^*F is unitarily equivalent to the bounded operator $\mathcal{M}$ on $L^2\Big([0, h]; l^2(\mathbf{Z})\Big)$ such that for any φ in $L^2\Big([0, h]; l^2(\mathbf{Z})\Big)$

$$\mathcal{M}\varphi(t) = M(t)\varphi(t) \tag{2.9}$$

for almost every $t \in [0, h]$. An operator of this form is said to be decomposable into the measurable field $t \mapsto M(t)$ of operators on $l^2(\mathbf{Z})$ [6]. From the general theory of decomposable operators it follows that the upper and lower bounds of the operator $\mathcal{M}$ coincide with

$$\operatorname*{ess\,inf}_{t\in[0,h]} \inf_{\|u\|=1} \langle M(t)u, u\rangle, \qquad \operatorname*{ess\,sup}_{t\in[0,h]} \sup_{\|u\|=1} \langle M(t)u, u\rangle , \tag{2.10}$$

respectively. Here $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ denote the inner product and the norm in $l^2(\mathbf{Z})$. Note that since $t \mapsto M(t)$ is t_o-periodic and $t_o \le h$, here we may replace the interval $[0, h]$ by $\mathbf{R}$.

By using (2.5), one can perform a similar analysis on the operator $\hat{F}^*\hat{F}$. Indeed for any $t \in \mathbf{R}$ let $M\hat{}(t)$ be the infinite matrix whose entries are

$$m_{\hat{jl}}(t) = \overline{S\hat{}_{l-j}}\left(t - l\frac{2\pi}{t_o}\right) \tag{2.11}$$

$j, l \in \mathbf{Z}$. Then $\hat{F}^*\hat{F}$ is unitarily equivalent to an operator $\mathcal{M}\hat{}$ on $L^2\Big([0, \frac{2\pi}{t_o}]; l^2(\mathbf{Z})\Big)$ which is decomposable into the measurable field of matrices $t \mapsto M\hat{}(t)$, $t \in [0, \frac{2\pi}{t_o}]$. This leads to bounds for $\mathcal{M}\hat{}$ analogous to (2.10).

For later reference we summarize these results in the following proposition (see also [9], page 44 and [15]).

Proposition 2.

Let g or $\hat{g}$ be $\in W(L^\infty, L^1)$. Then

$$\begin{aligned} AI &\le F^*F \le BI \iff AI \le M(t) \le BI \qquad (2.12)\\ AI &\le \hat{F}^*\hat{F} \le BI \iff AI \le M\hat{}(t) \le BI , \qquad (2.13)\end{aligned}$$

for a.e. $t \in \mathbf{R}$.

It is now an easy matter to deduce Ron and Shen's estimates of the frame bounds:

Proposition 3.

Let g or $\hat{g}$ be in $W(L^\infty, L^1)$. Then the spectral bounds A and B of F^*F satisfy the estimates

$$\begin{aligned} A &\ge \max\left\{\operatorname{ess\,inf}\left(S_0 - \sum_{k\neq 0}|S_k|\right), \operatorname{ess\,inf}\left(S\hat{_0} - \sum_{k\neq 0}|S\hat{_k}|\right)\right\}\\ B &\le \min\left\{\operatorname{ess\,sup}\left(S_0 + \sum_{k\neq 0}|S_k|\right), \operatorname{ess\,sup}\left(S\hat{_0} + \sum_{k\neq 0}|S\hat{_k}|\right)\right\} . \qquad (2.14)\end{aligned}$$

Proof. The operators F^*F and $\mathcal{M}$ have the same spectral bounds, which are, by (2.10), the essential infimum and the essential supremum, for $t \in [0, \frac{2\pi}{\omega_o}]$, of the spectral bounds for the matrix $M(t)$. Let P_N denote the orthogonal projection onto the $(2N+1)$-dimensional subspace of $l^2(\mathbf{Z})$ spanned by the elements $\{e_j, |j| \le N\}$ of the canonical basis. Let $M_N(t) = P_N M(t) P_N$. By Gershgorin's theorem, the lowest and highest eigenvalues $\lambda_N(t)$ and $\Lambda_N(t)$ of the matrix $M_N(t)$ satisfy the estimates

$$\begin{aligned}\min_{|j|\le N}\left(m_{jj}(t) - \sum_{\substack{|l|\le N\\ l\neq j}} \left|m_{jl}(t)\right|\right) &\le \lambda_N(t)\\ &\le \Lambda_N(t) \le \max_{|j|\le N}\left(m_{jj}(t) + \sum_{\substack{|l|\le N\\ l\neq j}} \left|m_{jl}(t)\right|\right)\end{aligned}$$

for a.e. $t \in \mathbf{R}$. Since $m_{jl}(t) = S_{l-j}(t - lh)$,

$$\begin{aligned}m_{jj}(t) - \sum_{\substack{|l|\le N\\ l\neq j}} \left|m_{jl}(t)\right| &\ge S_0(t - jh) - \sum_{\substack{|l|\le N\\ l\neq j}} \left|S_{l-j}(t - lh)\right|\\ &\ge \text{ess inf}\left(S_0 - \sum_{k\neq 0} |S_k|\right) = A_1 .\end{aligned}$$

Similarly,

$$m_{jj}(t) + \sum_{\substack{|l|\le N\\ l\neq j}} \left|m_{jl}(t)\right| \le \text{ess sup}\left(S_0 + \sum_{k\neq 0} |S_k|\right) = B_1 .$$

Hence, $A_1 \le \lambda_N(t) \le \Lambda_N(t) \le B_1$ for a.e. t. Since $M_N(t)$ converges to $M(t)$ in the strong operator topology and $\|M_N(t)\| \le \|M(t)\|$ for every $t \in \mathbf{R}$ and every integer N,

$$\lim_{N\to\infty} \langle M_N(t)u, u\rangle = \langle M(t)u, u\rangle ,$$

for every $u \in l^2(\mathbf{Z})$. Therefore, A_1 is a lower bound and B_1 is an upper bound of the spectrum of $M(t)$ for almost every $t \in \mathbf{R}$. Similarly, one can see that

$$\text{ess inf}\left(\hat{S_0} - \sum_{k\neq 0} \left|\hat{S_k}\right|\right) \qquad \text{and} \qquad \text{ess sup}\left(\hat{S_0} + \sum_{k\neq 0} \left|\hat{S_k}\right|\right)$$

are, respectively, a lower and an upper bound for the spectrum of $\hat{F}^*\hat{F}$. Since the spectral bounds of the operators F^*F and $\hat{F}^*\hat{F}$ are the same, the conclusion follows. □

3. Exponential Decay of the Dual Function

Properties of the dual function as a consequence of analogous assumptions on the window function g have been investigated by Janssen; in [8] it is proved that if ρ_o is rational, $\tilde{g}$ satisfies the Tolimieri–Orr condition if g does. In [10] the same author shows that if g is in the Schwartz space $\mathcal{S}$, then $\tilde{g}$ is in $\mathcal{S}$.

In this section we prove that if the function generating a frame has exponential decay, then the dual function $\tilde{g}$ also has exponential decay. The proof is obtained by combining some of the ideas in [10] with a representation of the dual function given in [9]. Let G and $\tilde{G}$ be the elements of $L^2\Big([0,h]; l^2(\mathbf{Z})\Big)$ defined by

$$G(t) = (g(t-nh))_{n\in\mathbf{Z}} \qquad\qquad \tilde{G}(t) = (\tilde{g}(t-nh))_{n\in\mathbf{Z}}$$

for a.e. $t \in [0,h]$, $h = \frac{2\pi}{\omega_o}$. Then, by (2.7), equation (1.4) is equivalent to

$$G(t) = M(t)\tilde{G}(t) \tag{3.1}$$

for a.e. t in $\mathbf{R}$. We shall prove that if g has exponential decay, then $M(t)$ is invertible for a.e. $t \in \mathbf{R}$ and the entries of $M^{-1}(t)$ have exponential decay. Hence,

$$\tilde{G}(t) = M^{-1}(t)G(t) \tag{3.2}$$

for almost every t in $\mathbf{R}$ and $\tilde{g}$ is an infinite linear combination of translates of g, with exponentially decaying coefficients. Note that this is essentially the representation of the dual function given in [9] [formula (1.3.21)].

The key step of the proof consists in showing that if the elements of a positive definite matrix U decay exponentially away from the main diagonal, then the elements of U^{-1} also have some exponential decay (see Theorem 1 below).

As in the previous section, we identify a bounded operator on $l^2(\mathbf{Z})$ with the matrix associated to it via the canonical basis. Hence, the norm $\|U\|$ of a matrix $U = (u_{ij})_{i,j\in\mathbf{Z}}$ is the norm of the corresponding operator. To measure the exponential decay of the entries away from the main diagonal, we consider a family of spaces $\{\mathcal{V}^\alpha\}$, $\alpha > 0$. For every $\alpha > 0$ let $\mathcal{V}^\alpha$ denote the space of all infinite matrices $U = (u_{i,j})$ such that

$$\|U\|_\alpha = \sup_{k\in\mathbf{Z}} e^{|k|\alpha} \sup_{|i-j|=k} \left|u_{ij}\right| < \infty\,. \tag{3.3}$$

It is easy to check that $\mathcal{V}^\alpha$ is a Banach space with respect to the norm $\|\cdot\|_\alpha$, but not a Banach algebra with respect to operator composition. However, we shall see that the product of a finite number of elements of $\mathcal{V}^\alpha$ is in $\mathcal{V}^\beta$ for any $\beta < \alpha$ (see Proposition 4 below). First we need a lemma.

Lemma 1.

Let σ and b be two positive constants. For any $0 < \tau < \sigma$,

$$\sum_{k\in\mathbf{Z}} e^{-\sigma|x-kb|}\, e^{-\tau|y-kb|} \le 3\,\frac{\sigma + b^{-1}}{\sigma-\tau}\, e^{-\tau|x-y|} \tag{3.4}$$

for all $x, y \in \mathbf{R}$.

Proof. By a scaling argument it is enough to prove the lemma for $b = 1$. Assume first that $x < y$. Decompose the sum in (3.4) into three parts:

$$\sum_{k\in\mathbf{Z}} e^{-\sigma|x-k|}\, e^{-\tau|y-k|} = \sum_{k<x} + \sum_{x\le k\le y} + \sum_{y<k} = \sum_1 + \sum_2 + \sum_3 \tag{3.5}$$

Since

$$\sum_{s\le j} e^{-\alpha j} \le \frac{e^{-\alpha s}}{1-e^{-\alpha}} \tag{3.6}$$

for $\alpha > 0$ and $s \in \mathbf{R}$, we have that

$$\sum_1 \leq e^{-\sigma x - \tau y} \sum_{j > -x} e^{-(\sigma+\tau)j} \leq \frac{e^{-\tau|x-y|}}{1-e^{-(\sigma+\tau)}},$$

$$\sum_2 \leq e^{\sigma x - \tau y} \sum_{x \leq k} e^{-(\sigma-\tau)j} \leq \frac{e^{-\tau|x-y|}}{1-e^{-(\sigma-\tau)}},$$

$$\sum_3 \leq e^{\sigma x + \tau y} \sum_{j > y} e^{-(\sigma+\tau)j} \leq \frac{e^{-\sigma|x-y|}}{1-e^{-(\sigma+\tau)}}.$$

Hence,

$$\sum_{k \in \mathbf{Z}} e^{-\sigma|x-k|} e^{-\tau|y-k|} \leq e^{-\tau|x-y|} \left(\frac{1}{1-e^{-(\sigma-\tau)}} + \frac{2}{1-e^{-(\sigma+\tau)}} \right).$$

The conclusion follows since $\frac{1}{1-e^{-z}} \leq \frac{1}{z} + 1$. If $x > y$ or $x = y$, the proof is similar. □

Proposition 4.
If $U_1\ U_2, \ldots, U_k \in \mathcal{V}^{\alpha}$, then $\prod_{j=1}^{k} U_j \in \mathcal{V}^{\beta}$ for any $0 < \beta < \alpha$ and

$$\left\| \prod_{j=1}^{k} U_j \right\|_{\beta} \leq \left(3 \frac{\alpha+1}{\alpha-\beta} \right)^{k-1} \prod_{j=1}^{k} \|U_j\|_{\alpha}. \tag{3.7}$$

Proof. If $U_1 = (u_{ij})$ and $U_2 = (v_{ij})$ are in $\mathcal{V}^{\alpha}$, then

$$|u_{i,j}| \leq \|U_1\|_{\alpha}\, e^{-\alpha|i-j|}, \qquad |v_{ij}| \leq \|U_2\|_{\alpha}\, e^{-\alpha|i-j|}.$$

By Lemma 1, the ijth entry of the matrix $U_1 U_2$ is bounded by

$$\sum_{k \in \mathbf{Z}} |u_{ik} v_{kj}| \leq 3 \frac{\alpha+1}{\alpha-\beta} \|U_1\|_{\alpha} \|U_2\|_{\alpha}\, e^{-\beta|i-j|}$$

for every $0 < \beta < \alpha$. Therefore,

$$\|U_1 U_2\|_{\beta} \leq 3 \frac{\alpha+1}{\alpha-\beta} \|U_1\|_{\alpha} \|U_2\|_{\alpha}.$$

This proves the lemma for $k = 2$. For general k, the thesis follows by induction. □

Given an infinite matrix $U = (u_{ij})$, for any integer N let U_N denote the N-band matrix whose entries are u_{ij} if $|i-j| \leq N$ and zero otherwise. Note that for any positive α and any integer N,

$$\|U_N\|_{\alpha} \leq e^{N\alpha} \|U_N\|. \tag{3.8}$$

Let $\widetilde{U}_N$ denote the matrix $U - U_N$.

Lemma 2.
If $U \in \mathcal{V}^{\alpha}$ for some $\alpha > 0$, then

$$\left\|\widetilde{U}_N\right\| \leq \frac{2}{\alpha} e^{-\alpha N} \|U\|_{\alpha}, \tag{3.9}$$

$$\left\|\widetilde{U}_N\right\|_{\tau} \leq e^{N(\tau-\alpha)} \|U\|_{\alpha}. \tag{3.10}$$

for any $0 < \tau < \alpha$, $N \in \mathbf{N}$.

Proof. We only prove (3.9). Let $U = (u_{ij})$, and let $U(k)$ denote the matrix whose entries are u_{ij} if $i - j = k$ and zero otherwise. Then

$$\|\widetilde{U}_N\| \leq \sum_{|k|\geq N+1} \|U(k)\| \leq \sum_{|k|\geq N+1} \max_{|i-j|=k} |u_{ij}| \leq \frac{2e^{-\alpha(N+1)}}{1-e^{-\alpha}} \|U\|_\alpha .$$

Since $\frac{e^{-\alpha}}{1-e^{-\alpha}} \leq \frac{1}{\alpha}$ the thesis follows. □

Lemma 3.
Let $U \in \mathcal{V}^\alpha$, $\alpha > 0$, be such that

$$AI \leq U \leq BI \tag{3.11}$$

for $0 < A \leq B < \infty$. Then there exists an integer N_o such that

$$\frac{A}{2} I \leq U_N \leq \left(B + \frac{A}{2}\right) I \tag{3.12}$$

for $N > N_o$. Moreover, if $0 < \beta < \frac{1}{N} \log \frac{A+B}{B}$,

$$\left\| U_N{}^{-1} \right\|_\beta \leq \frac{2}{A+B} \frac{1}{1 - e^{\beta N} \frac{B}{A+B}} . \tag{3.13}$$

Proof. Let N_o be an integer greater than $\frac{1}{\alpha} \log(4 \frac{1}{\alpha A} \|U\|_\alpha)$; by (3.9), $\|\widetilde{U}_N\| < \frac{A}{2}$ if $N > N_o$. Since for any N

$$\left(A - \left\|\widetilde{U}_N\right\|\right) I \leq U_N \leq \left(B + \left\|\widetilde{U}_N\right\|\right) I ,$$

(3.12) follows. To show the second part of the lemma, we fix $N > N_o$ and consider the N-band matrix $V_N = I - \frac{2}{A+B} U_N$, so that $\|V_N\| \leq \frac{B}{A+B}$ and

$$U_N = \frac{A+B}{2} (I - V_N) . \tag{3.14}$$

Therefore, we only need to show that the Neumann series $\sum_k V_N{}^k$ for $(I - V_N)^{-1}$ converges in $\mathcal{V}^\beta$ if β is sufficiently small. Since $V_N{}^k$ is a (kN)-band matrix, from (3.8) we get $\|V_N{}^k\|_\beta \leq e^{k\beta N} \|V_N{}^k\|$ for any positive β. It follows that $\|V_N{}^k\|_\beta \leq (e^{\beta N} \frac{B}{A+B})^k$. Therefore, if $\beta < \frac{1}{N} \log \frac{A+B}{B}$, the series $\sum_k \|V_N^k\|_\beta$ converges and is bounded by $(1 - e^{\beta N} \frac{B}{A+B})^{-1}$. □

Theorem 1.
Suppose that $U \in \mathcal{V}^\alpha$, $\alpha > 0$. If there exist $0 < A \leq B < \infty$ such that

$$AI \leq U \leq BI , \tag{3.15}$$

then $U^{-1} \in \mathcal{V}^\delta$ for some $\delta > 0$.

Proof. By Lemma 3, $U_N^{-1} \in \mathcal{V}^\beta$ if $N > N_o$ and $0 < \beta < \frac{1}{N} \log \frac{A+B}{B}$. Therefore, we can write

$$U = U_N \left(I + U_N^{-1} \widetilde{U}_N\right) .$$

We may assume that $\frac{1}{N} \log \frac{A+B}{B} < \alpha$. Moreover, by (3.10), $\widetilde{U}_N \in \mathcal{V}^\beta$. Thus, $(U_N^{-1} \widetilde{U}_N)^k$ is the product of $2k$ elements of $\mathcal{V}^\beta$; thus, by Proposition 4, (3.10), and (3.13)

$$\left\| \left(U_N^{-1} \widetilde{U}_N\right)^k \right\|_{\beta/2} \leq \mu^k$$

where

$$\mu = 36\left(1+\frac{1}{\beta}\right)^2 \left\|U_N^{-1}\right\|_\beta \|\widetilde{U}_N\|_\beta \le 36\left(1+\frac{1}{\beta}\right)^2 \frac{2}{A+B}$$
$$\left(1-e^{\beta N}\frac{B}{A+B}\right)^{-1} e^{N(\beta-\alpha)}\|U\|_\alpha .$$

For any N let β be such that

$$e^{\beta N}\frac{B}{A+B} = \frac{1}{2} .$$

Then $1+\frac{1}{\beta} = O(N)$. Therefore, we may choose N sufficiently large so that $\mu < 1$. Thus, the Neumann series $\sum_{k\ge o}(-1)^k(U_N^{-1}\widetilde{U}_N)^k$ converges to $(1+U_N^{-1}\widetilde{U}_N)^{-1}$ in $\mathcal{V}^{\beta/2}$. Since $U_N^{-1} \in \mathcal{V}^\beta$, by Proposition 4, we conclude that $U^{-1} = (1+U_N^{-1}\widetilde{U}_N)^{-1}\, U_N^{-1} \in \mathcal{V}^\delta$ for every $\delta < \frac{\beta}{2}$. □

We can now prove the main result of this section, namely that if the function g generating a frame has exponential decay, then the dual function also has exponential decay. We first show that the entries of the matrix $M(t) = (m_{jl}(t))$ in (3.1) have exponential decay.

Lemma 4.

Suppose $t_o \le h = \frac{2\pi}{\omega_o}$ and let $g \in L^2(\mathbf{R})$ be such that

$$|g(t)| \le Ce^{-\sigma|t|}$$

for some C and $\sigma > 0$. Then, for any $0 < \tau < \sigma$, the matrix $M(t)$ is in $\mathcal{V}^{\tau h}$ uniformly in t.

Proof. By (2.8)

$$|m_{jl}(t)| = h\left|\sum_n g(t-nt_o-lh)\,\overline{g}(t-nt_o-jh)\right| \le C^2\sum_n e^{-\sigma|t-lh-nt_o|}\, e^{-\sigma|t-jh-nt_o|} .$$

By applying Lemma 1 with $b = t_o$, $x = t-lh$, and $y = t-jh$, we obtain

$$|m_{jl}(t)| \le 3C^2\,\frac{\sigma+t_o^{-1}}{\sigma-\tau}\, e^{-\tau h|j-l|}$$

for any $t \in \mathbf{R}$. □

Theorem 2.

Suppose $t_o \le h = \frac{2\pi}{\omega_o}$ and let $g \in W(L^\infty, L^1)$ be such that $|g(t)| \le C\, e^{-\sigma|t|}$, for some C and $\sigma > 0$. If

$$AI \le F^*F \le BI ,$$

$0 < A \le B < \infty$, then $|\tilde{g}(t)| \le C_1\, e^{-\gamma|t|}$ for some C_1 and $\gamma > 0$.

Proof. By Lemma 4 the matrix $M(t)$ is in $\mathcal{V}^{\tau h}$ uniformly in t for any $\tau < \sigma$. By Proposition 2, the bounds A and B are also bounds of the matrix $M(t)$ for almost all $t \in \mathbf{R}$. Thus, we can apply Theorem 1 with $\alpha = \tau h$ and conclude that for almost every t the matrix $M(t)$ is invertible and $M^{-1}(t) = (\ell_{ij}(t))$ is in $\mathcal{V}^{\delta h}$ for some δ. Moreover, ess sup$\|M^{-1}(t)\|_{\delta h} < \infty$ since $M(t)$ is in $\mathcal{V}^{\tau h}$ uniformly in t. Hence, by (3.2),

$$|\tilde{g}(t)| \le \sum_k |\ell_{0k}(t)|\, |g(t-kh)| \le C'\sum_{k\in\mathbf{Z}} e^{-\delta h|k|}\, e^{-\sigma|t-kh|} .$$

By applying Lemma 1, we obtain the thesis with $\gamma = \min(\delta h, \sigma)$. □

We end this section by discussing briefly the question whether the dual function $\tilde{g}$ has compact support if the window g has compact support (see also [9]). Let the support of g be in $[-a, a]$. From Remark 1 in Section 2, it follows immediately that if $h \geq 2a$, $\tilde{g}$ has compact support. If $h < 2a$ and $\rho_o = \frac{2\pi}{t_o\omega_o}$ is rational, by using the Zak transform, it can be shown that $\tilde{g}$ cannot have compact support [5]. Indeed the Zak transform of $\tilde{g}$, $U_h\tilde{g}(t,s)$ $(t,s) \in [0,1] \times [0,1]$, extends to a meromorphic function $z \mapsto H(z,s)$, with poles off the real axis. Hence, $\tilde{g}$ cannot have compact support since the Zak transform of a function with compact support, as function of t, extends to an entire function.

4. The Computation of the Dual Function

In this section we present a numerical technique to find the dual function $\tilde{g}$ and, by using the results of the previous section, we estimate the error. Let $\|\cdot\|$ denote the $l^2(\mathbf{Z})$ norm. The method we propose is based on identity (3.2). Let $\{g_{m\omega_o,nt_o}\}$ be a frame with best frame bounds A and B. Let $t \in \mathbf{R}$ be fixed. Henceforth we shall suppress the dependence on t. Let $\mathbf{g}$ and $\tilde{\mathbf{g}}$, $j \in \mathbf{Z}$ denote the vectors

$$\mathbf{g} = (g(t - jh))_{j\in\mathbf{Z}} \qquad \tilde{\mathbf{g}} = (\tilde{g}(t - jh))_{j\in\mathbf{Z}} .$$

By (3.1),

$$M\tilde{\mathbf{g}} = \mathbf{g} . \tag{4.1}$$

If g has compact support in $[-a, a]$, then M is a band matrix with band width $N_o = [\frac{2a}{h}]$. Let m, n be positive integers. To solve (4.1), we replace it by the finite dimensional system

$$PMP\mathbf{v} = P\mathbf{g} \tag{4.2}$$

where $P = P(n, m)$ is the projection defined by:

$$(P\mathbf{u})_k = \begin{cases} u_k & \text{if } -n \leq k \leq m \\ 0 & \text{otherwise} , \end{cases} \qquad \text{for any } \mathbf{u} = (u_j)_{j\in\mathbf{Z}} .$$

Theorem 3.
Let g or $\tilde{g}$ be in $W(L^\infty, L^1)$, then equation (4.2) has a unique solution $\mathbf{v}_P$ in $Im\,P$, which satisfies the error estimate

$$\|\mathbf{v}_P - \tilde{\mathbf{g}}\| \leq \left(\frac{B}{A} + 1\right) \|(I - P)\tilde{\mathbf{g}}\| , \tag{4.3}$$

where A and B are the optimal frame bounds.

Proof. By Proposition 2 the restriction of $P\,M\,P$ to $\mathrm{Im}P$ is invertible and the norm of its inverse is $\leq A^{-1}$. Thus, equation (4.2) has a unique solution $\mathbf{v}_P$ in $\mathrm{Im}P$. Let $Q = I - P$, then

$$\|\mathbf{v}_P - \tilde{\mathbf{g}}\| \leq \|\mathbf{v}_P - P\tilde{\mathbf{g}}\| + \|Q\tilde{\mathbf{g}}\| . \tag{4.4}$$

To estimate the first summand, we observe that, from (4.1) and (4.2),

$$PMP(\mathbf{v}_P - P\tilde{\mathbf{g}}) = P\mathbf{g} - PM(I - Q)\tilde{\mathbf{g}} = P\mathbf{g} - PM\tilde{\mathbf{g}} + PMQ\tilde{\mathbf{g}} = PMQ\tilde{\mathbf{g}} . \tag{4.5}$$

Hence,

$$\|\mathbf{v}_P - P\tilde{\mathbf{g}}\| \leq \frac{1}{A}\|P\,M\,Q\|\,\|Q\tilde{\mathbf{g}}\| \leq \frac{B}{A}\,\|Q\tilde{\mathbf{g}}\| . \tag{4.6}$$

From (4.4) the thesis follows. □

By Theorem 2, if g has exponential decay, then $\tilde{g}$ also has exponential decay. In this case an explicit error estimate can be given.

Corollary 2.
Let g or $\tilde{g}$ be in $W(L^\infty, L^1)$. If $|\tilde{g}(t)| \le C_1 e^{-\gamma|t|}$ then for every $t \in \mathbf{R}$ and every $\delta > 0$,

$$\|\mathbf{v}_P - \tilde{\mathbf{g}}\| \le \frac{c}{\sqrt{h\gamma}} \left(\frac{B}{A} + 1\right) e^{-\gamma\delta},$$

provided $m > \frac{\delta+t}{h}$ and $n > \frac{\delta-t}{h}$.

Proof. To estimate $\|Q\tilde{\mathbf{g}}\|^2 = \sum_{k>m} |\tilde{g}(t-kh)|^2 + \sum_{k<-n} |\tilde{g}(t-kh)|^2$, we first let t be positive. Then the first sum is estimated by

$$C_1^2 \int_m^\infty |e^{-2\gamma|t-uh|}du \le \frac{C_1^2}{h} \int_{hm-t}^\infty e^{-2\gamma y} dy = \frac{C_1^2}{2\gamma h} e^{-2\gamma(hm-t)}$$

if $mh - t > 0$. Similarly, we find that the second sum in the expression of $\|Q\tilde{\mathbf{g}}\|$ is less than $\frac{C_1^2}{2\gamma h} e^{-2\gamma(t+nh)}$. Fix $\delta > 0$. If $t > 0$ and $m > \frac{\delta+t}{h}$ and $n > \frac{\delta-t}{h}$

$$\|Q\tilde{\mathbf{g}}\|^2 \le \frac{C_1^2}{2\gamma h} \left(e^{-2\gamma(mh-t)} + e^{-2\gamma(nh+t)}\right) \le \frac{C_1^2}{2\gamma h} e^{-2\gamma\delta}. \tag{4.7}$$

A similar argument shows that the conclusion holds if $t < 0$. □

5. Frames Generated by Cardinal Splines

In this section we shall use the previous results to study frames generated by the functions $g(t) = s_M(t + \frac{M}{2})$, where s_M is the cardinal B-spline of order M,

$$s_M(t) = \frac{1}{(M-1)!} \sum_{k=0}^{M} (-1)^k \binom{M}{k} (t-k)_+^{M-1},$$

for $0 \le t \le M$, and zero otherwise. So, for $M = 1$, g is the characteristic function χ of the interval $[-\frac{1}{2}, \frac{1}{2}]$, for $M = 2$, g is the "tent" function $g(t) = (1 - |t|)_+$. The Fourier transform of g is

$$\widehat{g}(\xi) = \frac{1}{\sqrt{2\pi}} \left(\frac{\sin(\xi/2)}{\xi/2}\right)^M. \tag{5.1}$$

Before presenting the numerical experiments, we review some facts from the theory of frames constructed by means of cardinal B-splines.

It is well known that if $\{g_{m\omega_o, nt_o}\}$ is a frame, then the frame parameters must satisfy the condition $t_o\omega_o \le 2\pi$. If $t_o\omega_o = 2\pi$, by the Balian–Low theorem, either $tg(t) \notin L^2(\mathbf{R})$ or $t\hat{g}(t) \notin L^2(\mathbf{R})$. Therefore, if $t_o\omega_o = 2\pi$, cardinal B-splines of order greater than 1 cannot generate frames, while it is well known that χ generates an orthonormal basis for $t_o = 1$ and $\omega_o = 2\pi$. Henceforth we shall suppose $t_o\omega_o < 2\pi$.

First we discuss briefly the existence of tight frames. Cardinal B-splines of order $M > 1$ cannot generate tight frames if $t_o\omega_o < 2\pi$, by Remark 1 at the end of Section 2. Indeed if $M > 1$, the sum $S_0(t) = \frac{2\pi}{\omega_o} \sum_{n\in\mathbf{Z}} g^2(t - nt_o)$ is not constant for any t_o. Tight frames can be constructed by taking as window the square root of a cardinal B-spline [2]. Indeed, since by Poisson summation formula $\sum_l s_M(t-l) = 1$, from Remark 2, it follows that for $t_o < M$ and $\omega_o < 2\pi/M$, the function

$g(t) = (t_o)^{-1/2} s_M{}^{1/2}(t_o{}^{-1}t)$, generates a tight frame. In particular, if $t_o \leq 1$, $t_o\omega_o \leq 2\pi$, and $g(t) = \frac{1}{\sqrt{t_o}}\chi(\frac{t}{t_o})$, then $\{g_{m\omega_o,nt_o}\}$ is a tight frame with frame bounds $\frac{2\pi}{t_o\omega_o}$. Tight frames can also be constructed from any frame, via a general procedure: if the family $\{g_{m\omega_o,nt_o}\}$ is a frame, F^*F the associated frame operator and $G = (F^*F)^{-1/2}g$, then $\{G_{m\omega_o,nt_o}\}$ is a tight frame (see [3]). If g has support $[-a, a]$ and $2\pi/\omega_o > 2a$, then, by Remark 2, G has compact support since $G = (\frac{\omega_o}{2\pi})^{1/2} g(\sum_k \tau_{kt_o} g^2)^{-1/2}$.

To find the points (t_o, ω_o) of the time-frequency plane such that s_M generates a frame, we note that, for the spline of order greater than 1, $\inf S_0 > 0$ if and only if $t_o \geq M$. Moreover, if $\omega_o \leq \frac{2\pi}{M}$, the frame operator reduces to the operator of multiplication bt S_0. Hence, by Remark 1, if $t_o \geq M > 1$, s_M does not generate a frame; on the contrary, if $t_o < M$ and $\omega_o \leq \frac{2\pi}{M}$, s_M generates a frame.

Proposition 5.

If ω_o is equal to $2j\pi$, $j \geq 2$ integer, the family $\{(s_M)_{m\omega_o,nt_o}\}$ is not a frame.

Proof. By Remark 1, it is enough to show that if $\omega_o = 2j\pi$, j integer ≥ 2, the infimum of the function

$$S_0\hat{}(\xi) = \sum_n (\widehat{s_M}(\xi - n\omega_o))^2$$

is zero. Indeed, by (5.1), all the summands vanish in 2π. □

Remark. We notice that $\omega_o = 2j\pi$, $j \geq 2$ integer, is a necessary condition for $\inf S_0\hat{} = 0$. Indeed if $\omega_o \neq 2j\pi$ $j \geq 2$ integer, the intersection of the sets of zeros of $\hat{s_M}$ and $\hat{s_M}(\cdot - \omega_o)$is empty. Thus, $S_0\hat{}(\xi) > 0$ for every ξ. Since $S_0\hat{}(\xi) > 0$ is continuous and periodic, its infimum is positive.

We can summarize this result and the previous considerations as follows:

Proposition 6.

Suppose $M > 1$. Then the family $\{(s_M)_{m\omega_o,nt_o}\}$

i) is a frame in the region $\{(t_o, \omega_o) : t_o < M,\ \omega_o \leq \frac{2\pi}{M}\}$,

ii) is not a frame in the region

$$\{(t_o, \omega_o) : t_o > 0, \omega_o = 2j\pi,\ j \geq 2\} \cup \{(t_o, \omega_o) : t_o \geq M, \omega_o > 0\}\ .$$

If $M = 1$ the same result holds, provided that, in condition *ii)*, we replace the region $\{(t_o, \omega_o) : t_o \geq M, \omega_o > 0\}$ with $\{(t_o, \omega_o) : t_o > 1, \omega_o > 0\}$.

Thus, it remains to investigate the region of the time-frequency plane

$$\left\{(t_o, \omega_o) : t_o < M,\ \omega_o > \frac{2\pi}{M},\ \omega_o \neq 2j\pi,\ j \geq 2\right\}\ .$$

In the rest of this section we shall deal with this question, describing some numerical experiments based on the frame bounds estimates given in Section 2. First we compare the sharpness of estimates (2.6) and (2.14) for the Gaussian and for the spline of order 2. Next, we investigate numerically the region of the time-frequency plane (t_o, ω_o) such that the family $\{(s_2)_{m\omega_o,nt_o}\}$ is a frame. Finally, by using the method described in Section 4, we find the dual function $\tilde{s_2}$.

We shall denote by A, B the frame bounds computed by using Daubechies' estimates (2.6) and by A_n, B_n the frame bounds computed via the new estimates (2.14).

If $t_o\omega_o = \frac{2\pi}{N}$, N a positive integer, it is possible to compute the exact values A_{ex}, B_{ex} of the optimal frame bounds. Indeed, the frame operator F^*F is unitarily equivalent to the operators of multiplication by the functions

$$K_o(t,s) = \sum_{l=0}^{N-1} \left| U_{t_o}\, g\left(t + \frac{l}{N}, s\right)\right|^2 \qquad L_o(t,s) = \sum_{l=0}^{N-1} \left| U_h\, g\left(t, s - \frac{l}{N}\right)\right|^2$$

acting on $L^2([0,1]^2)$. Here U_λ $\lambda \in \mathbf{R}^+$ denotes the Zak transform of g with parameter λ (see [1]). Therefore, it is possible to compare the computed frame bounds A_n and B_n with the exact values

$$A_{ex} = \inf K_o = \inf L_o, \qquad B_{ex} = \sup K_o = \sup L_o .$$

First we test the sharpness of the estimates on the Gaussian $\pi^{-1/4}e^{-x^2/2}$. It is well known that this function generates a frame if and only if $t_o\omega_o < 2\pi$ (see [13], [16]). For the Gaussian, Janssen found an analytic expression for the frame bounds in the case $t_o\omega_o = \frac{2\pi}{N}$ and computed the dual function for N even [12]. In Table 2.1 we compare the estimated and the exact lower frame bounds for this function in the case $t_o\omega_o = \pi$ for several values of t_o. In all these cases the upper frame bounds coincide to the fifth digit. The same calculations for other hyperbolas $t_o\omega_o = 2\pi/N$, $N > 2$, give similar results. So, for the Gaussian, Ron and Shen's estimates slightly improve Daubechies' estimates, and there is a very good agreement between the computed and the exact frame bounds.

TABLE 2.1

The lower frame bound for $t_o\omega_o = \pi$ for $g = \pi^{-1/4}e^{-x^2/2}$

t_o	A	A_n	A_{ex}
0.5	0.00073	0.00073	0.00073
1.0	0.60089	0.60119	0.60119
1.5	1.51863	1.53961	1.53961
2.0	1.57489	1.60014	1.60015
2.5	1.17168	1.17804	1.17804
3.0	0.71275	0.71341	0.71341
3.5	0.36939	0.36942	0.36942
4.0	0.16533	0.16534	0.16534

Next, we test the estimates on the spline of order 2; in Table 2.2 we list computed vs. exact frame bounds for $t_o\omega_o = \pi$.

To find the region of the time-frequency plane such that $\{g_{m\omega_o,nt_o}\}$ constitutes a frame, we shall compute the lower frame bounds for each fixed value of t_o in a range and for increasing values of ω_o, detecting the value $\omega_o{}^c = \omega_o{}^c(t_o)$ such that the computed lower frame bound becomes zero. This technique was used by Daubechies to investigate Gaussian frames. We shall also compare results obtained using estimates (2.6) and (2.14) [1]. In Table 2.3 we list, for $t_o = 0.2, 0.3 \ldots, 1.9$, the values $\omega_o{}^c$ obtained by using Daubechies's estimates and the values $\omega_o{}^c{}_n$ obtained with the new estimates.

In Figure 1 we have plotted these values and the hyperbola $t_o\omega_o = 2\pi$. We remind the reader that for $t_o\omega_o > 2\pi$ the family $\{g_{m\omega_o,nt_o}\}$ cannot be a frame. We can see that for $t_o \in [1,2]$ and ω_o close to $2\pi/t_o$ the estimates for the lower frame bound are excellent. We believe that, due to the poor localization of $\hat{g}$, the estimates for small t_o are not satisfactory.

In Figure 2 we have plotted the curve $(t_o, \omega_o{}^c{}_n)$, the hyperbolas $t_o\omega_o = 2\pi$, $t_o\omega_o = \pi$ and the lines $\omega_o = 2j\pi$, $j \geq 2$, where $A_{ex} = 0$, by Proposition 5. Although our estimates are not sufficiently sharp to investigate whether the family $\{g_{m\omega_o,nt_o}\}$ is a frame for the values of the time-frequency parameters between these lines, we have obtained $A_n > 0$ also in some points between the lines $\omega_o = 4\pi$ and $\omega_o = 6\pi$. Indeed A_{ex} is positive on some points on the hyperbola $t_o\omega_o = \pi$ that lie between these two lines.

Lastly, we have used the method described in Section 4 to compute the dual function of the spline of order 2. Figure 3 shows some sets of plots of $\tilde{g}$ for fixed ω_o. In the first set of plots $\omega_o = \pi$. In this case, we were able to compare the computed with the exact dual function since, for $\omega_o = \pi$, the dual

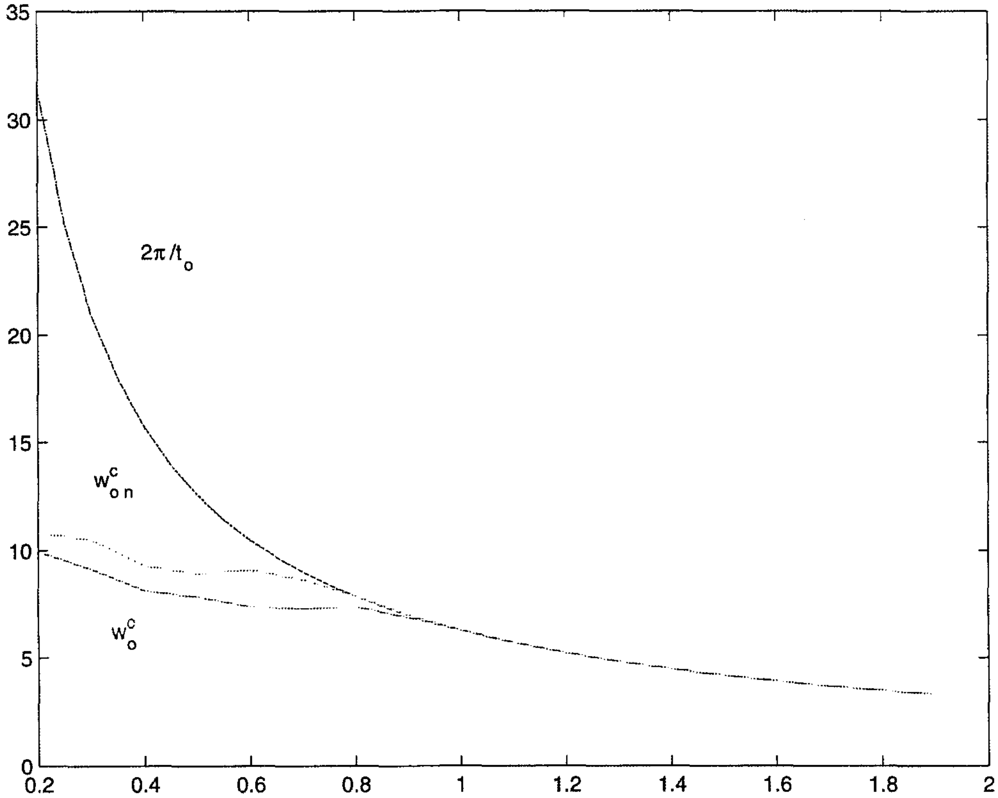

FIGURE 1 The hyperbola $t_o\omega_o = 2\pi$ and the values ω_o^c, ω_{on}^c.

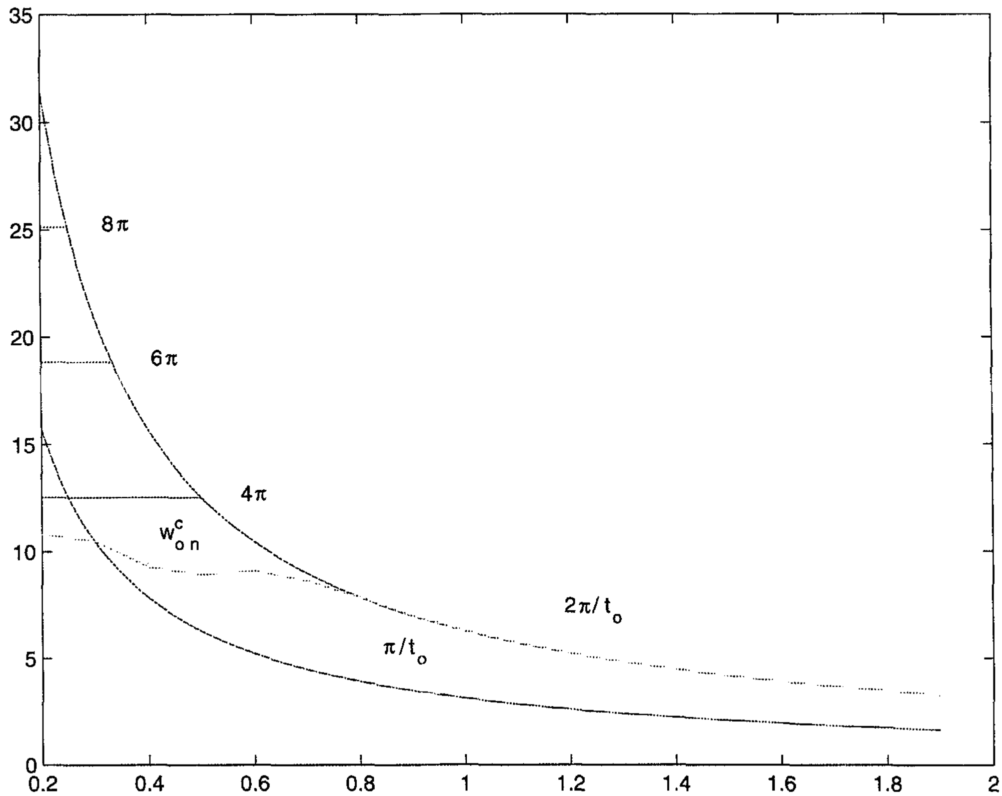

FIGURE 2 The values ω_{on}^c and the hyperbolas $t_o\omega_o = 2\pi$, $t_o\omega_o = \pi$.

TABLE 2.2

The frame bounds for $g = (1 - |x|)_+$ and $t_o\omega_o = \pi$. *Values obtained with the estimates for $\hat{g}$

t_o	A	A_n	A_{ex}	B	B_n	B_{ex}
0.2	−0.036*	0.004*	0.008	5.046*	5.008	5.008
0.25	−0.069*	0*	0	4.069	4.000	4.000
0.3	−0.101*	0.001*	0.003	3.450*	3.480	3.480
0.4	0.044*	0.218*	0.230	2.708*	2.720	2.720
0.5	0.5	0.5	0.5	2.250	2.000	2.000
0.6	0.816	0.864	0.864	1.968	1.968	1.968
0.7	0.931	1.120	1.120	1.904	1.904	1.904
0.8	1.024	1.152	1.152	1.856	1.856	1.856
0.9	1.053	1.089	1.089	1.872	1.872	1.872
1.0	1	1	1	2	2	2
1.2	0.768	0.768	0.768	2.4	2.4	2.4
1.4	0.504	0.504	0.504	2.8	2.8	2.8
1.6	0.256	0.256	0.256	3.2	3.2	3.2
1.8	0.072	0.072	0.072	3.6	3.6	3.6

TABLE 2.3

Values of w_o^c for $g = (1 - |x|)_+$

t_o	$\omega_o{}^c$	$2\pi/(\omega_o{}^c t_o) - 1$	$\omega_{o_n}^c$	$2\pi/(\omega_{o_n}^c t_o) - 1$
0.2	9.94000	2.16056	10.8197	1.9036
0.3	9.14195	1.2910	10.4991	$9.9483 \cdot 10^{-1}$
0.4	8.12088	$9.3427 \cdot 10^{-1}$	9.27700	$6.9322 \cdot 10^{-1}$
0.5	7.83451	$6.0398 \cdot 10^{-1}$	8.92778	$4.0755 \cdot 10^{-1}$
0.6	7.36071	$4.2269 \cdot 10^{-1}$	9.09486	$1.5141 \cdot 10^{-1}$
0.7	7.28531	$2.3207 \cdot 10^{-1}$	8.61735	$4.1617 \cdot 10^{-2}$
0.8	7.36071	$6.6783 \cdot 10^{-2}$	7.85239	$2.0245 \cdot 10^{-4}$
0.9	6.86549	$1.6871 \cdot 10^{-2}$	6.97859	$3.9148 \cdot 10^{-4}$
1	6.28319	$2.2204 \cdot 10^{-16}$	6.28319	$2.2204 \cdot 10^{-16}$
$1.1 \le t_o \le 1.9$	$6.28319/t_o$	$2.2204 \cdot 10^{-16}$	$6.28319/t_o$	$2.2204 \cdot 10^{-16}$

function $\tilde{g}$ is $S_o{}^{-1}g$ (see Remark 1 in Section 2). For $t_o = 0.2$, where the frame is close to a tight frame, the function $\tilde{g}$ is very similar to g. For growing values of t_o, the function $\tilde{g}$ becomes very different from g; for $t_o = 1.9$ the $\tilde{g}$ does not even look like a perturbed tent any more. In the second set of plots, where $\omega_o = 2\pi$, we can observe the same behavior: as we get closer to the hyperbola $t_o\omega_o = 2\pi$ the deviation of $\tilde{g}$ increases.

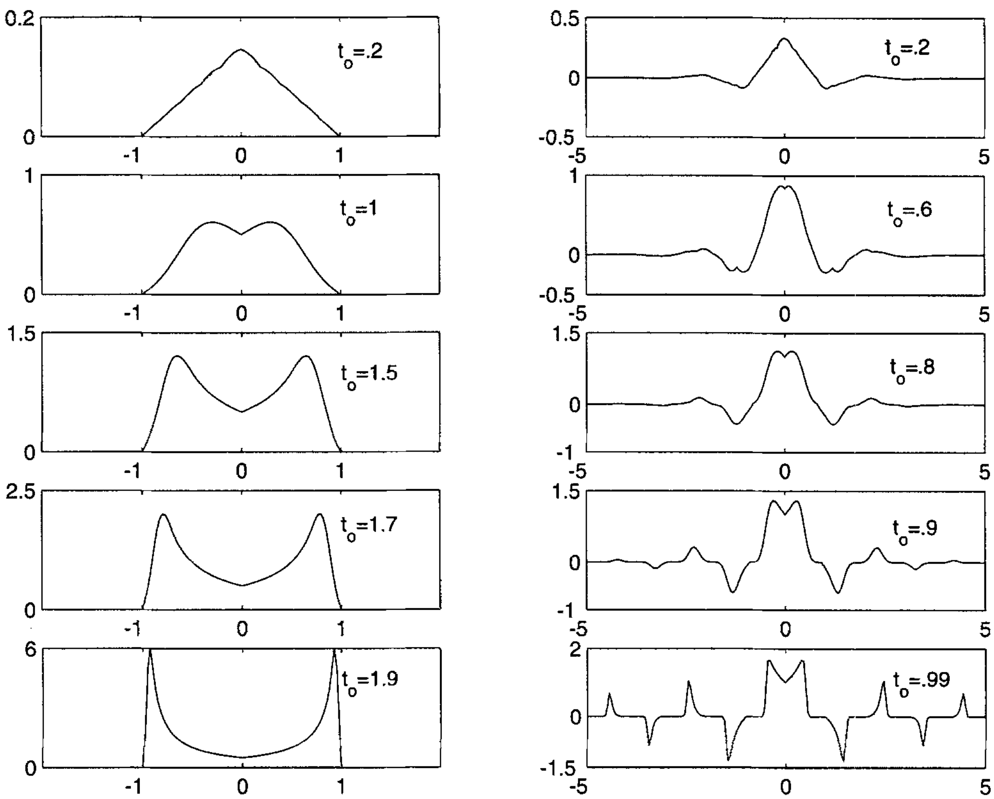

FIGURE 3 The dual function $\tilde{g}$. The parameter $h = \frac{2\pi}{\omega_o}$ is equal to 2 in the first column, and is equal to 1 in the second.

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