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# Analytic and Asymptotic Properties of Non-Symmetric Linnik's Probability Densities

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*ABSTRACT. The function*

$$\varphi_\alpha^\theta(t) = \frac{1}{1 + e^{-i\theta \operatorname{sgn} t} |t|^\alpha}, \; \alpha \in (0, 2), \; \theta \in (-\pi, \pi],$$

*is a characteristic function of a probability distribution iff* $|\theta| \le \min(\frac{\pi\alpha}{2}, \pi - \frac{\pi\alpha}{2})$. *This distribution is absolutely continuous; for* $\theta = 0$ *it is symmetric. The latter case was introduced by Linnik in 1953 [13] and several applications were found later. The case* $\theta \neq 0$ *was introduced by Klebanov, Maniya, and Melamed in 1984 [9], while some special cases were considered previously by Laha [12] and Pillai [18]. In 1994, Kotz, Ostrovskii and Hayfavi [10] carried out a detailed investigation of analytic and asymptotic properties of the density of the distribution for the symmetric case* $\theta = 0$. *We generalize their results to the non-symmetric case* $\theta \neq 0$. *As in the symmetric case, the arithmetical nature of the parameter* $\alpha$ *plays an important role, but several new phenomena appear.*

## 1. Introduction

In 1953, Linnik [13] proved that the function

$$\varphi_\alpha(t) = \frac{1}{1 + |t|^\alpha}, \; \alpha \in (0, 2), \tag{1.1}$$

is a characteristic function of a symmetric probability density $p_\alpha(x)$. Since then, the family of symmetric Linnik's densities $\{p_\alpha(x) : \alpha \in (0, 2)\}$ has had several probabilistic applications (see, e.g., [1]–[6]). In 1994, Kotz et al. [10] carried out a detailed investigation of analytic and asymptotic properties of $p_\alpha(x)$.

In 1984, Klebanov et al. [9], introduced the concept of geometric strict stability and proved that the family of geometrically strictly stable densities coincides with the family of densities with

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characteristic functions

$$\varphi_\alpha^\theta(t) = \frac{1}{1 + e^{i\theta \operatorname{sgn} t}|t|^\alpha}, \ \alpha \in (0,2), \ |\theta| \le \min\left(\frac{\pi\alpha}{2}, \pi - \frac{\pi\alpha}{2}\right). \tag{1.2}$$

A few years ago, it turned out that these densities have useful applications in modeling financial data [11, 14]. In 1992, Pakes [17] showed that, in some characterization problems of Mathematical Statistics, the probability densities with characteristic functions (1.2) play an important role. These densities can be viewed as generalizations of symmetric Linnik's densities. For $|\theta| = \min(\frac{\pi\alpha}{2}, \pi - \frac{\pi\alpha}{2})$, these densities appeared in the papers by Laha [12] and Pillai [18]. Therefore, the problem of the study of analytic and asymptotic properties of the densities with characteristic function given by (1.2) seems to be of interest.

As it was shown by Pakes [17], the function

$$\varphi_\alpha^\theta(t) = \frac{1}{1 + e^{-i\theta \operatorname{sgn} t}|t|^\alpha}, \ \alpha \in (0,2), \ \theta \in (-\pi, \pi], \tag{1.3}$$

is a characteristic function of a probability distribution iff

$$|\theta| \le \min\left(\frac{\pi\alpha}{2}, \pi - \frac{\pi\alpha}{2}\right). \tag{1.4}$$

The distribution is absolutely continuous. We denote its density by $p_\alpha^\theta(x)$. Clearly, for $\theta = 0$, $p_\alpha^\theta(x)$ coincides with symmetric Linnik's density $p_\alpha(x)$. For $\theta \ne 0$ we call $p_\alpha^\theta(x)$ non-symmetric Linnik's density. We study analytic and asymptotic properties of $p_\alpha^\theta(x)$ and obtain generalizations of the results of [10]. As in the symmetric case, convergence of series expansions of $p_\alpha^\theta(x)$ depends on the arithmetical nature of the parameter $\alpha$. However, several new phenomena appear connected with the non-symmetry parameter $\theta$.

## 2. Statement of Results

***Theorem 1.***

*The distribution function of the characteristic function $\varphi_\alpha^\theta(t)$ is absolutely continuous and its density $p_\alpha^\theta(x)$ can be represented in the form*
*(i) for $0 < \alpha < 1$, $0 \le \theta \le \pi\alpha/2$ and $1 \le \alpha < 2$, $0 \le \theta < \pi - \frac{\pi\alpha}{2}$;*

$$p_\alpha^\theta(x) = \frac{\sin\left(\frac{\pi\alpha}{2} + \theta \operatorname{sgn} x\right)}{\pi} \int_0^\infty \frac{e^{-y|x|} y^\alpha \, dy}{\left|1 + e^{i\theta \operatorname{sgn} x} y^\alpha e^{\frac{i\pi\alpha}{2}}\right|^2}, \ x \in \mathbb{R}, \tag{2.1}$$

*(ii) for $1 \le \alpha < 2$, $\theta = \pi - \frac{\pi\alpha}{2}$;*

$$p_\alpha^\theta(x) = \begin{cases} -\dfrac{\sin(\pi\alpha)}{\pi} \displaystyle\int_0^\infty \frac{e^{yx} y^\alpha \, dy}{\left|1 - e^{i\pi\alpha} y^\alpha\right|^2} & , x < 0, \\ e^{-x}/\alpha & , x > 0, \end{cases} \tag{2.2}$$

*(iii) for $-\pi < \theta < 0$, we have*

$$p_\alpha^\theta(x) = p_\alpha^{-\theta}(-x), \ x \in \mathbb{R}. \tag{2.3}$$

The set of all pairs $(\alpha, \theta)$ for which $\varphi_\alpha^\theta(t)$ is a characteristic function of a probability distribution is a diamond-shaped region described by (1.4) where $0 < \alpha < 2$. The points $(0, 0)$ and $(2, 0)$ are not included in the set. Note that the point $(2, 0)$ can be interpreted as the well-known Laplace distribution with the characteristic function $\varphi_2(t) = (1+t^2)^{-1}$ and the density $p_2(x) = e^{-|x|}/2$. We shall denote this set by $PD$ and call it *the parametrical domain.* Denote by $PD^+$ the part of $PD$ consisting of pairs $(\alpha, \theta)$ such that $\theta \geq 0$. Without loss of generality, we can restrict our study of $p_\alpha^\theta(x)$ to $PD^+$ since one can obtain $p_\alpha^\theta(x)$ for $(\alpha, \theta) \in PD \backslash PD^+$ from (2.3).

Recall that a function $f(x)$ defined on an interval $I \subset \mathbb{R}$ is called *completely monotonic* (resp., *absolutely monotonic*) if it is infinitely differentiable on $I$ and, moreover, for any $x \in I$ and any $k = 0, 1, \ldots$, $(-1)^k f^{(k)}(x) \geq 0$ (resp., $f^{(k)}(x) \geq 0$).

The following theorem related to analytic properties of $p_\alpha^\theta(x)$ was proved in the symmetric case $\theta = 0$ in [10].

***Theorem 2.***

*(i) For any $(\alpha, \theta) \in PD^+$, the function $p_\alpha^\theta(x)$ is completely monotonic on $(0, \infty)$ and absolutely monotonic on $(-\infty, 0)$.*
*(ii) For $1 < \alpha < 2$, $0 \leq \theta \leq \pi - \frac{\pi\alpha}{2}$, $p_\alpha^\theta(x)$ is a continuous function on $\mathbb{R}$ and*

$$p_\alpha^\theta(0) := \lim_{x\to 0^+} p_\alpha^\theta(x) = \lim_{x\to 0^-} p_\alpha^\theta(x) = \frac{1}{\alpha}\frac{\cos\frac{\theta}{\alpha}}{\sin\frac{\pi}{\alpha}} . \tag{2.4}$$

*For $0 < \alpha \leq 1$, $0 \leq \theta < \frac{\pi\alpha}{2}$, $\lim_{x\to 0^+} p_\alpha^\theta(x) = \lim_{x\to 0^-} p_\alpha^\theta(x) = +\infty$.*
*For $0 < \alpha \leq 1$, $\theta = \frac{\pi\alpha}{2}$, $\lim_{x\to 0^+} p_\alpha^\theta(x) = \infty$; $p_\alpha^\theta(x) = 0$, for $x < 0$.*
*(iii) For $1 < \alpha < 2$, $0 \leq \theta \leq \pi - \frac{\pi\alpha}{2}$ and $0 < \alpha \leq 1$, $0 \leq \theta < \frac{\pi\alpha}{2}$,*

$$\lim_{x\to 0^+} (-1)^k \left(p_\alpha^\theta(x)\right)^{(k)} = \infty , \quad \lim_{x\to 0^-} \left(p_\alpha^\theta(x)\right)^{(k)} = \infty , \; k = 1, 2, 3, \ldots$$

*The first of these equalities remains valid for $0 < \alpha \leq 1$, $\theta = \pi\alpha/2$.*

Recall that an absolutely continuous distribution is called *unimodal with mode 0* if its density is non-decreasing on $(-\infty, 0)$ and non-increasing on $(0, \infty)$. The following theorem is an immediate corollary of Theorem 2.

***Theorem 3.***

*For any $(\alpha, \theta) \in PD$, the distribution with the characteristic function (1.3) is unimodal with mode 0.*

Note that, in the case $\theta = \min(\frac{\pi\alpha}{2}, \pi - \frac{\pi\alpha}{2})$, this theorem was proved by Laha [12] in 1961.

The following theorem measures the non-symmetry of $p_\alpha^\theta(x)$. Surely, this non-symmetry increases with $|\theta|$.

***Theorem 4.***

*(i) For any $(\alpha, \theta) \in PD^+$,*

$$\int_0^\infty p_\alpha^\theta(\pm x)dx = \frac{1}{2} \pm \frac{\theta}{\pi\alpha} .$$

*(ii) For any $(\alpha, \theta) \in PD^+$, and any $k = 0, 1, 2, \ldots$,*

$$(-1)^k \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^k p_\alpha^\theta(x) \sin\left(\frac{\pi\alpha}{2} - \theta\right) \geq \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^k p_\alpha^\theta(-x) \sin\left(\frac{\pi\alpha}{2} + \theta\right) , \; x > 0 .$$

*In particular, $p_\alpha^\theta(x)\sin(\frac{\pi\alpha}{2} - \theta) \geq p_\alpha^\theta(-x)\sin(\frac{\pi\alpha}{2} + \theta)$, $x > 0$.*
*(iii) For any $(\alpha, \theta) \in PD^+$ such that $\alpha \in (0, 1)$ and any $k = 0, 1, 2, \ldots$,*

$$(-1)^k \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^k p_\alpha^\theta(x) \geq \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^k p_\alpha^\theta(-x) , \; x > 0 .$$

*In particular,* $p^\theta_\alpha(x) \geq p^\theta_\alpha(-x)$, $x > 0$.

*For any* $(\alpha, \theta) \in PD^+$ *such that* $\alpha \in (1, 2)$, $\theta > 0$, *the last assertion is false.*

*(iv) As a function of* $\theta$, $0 \leq \theta < \min(\frac{\pi\alpha}{2}, \frac{\pi}{2} - \frac{\pi\alpha}{2})$, $(-1)^k (\mathrm{d}/\mathrm{d}x)^k p^\theta_\alpha(x)$ *increases and* $(\mathrm{d}/\mathrm{d}x)^k p^\theta_\alpha(-x)$ *decreases for any fixed* $\alpha \in (0, 1)$, *any* $k = 0, 1, 2, \ldots$, *and* $x > 0$.

*For any* $(\alpha, \theta) \in PD^+$ *such that* $\alpha \in (1, 2)$, $\theta > 0$, *the last assertion is false.*

In Figure 1, there are graphs of $p^{\theta_1}_\alpha(x)$ and $p^{\theta_2}_\alpha(x)$: (i) for $0 < \alpha < 1$, $0 < \theta_1 < \theta_2 < \min(\frac{\pi\alpha}{2}, \frac{\pi}{2} - \frac{\pi\alpha}{2})$, (ii) for $1 < \alpha < 3/2$, $\frac{\pi\alpha}{2} - \frac{\pi}{2} < \theta_1 < \theta_2 < \pi - \frac{\pi\alpha}{2}$. The graphs of $p^{\theta_1}_\alpha(x)$ are shown by continuous lines, while the graphs of $p^{\theta_2}_\alpha(x)$ are shown by dotted lines.

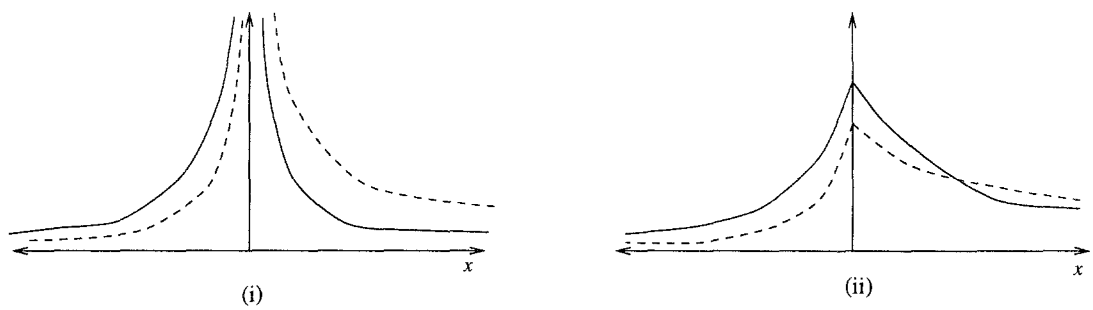


FIGURE 1

The following theorem characterizes the asymptotic behavior of $p^\theta_\alpha(x)$ at infinity. For $\theta = 0$, the result was proved in [10]. Denote by $PD^+_0$ the part of $PD^+$ which is obtained by removing the pairs $(\alpha, \theta)$ with $\theta = \min(\frac{\pi\alpha}{2}, \pi - \frac{\pi\alpha}{2})$.

***Theorem 5.***

*(i) For any* $(\alpha, \theta) \in PD^+_0$ *and* $N = 1, 2, 3, \ldots$,

$$p^\theta_\alpha(x) = \frac{1}{\pi} \sum_{k=1}^{N} \Gamma(1 + \alpha k)(-1)^{k+1} \sin\left(\frac{\pi\alpha k}{2} + k\theta \operatorname{sgn} x\right) |x|^{-1-\alpha k} + R_{N,\alpha}(x)\,, \tag{2.5}$$

*where*

$$\left|R_{N,\alpha}(x)\right| \leq \frac{\alpha\Gamma(1 + \alpha(N+1))}{\pi \left|\sin\left(\frac{\pi\alpha}{2} + \theta \operatorname{sgn} x\right)\right|} |x|^{-1-\alpha(N+1)}\,. \tag{2.6}$$

*(ii) This statement remains true both for* $\alpha \in (0, 1)$, $\theta = \pi\alpha/2$, $x > 0$, *and for* $\alpha \in [1, 2)$, $\theta = \pi - \pi\alpha/2$, $x < 0$. *For the remaining cases, we have the explicit representations* $p^\theta_\alpha(x) = 0$ *for* $\alpha \in (0, 1)$, $\theta = \pi\alpha/2$, $x < 0$; $p^\theta_\alpha(x) = e^{-x}/\alpha$ *for* $\alpha \in [1, 2)$, $\theta = \pi - \pi\alpha/2$, $x > 0$.

***Corollary 1.***

*For any* $(\alpha, \theta) \in PD^+_0$,

$$p^\theta_\alpha(x) = \frac{1}{\pi}\Gamma(1 + \alpha) \sin\left(\frac{\pi\alpha}{2} + \theta \operatorname{sgn} x\right) |x|^{-1-\alpha} + O\left(|x|^{-1-2\alpha}\right), |x| \to \infty\,.$$

***Corollary 2.***

*For any* $(\alpha, \theta) \in PD^+$,

$$\lim_{x \to \infty} \frac{p^\theta_\alpha(x)}{p^\theta_\alpha(-x)} = \frac{\sin\left(\frac{\pi\alpha}{2} + \theta\right)}{\sin\left(\frac{\pi\alpha}{2} - \theta\right)}\,, \; x > 0\,,$$

*(the right-hand side is equal to* $+\infty$ *for* $\theta = \frac{\pi\alpha}{2}$*).*

***Corollary 3.***
*For any* $(\alpha, \theta_1)$, $(\alpha, \theta_2) \in PD_0^+$,

$$\lim_{x\to\infty} \frac{p_\alpha^{\theta_1}(x)}{p_\alpha^{\theta_2}(x)} = \frac{\sin\left(\frac{\pi\alpha}{2}+\theta_1\right)}{\sin\left(\frac{\pi\alpha}{2}+\theta_2\right)}\ , \ x > 0 .$$

The analytic structure of $p_\alpha^\theta(x)$ depends on the arithmetic nature of the parameter $\alpha$. First we will deal with the case $\alpha = 1/n$, where $n$ is an integer.

Theorems 6 through 9 were proved for $\theta = 0$ in [10].

***Theorem 6.***
*For any* $n = 1, 2, 3, \ldots$, *and* $0 \le \theta < \frac{\pi}{2n}$,

$$\begin{aligned} p_{1/n}^\theta(x) &= \frac{1}{\pi}\sum_{k=1,\frac{k}{n}\notin\mathbb{N}}^{\infty} \Gamma\left(1-\frac{k}{n}\right)(-1)^{k+1}\sin\left(\frac{\pi k}{2n}+k\theta\,\mathrm{sgn}x\right)|x|^{\frac{k}{n}-1} \\ &+ \frac{1}{\pi}\sum_{j=1}^{\infty}(-1)^{(n+1)j}\frac{\Gamma'(j)}{\Gamma^2(j)}\sin\left(\frac{\pi j}{2}+\theta n j\,\mathrm{sgn}x\right)|x|^{j-1} \\ &+ \frac{n(-1)^n}{\pi}(\log|x|)e^{(x(-1)^n\sin(\theta n))}\cos\left(x\cos(\theta n)-\theta n(-1)^n\right) \\ &- (-1)^n\frac{2\theta n+\pi\,\mathrm{sgn}x}{2\pi}e^{(x(-1)^n\sin(\theta n))}\sin(x\cos(\theta n)+\theta n)\,. \end{aligned} \tag{2.7}$$

***Corollary 4.***
*For any* $0 \le \theta \le \frac{\pi}{2}$,

$$\begin{aligned} p_1^\theta(x) &= \frac{1}{\pi}\sum_{j=1}^{\infty}\frac{\Gamma\prime(j)}{\Gamma^2(j)}\sin\left(\frac{\pi j}{2}+\theta j\,\mathrm{sgn}x\right)|x|^{j-1} \\ &- \frac{1}{\pi}(\log|x|)e^{-x\sin\theta}\cos(x\cos\theta+\theta) - \frac{1}{\pi}\left(\frac{\pi\,\mathrm{sgn}x+2\theta}{2}\right)e^{-x\sin\theta}\sin(x\cos\theta+\theta)\,. \end{aligned}$$

The following theorem deals with the case of a rational $\alpha$:

***Theorem 7.***
*Let* $\alpha \in (0, 2)$ *be a rational number. Set* $\alpha = m/n$ *where* $m$ *and* $n$ *are relatively prime integers both greater than* 1. *For* $\alpha = m/n \in (0, 2)$ , $0 \le \theta < \min(\frac{\pi\alpha}{2}, \pi - \frac{\pi\alpha}{2})$,

$$\begin{aligned} p_\alpha^\theta(x) &= \sum_{k=1,\frac{k}{n}\notin\mathbb{N}}^{\infty}\frac{(-1)^{k+1}\sin\left(\frac{\pi k\alpha}{2}+k\theta\,\mathrm{sgn}x\right)}{\Gamma(k\alpha)\sin(\pi k\alpha)}|x|^{k\alpha-1} \\ &+ \frac{1}{\pi}\log\frac{1}{|x|}\sum_{t=1}^{\infty}\frac{(-1)^{(m+n)t}}{\Gamma(mt)}\sin\left(\frac{\pi mt}{2}+\theta nt\,\mathrm{sgn}x\right)|x|^{mt-1} \\ &- \left(\frac{\theta\,\mathrm{sgn}x}{\pi\alpha}+\frac{1}{2}\right)\sum_{t=1}^{\infty}\frac{(-1)^{(m+n)t}}{\Gamma(mt)}\cos\left(\frac{\pi mt}{2}+\theta nt\,\mathrm{sgn}x\right)|x|^{mt-1} \\ &+ \frac{1}{\alpha}\sum_{j=1,\frac{j}{m}\notin\mathbb{N}}^{\infty}\frac{(-1)^{j-1}}{\Gamma(j)}\frac{\sin\left(\frac{\pi j}{2}+\frac{\theta}{\alpha}j\,\mathrm{sgn}x\right)}{\sin\frac{\pi j}{\alpha}}|x|^{j-1} \end{aligned}$$

$$+\frac{1}{\pi}\sum_{t=1}^{\infty}(-1)^{(m+n)t}\frac{\Gamma\prime(mt)}{\Gamma^2(mt)}\sin\left(\frac{\pi mt}{2}+\theta nt\,\mathrm{sgn}x\right)|x|^{mt-1}. \tag{2.8}$$

All the series in (2.8) can be represented by entire functions. The following theorem is an immediate corollary of Theorem 7.

***Theorem 8.***

*Under the conditions of Theorem 8,*

$$p_\alpha^\theta(\pm x)=\frac{1}{|x|}A_\pm\left(|x|^\alpha\right)+\frac{1}{\pi}\log\frac{1}{|x|}B_\pm\left(|x|^m\right)+C_\pm(|x|),\ x>0\,,$$

*where $A_\pm(z)$, $B_\pm(z)$, $C_\pm(z)$ are entire functions of finite order.*

Note that the term with $\log|x|$ in (2.8) vanishes identically if $\theta=\pi l/n$, for some integer $l$ and, moreover, $m$ is even and $n$ is odd.

The following theorem deals with the case of an irrational $\alpha$:

***Theorem 9.***

*If the number $\alpha\in(0,2)$ is not a rational number, then for $0\le\theta\le\min(\frac{\pi\alpha}{2},\pi-\frac{\pi\alpha}{2})$,*

$$p_\alpha^\theta(x)=\frac{1}{|x|}\lim_{s\to\infty}\left\{\sum_{k=1}^{s}\frac{(-1)^{k+1}}{\Gamma(k\alpha)}\frac{\sin\left(\frac{\pi k\alpha}{2}+k\theta\,\mathrm{sgn}x\right)}{\sin(\pi k\alpha)}|x|^{k\alpha}\right.$$
$$\left.+\frac{1}{\alpha}\sum_{1\le k<\alpha(s+\frac{1}{2})}\frac{(-1)^{k+1}}{\Gamma(k)}\frac{\sin\left(\frac{\pi k}{2}+\frac{k\theta}{\alpha}\mathrm{sgn}x\right)}{\sin\left(\frac{\pi k}{\alpha}\right)}|x|^{k}\right\}. \tag{2.9}$$

*The limit is uniform with respect to $x$ on any compact subset of $\mathbb{R}$.*

The following theorem, which is immediate from Theorem 7 and Theorem 9, deals with the "extremely" non-symmetric case. In the case $0<\alpha<1$, it was proved by Pillai [18].

***Theorem 10.***

*The following representations are valid:*
*(i) for $0<\alpha<1$, $\theta=\frac{\pi\alpha}{2}$,*

$$p_\alpha^\theta(x)=0,\ x<0;\quad p_\alpha^\theta(x)=\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{\Gamma(k\alpha)}|x|^{k\alpha-1},\ x>0\,,$$

*(ii) for $1\le\alpha<2$, $\theta=\pi-\frac{\pi\alpha}{2}$,*

$$p_\alpha^\theta(x)=\frac{e^{-x}}{\alpha},\ x>0;\quad p_\alpha^\theta(x)=\frac{e^{-x}}{\alpha}-\sum_{k=1}^{\infty}\frac{|x|^{k\alpha-1}}{\Gamma(k\alpha)},\ x<0\,.$$

*The representations above can also be written in the following form:*
*(i) for $0<\alpha<1$, $\theta=\frac{\pi\alpha}{2}$,*

$$p_\alpha^\theta(x)=-\frac{1+\mathrm{sgn}x}{2}\left(E_\alpha\left(-x^\alpha\right)\right)'\,,$$

*(ii) for $1\le\alpha<2$, $\theta=\pi-\frac{\pi\alpha}{2}$,*

$$p_\alpha^\theta(x)=\frac{e^{-x}}{\alpha}+\frac{1-\mathrm{sgn}x}{2}\left(E_\alpha\left(|x|^\alpha\right)\right)'\,,$$

*where $E_\alpha(z)$ is the Mittag–Leffler function defined as* $E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1+\alpha k)}$.

It is natural to ask whether the limits of each of the two sums in the right-hand side of (2.9) exist. We prove that this is the case for almost all $(\alpha, \theta) \in PD$ in the sense of planar Lebesgue measure. To describe this set we use Liouville numbers (see, e.g., [16]). We denote the set of all Liouville numbers by $L$. By the famous Liouville theorem ([16], p. 7), all numbers in $L$ are transcendental. Moreover ([16], p. 8), the set $L$ has Lebesgue measure zero.

***Theorem 11.***

*If* $(\alpha, \theta) \in \{(\alpha, \theta) \in PD : \alpha \notin L \bigcup \mathbb{Q}\}$, *then*

$$p_\alpha^\theta(x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\Gamma(k\alpha)} \frac{\sin\left(\frac{\pi k \alpha}{2} + k\theta \operatorname{sgn} x\right)}{\sin(\pi k \alpha)} |x|^{k\alpha - 1} + \frac{1}{\alpha} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\Gamma(k)} \frac{\sin\left(\frac{\pi k}{2} + \frac{k\theta}{\alpha} \operatorname{sgn} x\right)}{\sin\left(\frac{\pi k}{\alpha}\right)} |x|^{k-1} \tag{2.10}$$

*where both of the series converge absolutely and uniformly on any compact set.*

The following theorem is an immediate corollary of Theorem 11.

***Theorem 12.***

*If* $(\alpha, \theta) \in \{(\alpha, \theta) \in PD : \alpha \notin L \bigcup \mathbb{Q}\}$, *then the following representation holds for* $x > 0$

$$p_\alpha^\theta(\pm x) = \frac{1}{|x|} G_\pm\left(|x|^\alpha\right) + \frac{1}{\alpha} H_\pm(|x|)$$

*where $G_\pm(z)$, $H_\pm(z)$ are entire functions of finite order.*

Since the set $L \bigcup \mathbb{Q}$ has zero linear Lebesgue measure, the set $\{(\alpha, \theta) \in PD : \alpha \notin L \bigcup \mathbb{Q}\}$ is of full measure in $PD$. Thus, (2.10) is valid almost everywhere in $PD$. But, it turns out that the set where both of the series in the right-hand side of (2.10) diverge is non-empty. Moreover, this set is large in some sense.

***Theorem 13.***

*Both of the series in (2.10) diverge on a dense subset of $PD$ which has cardinality of the continuum.*

This theorem is a generalization of a theorem of Ostrovskii [15] related to the case $\theta = 0$ [when the role of $PD$ is played by the interval $(0, 2)$].

## 3. Integral Representation and Analytic Properties of Non-Symmetric Linnik's Probability Densities

**Proof of Theorem 1.** Case (i): $0 < \alpha < 1$, $0 \le \theta \le \pi\alpha/2$ and $1 \le \alpha < 2$, $0 \le \theta < \pi - \frac{\pi\alpha}{2}$.

We define $p_\alpha^\theta$ by (2.1). Evidently $p_\alpha^\theta$ is non-negative and $p_\alpha^\theta \in L^1(\mathbb{R})$; hence, it is enough to prove that its Fourier transform is equal to $\varphi_\alpha^\theta$.

Using Fubini's theorem, we derive

$$\int_{-\infty}^{\infty} e^{itx} p_\alpha^\theta(x) dx = \frac{1}{\pi} \operatorname{Im} \left\{ \int_0^\infty \frac{y dy}{\left(1 + e^{i\theta} y^\alpha e^{-i\frac{\pi\alpha}{2}}\right)\left(y^2 + t^2\right)} \right\}$$

$$+\frac{1}{\pi}\mathrm{Im}\left\{\int_0^\infty \frac{ydy}{\left(1+e^{-i\theta}y^\alpha e^{-i\frac{\pi\alpha}{2}}\right)(y^2+t^2)}\right\}$$

$$-\frac{it}{\pi}\mathrm{Im}\left\{\int_0^\infty \frac{dy}{\left(1+e^{i\theta}y^\alpha e^{-i\frac{\pi\alpha}{2}}\right)(y^2+t^2)}\right\}$$

$$+\frac{it}{\pi}\mathrm{Im}\left\{\int_0^\infty \frac{ydy}{\left(1+e^{-i\theta}y^\alpha e^{-i\frac{\pi\alpha}{2}}\right)(y^2+t^2)}\right\}$$

$$= \; : \frac{1}{\pi}\left[\mathrm{Im}A+\mathrm{Im}B-it\mathrm{Im}C+it\mathrm{Im}D\right] . \tag{3.1}$$

In the complex $y$-plane, we consider the region

$$G_R=\{y=\xi+i\eta : |y|<R, \eta>0\} \ , \ R>|t| \tag{3.2}$$

and define the branch of multivalued function $y^\alpha$ as

$$y^\alpha=|y|^\alpha e^{i\alpha \arg y} \ , \ 0\le \arg y\le \pi \ . \tag{3.3}$$

The integrands of A and C are analytic in the closure of $G_R$ except the simple pole at $y=i|t|$. By Cauchy's residue theorem, we have

$$\oint_{\partial G_R} \frac{y\ dy}{\left(1+e^{i\theta}y^\alpha e^{-i\frac{\pi\alpha}{2}}\right)(y^2+t^2)} = 2\pi i\mathrm{Res}_{i|t|} = \frac{\pi i}{1+e^{i\theta}|t|^\alpha} \ .$$

Letting $R\to\infty$ and using the notation $A$ and $B$, we obtain

$$\mathrm{Im}A+\mathrm{Im}B=\pi\mathrm{Re}\frac{1}{1+e^{i\theta}|t|^\alpha} \ . \tag{3.4}$$

We have in the similar way:

$$-\mathrm{Im}C+\mathrm{Im}D=-\frac{\pi}{|t|}\mathrm{Im}\frac{1}{1+e^{i\theta}|t|^\alpha} \ . \tag{3.5}$$

Substituting (3.4) and (3.5) into (3.1), we have

$$\int_{-\infty}^{\infty} e^{itx}\, p_\alpha^\theta(x)\,dx=\mathrm{Re}\,\frac{1}{1+e^{i\theta}\,|t|^\alpha}+i\,\mathrm{sgn}t\,\mathrm{Im}\,\frac{1}{1+e^{i\theta}\,|t|^\alpha}=\varphi_\alpha^\theta(t) \ .$$

Case (ii): $1\le\alpha<2$, $\theta=\pi-\frac{\pi\alpha}{2}$.

We define $p_\alpha^\theta$ by (2.2). Similarly, $p_\alpha^\theta$ is non-negative and $p_\alpha^\theta\in L^1(\mathbb{R})$. We prove that its Fourier transform coincides with $\varphi_\alpha^\theta$.

From (2.2) we have

$$\int_{-\infty}^{\infty} e^{itx}p_\alpha^\theta(x)dx = \frac{1}{\alpha(1-it)}+\frac{1}{\pi}\mathrm{Im}\left\{\int_0^\infty \frac{y\ dy}{\left(1-e^{-i\pi\alpha}y^\alpha\right)(y^2+t^2)}\right\}$$

$$-\frac{(it)}{\pi}\mathrm{Im}\left\{\int_0^\infty \frac{dy}{\left(1-e^{-i\pi\alpha}y^\alpha\right)(y^2+t^2)}\right\}$$

$$=: \frac{1}{\alpha(1-it)}+\frac{1}{\pi}\mathrm{Im}A-\frac{it}{\pi}\mathrm{Im}B \ . \tag{3.6}$$

Having defined the region $G_R$ by (3.2) and the branch of $y^\alpha$ by (3.3), we have

$$\begin{aligned}\text{v.p.} \oint_{\partial G_R} \frac{y\,dy}{\left(1-e^{-i\pi\alpha}y^\alpha\right)\left(y^2+t^2\right)} &= 2\pi i \mathrm{Res}_{i|t|} + \pi i \mathrm{Res}_{-1} \\ &= \frac{\pi i}{1-e^{-\frac{i\pi\alpha}{2}}|t|^\alpha} - \frac{\pi i}{\alpha\left(1+t^2\right)},\end{aligned}$$

whence, letting $R \to \infty$, we obtain

$$A - \text{v.p.} \int_0^\infty \frac{\xi\,d\xi}{(1-\xi^\alpha)\left(\xi^2+t^2\right)} = \frac{\pi i}{1-e^{-\frac{i\pi\alpha}{2}}|t|^\alpha} - \frac{\pi i}{\alpha\left(1+t^2\right)}. \tag{3.7}$$

Similarly, we obtain

$$B + \text{v.p.} \int_0^\infty \frac{d\xi}{(1-\xi^\alpha)\left(\xi^2+t^2\right)} = \frac{\pi}{\left(1-e^{-\frac{i\pi\alpha}{2}}|t|^\alpha\right)|t|} + \frac{\pi i}{\alpha\left(1+t^2\right)}. \tag{3.8}$$

From (3.7) and (3.8) we obtain

$$\begin{aligned}\frac{1}{\pi}\mathrm{Im}A - \frac{it}{\pi}\mathrm{Im}B &= \mathrm{Re}\frac{1}{1-e^{-\frac{i\pi\alpha}{2}}|t|^\alpha} - i\,\mathrm{sgn}t\,\mathrm{Im}\frac{1}{1-e^{-\frac{i\pi\alpha}{2}}|t|^\alpha} \\ &\quad - \frac{1}{\alpha\left(1+t^2\right)} - \frac{it}{\alpha\left(1+t^2\right)}.\end{aligned} \tag{3.9}$$

Substituting (3.9) into (3.6) we have

$$\int_{-\infty}^\infty e^{itx} p_\alpha^\theta(x)dx = \frac{1}{1-e^{\frac{i\pi\alpha}{2}\mathrm{sgn}t}|t|^\alpha} = \varphi_\alpha^\theta(t).$$

Case (iii): $-\pi < \theta < 0$;
From (1.3) it is evident that $\varphi_\alpha^\theta(t) = \varphi_\alpha^{-\theta}(-t)$. Hence, (2.3) follows from the Levy unicity theorem.
□

**Proof of Theorem 2.** (i) It is obvious that for any $|x| > 0$, $(\alpha,\theta) \in PD_0^+$ and for any $k = 0, 1, 2, 3\ldots$ the integral in formula (2.1) is $k$-times differentiable and

$$(-1)^k\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^k p_\alpha^\theta(x) = \frac{\sin\left(\frac{\pi\alpha}{2}+\theta\right)}{\pi}\int_0^\infty \frac{e^{-yx}y^{\alpha+k}dy}{1+2\cos\left(\frac{\pi\alpha}{2}+\theta\right)y^\alpha + y^{2\alpha}} > 0,\ x > 0, \tag{3.10}$$

$$\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^k p_\alpha^\theta(x) = \frac{\sin\left(\frac{\pi\alpha}{2}-\theta\right)}{\pi}\int_0^\infty \frac{e^{yx}y^{\alpha+k}dy}{1+2\cos\left(\frac{\pi\alpha}{2}-\theta\right)y^\alpha + y^{2\alpha}} > 0,\ x < 0. \tag{3.11}$$

Hence, $p_\alpha^\theta(x)$ is completely monotonic on $(0,\infty)$ and absolutely monotonic on $(-\infty,0)$ for $(\alpha,\theta) \in PD_0^+$. The proof is similar for $\theta = \min(\frac{\pi\alpha}{2}, \pi - \frac{\pi\alpha}{2})$.
(ii) By the monotonic convergence theorem, we have from (2.1)

$$\lim_{x\to 0\pm} p_\alpha^\theta(x) = \frac{\sin\left(\frac{\pi\alpha}{2}\pm\theta\right)}{\pi}\int_0^\infty \frac{y^\alpha dy}{\left|1+e^{i\frac{\pi\alpha}{2}\pm i\theta}y^\alpha\right|^2}.$$

Evidently, the integral in the right-hand side is divergent for $0 < \alpha \le 1$ and convergent for $1 < \alpha < 2$, and in the latter case we have (2.4). For the $(\alpha,\theta)$ located on the boundary of the $PD$, the proof is obvious.

(iii) For $0 \le \theta < \frac{\pi\alpha}{2}$, the proof is obvious by applying monotonic convergence theorem to (3.10) and (3.11). For $\theta = \pi\alpha/2$, it follows from (2.1) immediately. □

**Proof of Theorem 4.** (i) For $(\alpha, \theta) \in PD_0^+$, we have from (2.1), by applying Fubini's theorem

$$\begin{aligned}\int_0^\infty p_\alpha^\theta(x)dx &= \frac{\sin\left(\frac{\pi\alpha}{2}+\theta\right)}{\pi}\int_0^\infty \frac{y^{\alpha-1}\,dy}{\left|1+e^{i\theta+i\frac{\pi\alpha}{2}}y^\alpha\right|^2}\\ &= \frac{\sin\left(\frac{\pi\alpha}{2}+\theta\right)}{\pi\alpha}\int_0^\infty \frac{du}{1+2\cos\left(\frac{\pi\alpha}{2}+\theta\right)u+u^2} = \frac{1}{2}+\frac{\theta}{\pi\alpha}\,.\end{aligned}$$

For $\theta = \min(\frac{\pi\alpha}{2}, \pi - \frac{\pi\alpha}{2})$, the proof is evident from (2.1) and (2.2).
(ii) For the pairs $(\alpha, \theta) \in PD_0^+$, from (3.10) and (3.11) we have for $x > 0$

$$\begin{aligned}&\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^k\left((-1)^k\frac{p_\alpha^\theta(x)}{\sin\left(\frac{\pi\alpha}{2}+\theta\right)}-\frac{p_\alpha^\theta(-x)}{\sin\left(\frac{\pi\alpha}{2}-\theta\right)}\right)\\ &=\frac{1}{\pi}\int_0^\infty \frac{e^{-yx}4y^{2\alpha+k}\sin\frac{\pi\alpha}{2}\sin\theta\,dy}{\left|1+e^{i\theta+i\frac{\pi\alpha}{2}}y^\alpha\right|^2\left|1+e^{-i\theta+i\frac{\pi\alpha}{2}}y^\alpha\right|^2}\ge 0\,.\end{aligned}$$

For $\theta = \min(\frac{\pi\alpha}{2}, \pi - \frac{\pi\alpha}{2})$, the proof is evident from (2.1) and (2.2).
(iii) The positive assertion is an immediate corollary of (ii). Using Corollary 2 of Theorem 5, we conclude that $p_\alpha^\theta(x) < p_\alpha^\theta(-x)$ for $x$ being large enough if $(\alpha, \theta) \in PD^+$, $\alpha \in (1, 2)$, $\theta > 0$.
(iv) Consider formulas (3.10) and (3.11). For $0 \le \theta < \min(\frac{\pi\alpha}{2}, \frac{\pi}{2} - \frac{\pi\alpha}{2})$, both $\frac{\pi\alpha}{2} + \theta$ and $\frac{\pi\alpha}{2} - \theta$ are in between 0 and $\frac{\pi}{2}$. Thus, as $\theta$ increases $\sin(\frac{\pi\alpha}{2} + \theta)$ increases and $\cos(\frac{\pi\alpha}{2} + \theta)$ decreases; hence, $(-1)^k(\mathrm{d}/\mathrm{d}x)^k p_\alpha^\theta(x)$ increases for fixed $x > 0$. Similarly $(\mathrm{d}/\mathrm{d}x)^k p_\alpha^\theta(x)$ decreases for fixed $x < 0$.

For $\alpha \in (1, 2)$, $p_\alpha^\theta(x)$ is a continuous function of $x$ on $\mathbb{R}$ by Theorem 2 (ii). Moreover, for fixed $\alpha \in (1, 2)$, $p_\alpha^\theta(0)$ decreases as $\theta$ increases. Hence, $p_\alpha^\theta(x)$ cannot increase with $\theta$ for $x > 0$ small enough. □

Note that (3.11) yields that $p_\alpha^\theta(x)$ is decreasing with $\theta \in (0, \min(\frac{\pi\alpha}{2}, \frac{\pi}{2} - \frac{\pi\alpha}{2}))$ for any fixed $\alpha \in (1, 3/2)$ and $x < 0$. Corollary 3 of Theorem 5 shows that, for $\alpha \in (1, 2)$, $p_\alpha^\theta(x)$ is increasing with $\theta$ for fixed $x > 0$ being large enough. This justifies the graphs in Figure 1.

## 4. Representation by a Cauchy Type Integral

Consider the Cauchy type integral

$$f_\alpha(z) = \frac{1}{\pi}\int_0^\infty \frac{e^{-v^{1/\alpha}}\,v^{1/\alpha}\,dv}{v-z}, \quad 0 < \alpha < 2\,. \tag{4.1}$$

The function is analytic in the region $C := \{z : 0 < \arg z < 2\pi\}$. Evidently the function $e^{-v^{1/\alpha}}\,v^{1/\alpha}$ satisfies Lipschitz condition on any ray $[a, \infty)$, $a > 0$. Therefore, by the well-known properties of Cauchy type integrals (see, e.g., [8], p. 25), $f_\alpha(z)$ has boundary values $f_\alpha(x + i0)$ and $f_\alpha(x - i0)$ for any $x > 0$. Henceforth, it will be convenient to write $f_\alpha(x)$ instead of $f_\alpha(x + i0)$ for $x > 0$.

The following lemma is a generalization of Lemma 4.1 of [10], which can be obtained from ours by setting $\theta = 0$.

***Lemma 1.***

*For any $(\alpha,\theta) \in PD^+$, the following representation is valid:*

$$|x|^{1/\alpha}\, p_\alpha^\theta\left(\mathrm{sgn}x\,|x|^{1/\alpha}\right) = \frac{1}{\alpha}\,\mathrm{Im} f_\alpha\left(|x|\, e^{i(\pi-\frac{\pi\alpha}{2}-\theta\mathrm{sgn}x)}\right). \tag{4.2}$$

**Proof.** For all $(\alpha,\theta) \in PD^+$, except when $\alpha \in (1,2)$, $\theta = \pi - \frac{\pi\alpha}{2}$, $x > 0$, we have $|x|\, e^{i(\pi-\frac{\pi\alpha}{2}+\theta\mathrm{sgn}x)} \in C$. We first prove the result for these values of parameters $\alpha$, $\theta$, and $x$.

Replacing $\mathrm{sgn}x|x|^{1/\alpha}$ with $x$ in (2.1) and multiplying by $|x|^{1/\alpha}$, we obtain

$$\begin{aligned}
|x|^{1/\alpha}\, p_\alpha^\theta\left(\mathrm{sgn}x\,|x|^{1/\alpha}\right) &= \frac{\sin\left(\frac{\pi\alpha}{2}+\theta\mathrm{sgn}x\right)}{\pi}\int_0^\infty \frac{e^{-y|x|^{1/\alpha}}\, y^\alpha\, |x|^{1/\alpha}\, dy}{\left|1+e^{i\theta\mathrm{sgn}x}\, y^\alpha\, e^{i\frac{\pi\alpha}{2}}\right|^2} \\
&= \frac{\sin\left(\frac{\pi\alpha}{2}+\theta\mathrm{sgn}x\right)}{\pi\alpha}\int_0^\infty \frac{e^{-v^{1/\alpha}}\, v^{1/\alpha}\, |x|\, dv}{\left||x|+e^{i\theta\mathrm{sgn}x}\, v\, e^{i\frac{\pi\alpha}{2}}\right|^2} \\
&= \frac{1}{\alpha}\,\mathrm{Im}\frac{1}{\pi}\int_0^\infty \frac{e^{-v^{1/\alpha}}\, v^{1/\alpha}\, dv}{v+|x|\, e^{-i\theta\mathrm{sgn}x}\, e^{-i\frac{\pi\alpha}{2}}} \\
&= \frac{1}{\alpha}\,\mathrm{Im} f_\alpha\left(|x|\, e^{i(\pi-\frac{\pi\alpha}{2}-\theta\mathrm{sgn}x)}\right).
\end{aligned} \tag{4.3}$$

For the exceptional values of $(\alpha,\theta)$ and $x > 0$, we have from (2.2)

$$x^{1/\alpha}\, p_\alpha^\theta\left(-x^{1/\alpha}\right) = x^{1/\alpha}\,\frac{e^{-x^{1/\alpha}}}{\alpha}. \tag{4.4}$$

By the Plemelj–Sokhotski theorem ([8], p. 25), the following equality holds

$$f_\alpha(x+i0) - f_\alpha(x-i0) = 2ie^{-x^{1/\alpha}}\, x^{1/\alpha}. \tag{4.5}$$

Evidently, for any $x, y \in \mathbb{R}$, $y \neq 0$, $f_\alpha(x+iy)$ and $f_\alpha(x-iy)$ are complex conjugates. Hence, $f_\alpha(x)$ ($:= f_\alpha(x+i0)$) and $f_\alpha(x-i0)$ are also complex conjugates, and (4.5) can be rewritten in the form

$$\mathrm{Im}\, f_\alpha(x) = e^{-x^{1/\alpha}}\, x^{1/\alpha}.$$

Comparing with (4.4), we obtain

$$x^{1/\alpha}\, p_\alpha^\theta\left(-x^{1/\alpha}\right) = \frac{1}{\alpha}\,\mathrm{Im}\, f_\alpha(x).$$

which coincides with (4.2) in this case. □

## 5. Asymptotic Behavior at Infinity

**Proof of Theorem 5.** (i) As it was shown in [10], the function $f_\alpha(z)$ defined by (4.1) can be represented in the form

$$f_\alpha(z) = -\frac{\alpha}{\pi}\sum_{k=1}^{N}\frac{\Gamma(1+\alpha k)}{z^k} + f_{\alpha,N}(z), \tag{5.1}$$

where

$$\left|f_{\alpha,N}(z)\right| \le \frac{\alpha\Gamma(1+\alpha(N+1))}{\pi|z|^{N+1}|\sin(\phi)|}, \quad \phi = \arg z$$

for $N = 1, 2, 3, \ldots$. By Lemma 1, we obtain

$$|x|^{1/\alpha} p_\alpha^\theta \left(\mathrm{sgn}x\,|x|^{1/\alpha}\right) = \frac{1}{\pi}\sum_{k=1}^{N} \frac{\Gamma(1+\alpha k)}{|x|^k}(-1)^{k+1}\sin\left(\frac{\pi\alpha k}{2} + k\theta\mathrm{sgn}x\right) + \frac{1}{\alpha}\mathrm{Im} f_{\alpha,N}\left(|x|e^{i(\pi-\frac{\pi\alpha}{2}-\theta\mathrm{sgn}x)}\right)$$

where

$$\left|\mathrm{Im} f_{\alpha,N}\left(|x|e^{i(\pi-\frac{\pi\alpha}{2}-\theta\mathrm{sgn}x)}\right)\right| \le \frac{\alpha\Gamma(1+\alpha(N+1))}{\pi|x|^{N+1}\left|\sin\left(\frac{\pi\alpha}{2}+\theta\mathrm{sgn}x\right)\right|}.$$

Putting $|x|$ instead of $|x|^{1/\alpha}$, we obtain (2.5) and (2.6) for any $(\alpha, \theta) \in PD_0^+$.
(ii) Evidently, the above proof remains valid for both cases $\alpha \in (0, 1)$, $\theta = \pi\alpha/2$, $x > 0$ and $\alpha \in [1, 2)$, $\theta = \pi - \pi\alpha/2$, $x < 0$. In the remaining cases, we obtain the desired assertion from (2.1) and (2.2) immediately. □

## 6. Analytic Structure of $p_\alpha^\theta(x)$

Proof of the theorems concerning the analytic structure of $p_\alpha^\theta(x)$ for the rational values of $\alpha$ are based on the following facts about the analytic structure of the Cauchy type integral (4.1):

***Theorem 14.***
*In $C = \{z : 0 < \arg z < 2\pi\}$ the following representation is valid:*

$$f_{1/n}(z) = \frac{1}{n\pi}\sum_{k=0}^{n-1} z^k \Gamma\left(1 - \frac{k}{n}\right) + z^n A_n(z) + z^n B_n(z) . \tag{6.1}$$

*Here*

$$A_n(z) = \frac{1}{\pi} e^{-z^n}\left[\log\frac{1}{z} + \pi i\right], \tag{6.2}$$

*(the branch of the logarithm is defined by the condition $0 < \arg z < 2\pi$); $B_n(z)$ is an entire function representable by the power series*

$$B_n(z) = \sum_{k=0}^{\infty} \beta_k^{(n)} z^k$$

*where*

$$\beta_k^{(n)} = \begin{cases} \Gamma(-kn)/\pi n & , k/n \notin \mathbb{N} \\ \dfrac{(-1)^j}{\pi n}\dfrac{\Gamma'(1+j)}{\Gamma^2(1+j)} & , k/n = j\ , \ j = 0, 1, 2, \ldots . \end{cases} \tag{6.3}$$

***Theorem 15.***
*Assume $\alpha \in (0, 2)$ is represented in the form $\alpha = m/n$, where $m$, $n$ are relatively prime integers. The following formula is valid in $C = \{z : 0 < \arg z < 2\pi\}$;*

$$f_{m/n}(z) = \frac{m}{n\pi}\sum_{k=0}^{q}\Gamma\left(1 - \frac{km}{n}\right)z^k + \left[\log\frac{1}{|z|} + i(\pi - \arg z)\right]\sum_{s=1}^{\infty}\xi_{ms-r-1}^{(n)} z^{s+q}$$
$$- \pi \sum_{\substack{k=0 \\ k\notin\{ms-r-1\}_{s=1}^{\infty}}}^{\infty} \xi_k^{(n)} \frac{e^{-\frac{\pi i}{m}(k+r+1)}}{\sin\left(\frac{\pi}{m}(k+r+1)\right)} z^{\frac{k+r+1}{m}+q}$$
$$+ m\sum_{s=1}^{\infty}\beta_{ms-r-1}^{(n)} z^{s+q} \tag{6.4}$$

*where $q$ is the greatest integer strictly less then $n/m$, $r = n - qm - 1$ and*

$$\xi_k^{(n)} = \begin{cases} 0 & , k/n \notin \mathbb{N} \\ \dfrac{(-1)^j}{\pi j!} & , k/n = j\ ,\ j = 0, 1, 2, \dots \end{cases} \tag{6.5}$$

*$\beta_k^{(n)}$ was defined by (6.3).*

Theorem 14 is a combination of Lemma 6.1 and Lemma 7.1 of [10]. Theorem 15 is a combination of Lemma 10.1 and Lemma 11.1 of [10].

**Proof of Theorem 6.** From (4.2) and (6.1) we have

$$\begin{aligned} |x|^n p_{1/n}^{\theta}(\mathrm{sgn} x|x|^n) &= \frac{1}{\pi}\sum_{k=0}^{n-1}(-1)^{k+1}\Gamma\left(1-\frac{k}{n}\right)\sin\left(\frac{\pi k}{2n}+\theta k\,\mathrm{sgn} x\right)|x|^k \\ &\quad - n(-|x|)^n\cos(\theta n)\mathrm{Re}\left\{A_n\left(|x|e^{i(\pi-\frac{\pi}{2n}-\theta\,\mathrm{sgn} x)}\right)\right\} \\ &\quad - n(-|x|)^n\cos(\theta n)\mathrm{Re}\left\{B_n\left(|x|e^{i(\pi-\frac{\pi}{2n}-\theta\,\mathrm{sgn} x)}\right)\right\} \\ &\quad - n(-|x|)^n\sin(\theta n\,\mathrm{sgn} x)\mathrm{Im}\left\{A_n\left(|x|e^{i(\pi-\frac{\pi}{2n}-\theta\,\mathrm{sgn} x)}\right)\right\} \\ &\quad - n(-|x|)^n\sin(\theta n\,\mathrm{sgn} x)\mathrm{Im}\left\{B_n\left(|x|e^{i(\pi-\frac{\pi}{2n}-\theta\,\mathrm{sgn} x)}\right)\right\} \\ &=: \Sigma + R_A + R_B + I_A + I_B\ . \end{aligned} \tag{6.6}$$

Utilizing (6.2), we obtain

$$\begin{aligned} R_A + I_A &= \frac{n(-|x|)^n}{\pi}\exp\left((-|x|)^n\mathrm{sgn} x\sin(\theta n)\right)\log|x|\cos\left((-|x|)^n\cos(\theta n)-\theta n\,\mathrm{sgn} x\right) \\ &\quad - \frac{n(-|x|)^n}{\pi}\exp\left((-|x|)^n\mathrm{sgn} x\sin(\theta n)\right)\left(\theta\,\mathrm{sgn} x+\frac{\pi}{2n}\right) \\ &\quad \sin\left(|x|^n\cos(\theta n)+\theta n\,\mathrm{sgn} x\right)\ . \end{aligned} \tag{6.7}$$

Utilizing (6.3), we obtain

$$\begin{aligned} R_B + I_B &= \frac{1}{\pi}\sum_{k=1,\frac{k}{n}\notin\mathbb{N}}^{\infty}|x|^{k+n}(-1)^{k+n+1}\Gamma\left(-\frac{k}{n}\right)\sin\left((k+n)\left(\frac{\pi}{2n}+\theta\,\mathrm{sgn} x\right)\right) \\ &\quad + \frac{1}{\pi}\sum_{j=0}^{\infty}|x|^{nj+n}(-1)^{(j+1)(n+1)}\frac{\Gamma'(1+j)}{\Gamma^2(1+j)}\sin\left((j+1)\left(\frac{\pi}{2}+\theta n\,\mathrm{sgn} x\right)\right)\ . \end{aligned}$$

Putting $s = k + n$ in the first sum and substituting $j + 1$ for $j$ in the second one, we have

$$\begin{aligned} R_B + I_B &= \frac{1}{\pi}\sum_{s=n+1,\frac{s}{n}\notin\mathbb{N}}^{\infty}|x|^{s}(-1)^{s+1}\Gamma\left(1-\frac{s}{n}\right)\sin\left(\frac{\pi s}{2n}+\theta s\,\mathrm{sgn} x\right) \\ &\quad + \frac{1}{\pi}\sum_{j=0}^{\infty}|x|^{nj}(-1)^{j(n+1)}\frac{\Gamma'(j)}{\Gamma^2(j)}\sin\left(\frac{\pi j}{2}+\theta j n\,\mathrm{sgn} x\right)\ . \end{aligned} \tag{6.8}$$

Putting (6.7) and (6.8) into (6.6) and substituting $|x|$ for $|x|^n$, we obtain (2.7). □

**Proof of Theorem 7.** From (4.2) and (6.4) we obtain

$$
\begin{aligned}
&|x|^{n/m} p^{\theta}_{m/n}\left(\mathrm{sgn}x\,|x|^{n/m}\right)\\
&= \frac{1}{\pi}\sum_{k=0}^{q}\Gamma\left(1-\frac{km}{n}\right)(-1)^{k+1}\sin\left(\frac{\pi mk}{2n}+\theta k\,\mathrm{sgn}x\right)|x|^{k}\\
&+\frac{n}{m}\log\frac{1}{|x|}\sum_{s=1}^{\infty}\xi^{(n)}_{ms-r-1}(-1)^{s+q+1}\sin\left((s+q)\left(\frac{\pi m}{2n}+\theta\,\mathrm{sgn}x\right)\right)|x|^{s+q}\\
&+\left(\frac{\pi}{2}+\frac{n\theta\,\mathrm{sgn}x}{m}\right)\sum_{s=1}^{\infty}\xi^{(n)}_{ms-r-1}(-1)^{s+q}\cos\left((s+q)\left(\frac{\pi m}{2n}+\theta\,\mathrm{sgn}x\right)\right)|x|^{s+q}\\
&-\frac{\pi n}{m}\sum_{\substack{k=0\\ k\notin\{ms-r-1\}_{s=1}^{\infty}}}^{\infty}\xi^{(n)}_{k}\frac{\sin\left(\pi q-\left(\theta\,\mathrm{sgn}x+\frac{\pi m}{2n}\right)\left(\frac{k+r+1}{m}+q\right)\right)}{\sin\left(\frac{\pi}{m}(k+r+1)\right)}|x|^{\frac{k+r+1}{m}+q}\\
&+n\sum_{s=1}^{\infty}\beta^{(n)}_{ms-r-1}(-1)^{s+q+1}\sin\left((s+q)\left(\frac{\pi m}{2n}+\theta\,\mathrm{sgn}x\right)\right)|x|^{s+q}\\
&=: \Sigma_1+\frac{n}{m}\log\frac{1}{|x|}\Sigma_2+\left(\frac{\pi}{2}+\frac{n\theta\,\mathrm{sgn}x}{m}\right)\Sigma_3-\frac{n\pi}{m}\Sigma_4+n\Sigma_5\,, \qquad (6.9)
\end{aligned}
$$

say. Now we shall transform $\Sigma_2$, $\Sigma_3$, $\Sigma_4$, $\Sigma_5$ by substituting $\xi^{(n)}_k$, $\beta^{(n)}_k$.

The coefficients $\xi^{(n)}_k$ differ from zero only if $k/n$ is an integer; hence, $\xi^{(n)}_{ms-r-1}$ is nonzero iff $(ms-r-1)/n$ is an integer. Remembering the definition of $r$ we have

$$\frac{ms-r-1}{n}=\frac{m(s+q)}{n}-1\,.$$

Since $m$, $n$ are relatively prime, $(ms-r-1)/n$ is an integer iff $(s+q)/n$ is. Hence, $\xi^{(n)}_{ms-r-1}\neq 0$ iff $s\in\{nt-q\}_{t=1}^{\infty}$. When $s=nt-q$, using (6.5), we obtain

$$\xi^{(n)}_{ms-r-1}=\xi^{(n)}_{n(mt-1)}=\frac{(-1)^{mt-1}}{\pi(mt-1)!}\,,\ t=1,2,\dots\,.$$

Thus,

$$\Sigma_2=\frac{1}{\pi}\sum_{t=1}^{\infty}\frac{(-1)^{(m+n)t}}{(mt-1)!}\sin\left(\frac{\pi mt}{2}+\theta nt\,\mathrm{sgn}x\right)\Big)|x|^{nt}\,. \qquad (6.10)$$

Similarly,

$$\Sigma_3=\frac{1}{\pi}\sum_{t=1}^{\infty}\frac{(-1)^{(m+n)t-1}}{(mt-1)!}\cos\left(\frac{\pi mt}{2}+\theta nt\,\mathrm{sgn}x\right)\Big)|x|^{nt}\,. \qquad (6.11)$$

Substituting $r=n-qm-1$, we obtain

$$\Sigma_4=-\sum_{\substack{k=0\\ k\notin\{ms-r-1\}_{s=1}^{\infty}}}^{\infty}\xi^{(n)}_{k}\frac{\sin\left(\left(\frac{\pi m}{2n}+\theta\,\mathrm{sgn}x\right)\left(\frac{k+n}{m}\right)\right)}{\sin\left(\frac{\pi(k+n)}{m}\right)}|x|^{\frac{k+n}{m}}$$

This sum is taken over the values of $k$ such that $k\notin\{ms-r-1\}_{s=1}^{\infty}$, and the summand vanishes if $k/n$ is of the form $k=nj$, $j\in\mathbb{N}$. The relation $k\notin\{ms-r-1\}_{s=1}^{\infty}$ is equivalent to $nj\notin\{ms-r-1\}_{s=1}^{\infty}$ which is the same as $j\notin\{\frac{m(s+q)}{n}-1\}_{s=1}^{\infty}$. But the numbers $\frac{m(s+q)}{n}-1$ are integers iff $s=nt-q$

for some $t \in \mathbb{N}^+$; hence, the relation $nj \notin \{ms - r - 1\}_{s=1}^{\infty}$ is equivalent to $j \notin \{mt - 1\}_{s=1}^{\infty}$. Using (6.5), we can rewrite $\Sigma_4$

$$\Sigma_4 = - \sum_{\substack{j=0 \\ j \notin \{mt-1\}_{t=1}^{\infty}}}^{\infty} \frac{(-1)^j}{j!} \frac{\sin\left((j+1)\left(\frac{\pi}{2} + \frac{\theta n}{m}\mathrm{sgn}x\right)\right)}{\sin\left(\frac{\pi n}{m}(j+1)\right)} |x|^{\frac{n(j+1)}{m}} .$$

Substituting $j+1$ for $j$, we obtain

$$\Sigma_4 = - \sum_{j=1, \frac{j}{m} \notin \mathbb{N}}^{\infty} \frac{(-1)^{j-1}}{(j-1)!} \frac{\sin\left(\frac{\pi j}{2} + \frac{\theta n j}{m}\mathrm{sgn}x\right)}{\sin\left(\frac{\pi n j}{m}\right)} |x|^{\frac{(nj)}{m}} . \tag{6.12}$$

Using the same argument, we shall divide $\Sigma_5$ into two parts. The first summation is taken over the values of $s$ for which $(ms - r - 1)/n$ is an integer, i.e., $s = nt - q$ , $t = 1, 2, 3 \ldots$. The second summation is taken over the values of $s$ for which $(ms - r - 1)/n$ is non-integer, i.e., the values of $s$ for which $(s+q)/n \notin \mathbb{N}$. Remembering the formula for $\beta_k^{(n)}$, we can rewrite $\Sigma_5$ in the form:

$$\begin{aligned} \Sigma_5 &= \sum_{t=1}^{\infty} \frac{(-1)^{mt+nt}}{\pi n} \frac{\Gamma'(mt)}{\Gamma^2(mt)} \sin\left(\frac{\pi m t}{2} + \theta n t \mathrm{sgn}x\right) |x|^{nt} \\ &+ \sum_{\substack{s=1 \\ \frac{s+q}{n} \notin \mathbb{N}}}^{\infty} \frac{(-1)^{s+q+1}}{\pi n} \Gamma\left(1 - \frac{m(s+q)}{n}\right) \sin\left((s+q)\left(\frac{m\pi}{2n} + \theta \mathrm{sgn}x\right)\right) |x|^{s+q} . \end{aligned}$$

Putting $p = s + q$ in the second sum we have,

$$\begin{aligned} \Sigma_5 &= \sum_{t=1}^{\infty} \frac{(-1)^{mt+nt}}{\pi n} \frac{\Gamma'(mt)}{\Gamma^2(mt)} \sin\left(\frac{\pi m t}{2} + \theta n t \mathrm{sgn}x\right) |x|^{nt} \\ &+ \sum_{p=q+1, \frac{p}{n} \notin \mathbb{N}}^{\infty} \frac{(-1)^{p+1}}{\pi n} \Gamma\left(1 - \frac{mp}{n}\right) \sin\left(\frac{mp\pi}{2n} + \theta p \mathrm{sgn}x\right) |x|^p . \end{aligned} \tag{6.13}$$

Substituting (6.10), (6.11), (6.12), and (6.13) into (6.9) and using the well-known equality $\Gamma(1 - z)\Gamma(z) = \pi / \sin(\pi z)$, we obtain (2.8). □

## 7. Representation of $p_\alpha^\theta(x)$ by a Contour Integral

In this section we shall represent $p_\alpha^\theta(x)$ by a contour integral. This representation plays the key role in the transition from rational to irrational $\alpha$'s. For $\theta = 0$, this representation was obtained in [10].

Fix a positive $\delta < \frac{1}{2}$ and consider the integral

$$I_\delta(x; \alpha, \theta) = \frac{i}{2\alpha} \int_{L(\delta)} \frac{e^{z \log|x|} \sin\left(\frac{\pi z}{2} + \frac{z\theta}{\alpha}\mathrm{sgn}x\right) dz}{\Gamma(z) \sin\frac{\pi z}{\alpha} \sin \pi z} \tag{7.1}$$

where $L(\delta)$ is the boundary of the region $G(\delta) := \{z : |z| > \frac{1}{2}\delta , \ |\arg z| < \frac{\pi}{4}\}$. $L(\delta)$ is traversed so that $G(\delta)$ remains to the left.

***Theorem 16.***

*The following representation is valid for* $(\alpha, \theta) \in PD^+ \backslash \{(\alpha,\theta) : \theta = \pi - \pi\alpha/2\}$, $x > 0$, *and for* $(\alpha,\theta) \in PD^+ \backslash \{(\alpha,\theta) : \theta = \pi\alpha/2\}$, $x < 0$:

$$p_\alpha^\theta(x) = \frac{1}{|x|} I_\delta(x;\alpha,\theta) \tag{7.2}$$

*where* $\delta$ *is such that* $\alpha \in [\delta, 2-\delta]$.

This theorem is a generalization of Theorem 13.1 of [10], which can be obtained from ours by setting $\theta = 0$.

We first prove the following two lemmas.

***Lemma 2.***

*For any fixed* $0 \le \theta \le \min(\frac{\pi\alpha}{2}, \pi - \frac{\pi\alpha}{2})$, $0 < \delta < \frac{1}{2}$, $1 < M < \infty$, *the integral* $I_\delta(x;\alpha,\theta)$ *converges absolutely and uniformly with respect to both* $\alpha \in [\delta, 2-\delta]$ *and* $|x| < M$.

**Proof.** Note that $|\sin(\pi z/\alpha)| \ge \sinh(\pi|\mathrm{Im}z|/\alpha)$, $|\sin(\pi z)| \ge \sinh(\pi|\mathrm{Im}z|)$. Moreover, on the rays $\{z : |z| \ge \delta/2, \arg z = \mp\pi/4\}$, we have $|\mathrm{Im}z| \ge \delta/2\sqrt{2}$. Hence, $|\sin(\pi z/\alpha)|$, $|\sin(\pi z)|$ are bounded on $L(\delta)$ from below by a positive constant $C$ not depending on $\alpha \in [\delta, 2-\delta]$.

Using the Stirling formula ([19], p. 249), we see that there are positive constants $\varepsilon$ and $B$ such that $|\Gamma(z)| \ge B\exp(\varepsilon|z|\log|z|)$, $z \in G(\delta)$.

Noting that

$$\left|\sin\left(\frac{\pi z}{2} + \frac{z\theta}{\alpha}\mathrm{sgn}x\right)\right| \le e^{|\frac{\pi}{2}+\frac{\theta}{\alpha}\mathrm{sgn}x|.|\mathrm{Im}z|} \le e^{\pi|\mathrm{Im}z|},$$

we see that for $|x| < M$ the integrand in (7.1) can be estimated as follows:

$$\left|\frac{e^{z\log|x|}\sin\left(\frac{\pi z}{2} + \frac{z\theta}{\alpha}\mathrm{sgn}x\right)}{\Gamma(z)\,\sin\frac{\pi z}{\alpha}\,\sin\pi z}\right| \le \frac{\exp\left(\mathrm{Re}z\log|x| + \pi|\mathrm{Im}z|\right)}{Be^{\varepsilon|z|\log|z|}C^2}. \tag{7.3}$$

This gives the assertion of the lemma. □

***Lemma 3.***

$p_\alpha^\theta(x)$ *is a continuous function of*
*(i)* $\alpha$ *on* $[\frac{2\theta}{\pi}, 2 - \frac{2\theta}{\pi})$ *for any fixed* $\theta \in (0, \frac{\pi}{2})$ *and* $x > 0$,
*(ii)* $\alpha$ *on* $(\frac{2\theta}{\pi}, 2 - \frac{2\theta}{\pi}]$ *for any fixed* $\theta \in (0, \frac{\pi}{2})$ *and* $x < 0$,
*(iii)* $\alpha$ *on* $(0, 2)$ *for* $\theta = 0$, *and any fixed* $x \ne 0$.

**Proof.** Comparing (2.1) and (2.2), we see that formula (2.1) is valid for the intervals in the statement of the lemma.
(i) Take $0 < \delta < \theta$ and consider $\alpha \in [\frac{2\theta}{\pi}, 2 - \frac{2(\theta+\delta)}{\pi}]$ we have the following bound for the integrand on the right-hand side of Equation (2.1)

$$\frac{e^{-yx}y^\alpha}{\left|1 + e^{i\theta}y^\alpha e^{\frac{i\pi\alpha}{2}}\right|^2} \le \frac{e^{-yx}y^\alpha}{\left(\sin\left(\frac{\pi\alpha}{2} + \theta\right)\right)^2} \le \frac{e^{-yx}(1+y)^2}{(\sin\delta)^2}.$$

Therefore, the integral in (2.1) converges uniformly with respect to $\alpha \in [\frac{2\theta}{\pi}, 2 - \frac{2(\theta+\delta)}{\pi}]$ for fixed $\theta \in (0, \pi/2)$ and fixed $x > 0$. Hence, the integral is a continuous function of $\alpha$. The proof of (ii) is similar to the proof of (i).
(iii) This was proved in [10]. □

**Proof of Theorem 16.** We first prove the validity of the formula (7.2) for rational $\alpha$'s. Since the rational numbers are dense in $(0, 2)$, by the continuity of $p_\alpha^\theta(x)$ and of the integral $I_\delta(x;\alpha,\theta)$

as functions of $\alpha$, the theorem will be proved for the triples $(\alpha\,,\,\theta\,,\,x)$ as in the statement of the theorem.

Since $\alpha$ is rational, it has a unique representation $\alpha = m/n$ where $m$ and $n$ are relatively prime integers. The functions $\sin\frac{\pi z}{\alpha}$ , $\sin \pi z$ vanish on the set $\alpha\mathbb{Z}$ and $\mathbb{Z}$, respectively. The intersection of these sets is $m\mathbb{Z}$, and they are contained in the set $\frac{1}{n}\mathbb{Z}$. Taking a positive $\nu < 1/2n$, both functions are bounded from below on the set $\mathbb{C}\setminus\bigcup_{s=-\infty}^{\infty}\{z : |z - s/n| < \nu\}$ by a positive constant $C$.

Set $X_s = s/n + 1/2n\,,\ s = 1, 2, \ldots$ and consider the integral

$$I_\delta\,(x;\alpha,\theta,X_s) = \frac{i}{2\alpha}\int_{L(\delta,X_s)} \frac{e^{z\log|x|}\sin\left(\frac{\pi z}{2} + \frac{z\theta}{\alpha}\mathsf{sgn}x\right)\,dz}{\Gamma(z)\,\sin\frac{\pi z}{\alpha}\,\sin\pi z} \tag{7.4}$$

where $L(\delta, X_s)$ is the boundary of the region $G(\delta)\bigcap\{z : \mathsf{Re}z < X_s\}$.

By Cauchy's residue theorem we have

$$\begin{aligned} I_\delta\,(x;\alpha,\theta,X_s) &= -\frac{\pi}{\alpha}\sum_{\substack{\alpha\le k\alpha<X_s\\ k/n\notin\mathbb{N}}}(\text{Residue at } z = k\alpha)\\ &\quad -\frac{\pi}{\alpha}\sum_{\substack{1\le k<X_s\\ k/m\notin\mathbb{N}}}(\text{Residue at } z = k) - \frac{\pi}{\alpha}\sum_{1\le mt<X_s}(\text{Residue at } z = mt)\\ &=: -\frac{\pi}{\alpha}\left(\Sigma_1 + \Sigma_2 + \Sigma_3\right)\,. \end{aligned} \tag{7.5}$$

Calculating the residues, we have

$$\Sigma_1 = -\frac{\alpha}{\pi}\sum_{\substack{\alpha\le k\alpha<X_s\\ k/n\notin\mathbb{N}}}\frac{(-1)^{k+1}\sin\left(\frac{\pi\alpha k}{2} + k\theta\mathsf{sgn}x\right)|x|^{k\alpha}}{\Gamma(k\alpha)\sin\pi k\alpha}\,, \tag{7.6}$$

$$\Sigma_2 = -\frac{1}{\pi}\sum_{\substack{1\le k<X_s\\ k/m\notin\mathbb{N}}}\frac{(-1)^{k+1}\sin\left(\frac{\pi k}{2} + \frac{k\theta}{\alpha}\mathsf{sgn}x\right)|x|^{k-1}}{\Gamma(k)\sin\frac{\pi k}{\alpha}}\,. \tag{7.7}$$

To evaluate the residues at the points $z = mt\,,\ t = 1, 2, \ldots,$ put

$$f(z) := \frac{\sin\left(\frac{\pi z}{2} + \frac{z\theta}{\alpha}\mathsf{sgn}x\right)|x|^z}{\Gamma(z)}\,.$$

Evidently, $f(z)$ is analytic at $z = mt\,,\ t = 1, 2, \ldots,$ and we have

$$\begin{aligned} \mathsf{Res}_{mt} &= \lim_{z\to mt}\left[\frac{(z-mt)^2 f(z)}{\sin\frac{\pi z}{\alpha}\sin\pi z}\right]'\\ &= \frac{\alpha}{\pi^2}(-1)^{(m+n)t}\left(\frac{\pi}{2} + \frac{\theta}{\alpha}\mathsf{sgn}x\right)\frac{\cos\left(\frac{\pi mt}{2} + \theta nt\mathsf{sgn}x\right)|x|^{mt}}{\Gamma(mt)}\\ &\quad + \frac{\alpha}{\pi^2}(-1)^{(m+n)t}\log|x|\frac{\sin\left(\frac{\pi mt}{2} + \theta nt\mathsf{sgn}x\right)|x|^{mt}}{\Gamma(mt)}\\ &\quad - \frac{\alpha}{\pi^2}(-1)^{(m+n)t}\frac{\Gamma'(mt)}{\Gamma^2(mt)}\sin\left(\frac{\pi mt}{2} + \theta nt\mathsf{sgn}x\right)|x|^{mt}\,. \end{aligned}$$

Hence,

$$\Sigma_3 = \left(\frac{\alpha}{2\pi} + \frac{\theta}{\pi^2}\mathsf{sgn}x\right)\sum_{1\le mt<X_s}\frac{(-1)^{(m+n)t}\cos\left(\frac{\pi mt}{2} + \theta nt\mathsf{sgn}x\right)|x|^{mt}}{\Gamma(mt)}$$

$$+\frac{\alpha}{\pi^2}\log|x| \sum_{1\le mt<X_s} \frac{(-1)^{(m+n)t}\sin\left(\frac{\pi mt}{2}+\theta nt\,\mathrm{sgn}x\right)|x|^{mt}}{\Gamma(mt)}$$

$$-\frac{\alpha}{\pi^2}\sum_{1\le mt<X_s}\frac{(-1)^{(m+n)t}\Gamma'(mt)}{\Gamma^2(mt)}\sin\left(\frac{\pi mt}{2}+\theta nt\,\mathrm{sgn}x\right)|x|^{mt}. \quad (7.8)$$

Substituting (7.6), (7.7), and (7.8) into (7.5), we obtain

$$\begin{aligned} I_\delta(x;\alpha,\theta,X_s) &= \sum_{\substack{\alpha\le k\alpha<X_s\\ k/n\notin\mathbb{N}}} \frac{(-1)^{k+1}\sin\left(\frac{\pi k\alpha}{2}+k\theta\,\mathrm{sgn}x\right)}{\Gamma(k\alpha)\sin(\pi k\alpha)}|x|^{k\alpha} \\ &\quad+\frac{1}{\pi}\log\frac{1}{|x|}\sum_{1\le mt<X_s}\frac{(-1)^{(m+n)t}}{\Gamma(mt)}\sin\left(\frac{\pi mt}{2}+\theta nt\,\mathrm{sgn}x\right)|x|^{mt} \\ &\quad-\left(\frac{\theta\,\mathrm{sgn}x}{\pi\alpha}+\frac{1}{2}\right)\sum_{1\le mt<X_s}\frac{(-1)^{(m+n)t}}{\Gamma(mt)}\cos\left(\frac{\pi mt}{2}+\theta nt\,\mathrm{sgn}x\right)|x|^{mt} \\ &\quad+\frac{1}{\alpha}\sum_{\substack{1\le k<X_s\\ k/m\notin\mathbb{N}}}\frac{(-1)^{k+1}}{\Gamma(k)}\frac{\sin\left(\frac{\pi k}{2}+\frac{k\theta}{\alpha}\mathrm{sgn}x\right)}{\sin\frac{\pi k}{\alpha}}|x|^{k} \\ &\quad+\frac{1}{\pi}\sum_{1\le mt<X_s}(-1)^{(m+n)t}\frac{\Gamma'(mt)}{\Gamma^2(mt)}\sin\left(\frac{\pi mt}{2}+\theta nt\,\mathrm{sgn}x\right)|x|^{mt}. \end{aligned} \quad (7.9)$$

On the other hand, we have

$$I_\delta(x;\alpha,\theta,X_s)=\frac{i}{2\alpha}\left\{\int_{L(\delta,X_s)}+\int_{C(X_s)}\right\}\frac{e^{z\log|x|}\sin\left(\frac{\pi z}{2}+\frac{z\theta}{\alpha}\mathrm{sgn}x\right)dz}{\Gamma(z)\sin\frac{\pi z}{\alpha}\sin\pi z} \quad (7.10)$$

where

$$\begin{aligned} L(\delta,X_s) &= L(\delta)\bigcap\{z:\mathrm{Re}z<X_s\}, \\ C(X_s) &= \left\{z:\mathrm{Re}z=X_s,\ |\arg z|\le\frac{\pi}{4}\right\}. \end{aligned} \quad (7.11)$$

Using the bound (7.3), we have

$$\left|\frac{e^{z\log|x|}\sin\left(\frac{\pi z}{2}+\frac{z\theta}{\alpha}\mathrm{sgn}x\right)}{\Gamma(z)\sin\frac{\pi z}{\alpha}\sin\pi z}\right|\le\frac{\exp\left(X_s(\log|x|+\pi)\right)}{Be^{\varepsilon X_s\log X_s}C^2},\ z\in C(X_s) \quad (7.12)$$

Hence, the integral along $C(X_s)$ tends to zero as $s\to\infty$. Therefore,

$$\lim_{s\to\infty}I_\delta(x;\alpha,\theta,X_s)=I_\delta(x;\alpha,\theta)$$

Taking the limit in (7.9) and using (2.8), we obtain

$$I_\delta(x;\alpha,\theta)=\frac{1}{|x|}p_\alpha^\theta(x),\ \alpha=\frac{m}{n}. \qquad \square$$

## 8. The Case of Irrational $\alpha$

**Proof of Theorem 9.** We shall evaluate the integral $I_\delta(x;\alpha,\theta)$ by means of the Cauchy residue theorem and obtain (2.9) using Theorem 16.

Since $\alpha$ is irrational, the intersection of the sets $\alpha\mathbb{Z}$, $\mathbb{Z}$ is empty. We construct a sequence $\{Q_s\}_{s=1}^{\infty}$ which plays the role that $\{X_s\}_{s=1}^{\infty}$ did in the proof of Theorem 16. Since $\alpha \in (0, 2)$, each of the intervals $(s\alpha, (s+1)\alpha)$ contains none, one, or two points from $\{k\}_{k=1}^{\infty}$. In the first case, we define $Q_s = (s + \frac{1}{2})\alpha$. In the second and third cases, we choose $Q_s \in (s\alpha, (s+1)\alpha)$ so that the distance from $Q_s$ to the nearest of the three points $s\alpha$, $(s+1)\alpha$, $k \in (s\alpha, (s+1)\alpha)$ is at least $\alpha/4$.

Taking a positive $\nu < \alpha/4$, we observe that the modulus of the functions $\sin \frac{\pi z}{\alpha}$, $\sin \pi z$ are bounded from below by a positive constant $C$ on the set

$$\mathbb{C} \setminus \bigcup_{k=-\infty}^{\infty} \left[\{z : |z - k\alpha| < \nu\} \bigcup \{z : |z - k| < \nu\}\right].$$

The vertical lines $\{z : \mathrm{Re} z = Q_s\}$, $s = 1, 2, \ldots$, are located in the interior of this set.

Consider the integral $I_\delta(x; \alpha, \theta, Q_s)$ defined by (7.4) with $Q_s$ instead of $X_s$. Analogously to (7.9) and (7.10), we have

$$\begin{aligned} I_\delta(x; \alpha, \theta, Q_s) &= \sum_{\alpha \le k\alpha < Q_s} \frac{(-1)^{k+1} \sin\left(\frac{\pi k \alpha}{2} + k\theta \mathrm{sgn} x\right)}{\Gamma(k\alpha) \sin \pi k \alpha} |x|^{k\alpha} \\ &\quad + \frac{1}{\alpha} \sum_{0<k<Q_s} \frac{(-1)^{k+1} \sin\left(\frac{\pi k}{2} + \frac{k\theta}{\alpha} \mathrm{sgn} x\right)}{\Gamma(k) \sin \frac{\pi k}{\alpha}} |x|^k, \end{aligned} \tag{8.1}$$

$$I_\delta(x; \alpha, \theta, Q_s) = \frac{i}{2\alpha} \left\{ \int_{L(\delta, Q_s)} + \int_{C(Q_s)} \right\} \frac{e^{z \log |x|} \sin\left(\frac{\pi z}{2} + \frac{z\theta}{\alpha} \mathrm{sgn} x\right) dz}{\Gamma(z) \sin \frac{\pi z}{\alpha} \sin \pi z} \tag{8.2}$$

where $L(\delta, Q_s)$, $C(Q_s)$ are defined by (7.11) with $Q_s$ instead of $X_s$. Obviously, the inequality (7.12) is valid with $Q_s$ instead of $X_s$. The integral along $C(Q_s)$ tends to zero uniformly with respect to $x$ as $s \to \infty$. By Lemma 2, the integral along $L(\delta, Q_s)$ approaches the integral along $L(\delta)$ as $s \to \infty$ uniformly with respect to $x$ on any compact subset of $\mathbb{R}$. Taking the limits as $s \to \infty$ in (8.1) and (8.2), we arrive at the assertion of Theorem 9 except the cases $\alpha \in (1, 2) \setminus \mathbb{Q}$, $\theta = \pi - \frac{\pi\alpha}{2}$, $x > 0$ and $\alpha \in (0, 1) \setminus \mathbb{Q}$, $\theta = \frac{\pi\alpha}{2}$, $x < 0$. But by comparing the series expansion in (2.9) with (2.1) and (2.2) for these exceptional values of $(\alpha, \theta, x)$, we see that the series expansion in (2.9) is valid. □

**Proof of Theorem 11.** For any integer $k \ge 2$, there exists an integer $l_k$ such that

$$\left|\alpha - \frac{l_k}{k}\right| < \frac{1}{2k}. \tag{8.3}$$

Since $\alpha$ is not a Liouville number, there is an integer $r \ge 2$ such that for any pair of integers $p$, $q \ge 2$, $|\alpha - p/q| \ge 1/q^r$. Thus,

$$\left|\alpha - \frac{l_k}{k}\right| \ge k^{-r}. \tag{8.4}$$

From (8.3) and (8.4), we obtain $k^{1-r} \le |\alpha k - l_k| < 1/2$. Using the inequality

$$\sin x \ge \frac{2}{\pi} x, \quad 0 \le x \le \frac{\pi}{2}, \tag{8.5}$$

we obtain $|\sin \pi k\alpha| = |\sin \pi(k\alpha - l_k)| \ge 2|k\alpha - l_k| \ge 2k^{1-r}$. Hence, the first of the series in (2.9) converges absolutely and uniformly on any compact subset of $\mathbb{R}$.

Similarly, as above, for any integer $k \ge 2$, there exists an integer $l_k$ such that

$$\left|\frac{1}{\alpha} - \frac{l_k}{k}\right| < \frac{1}{2k}. \tag{8.6}$$

It follows that

$$\frac{l_k}{k} \le \frac{1}{\alpha} + \left|\frac{1}{\alpha} - \frac{l_k}{k}\right| < \frac{1}{\alpha} + \frac{1}{2k} < \frac{2}{\alpha}\,;$$

hence,

$$l_k \le \frac{2k}{\alpha}\,. \tag{8.7}$$

Since $\alpha$ is not a Liouville number, we have $|\alpha - k/l_k| \ge l_k^{-r}$. Multiplying the inequality by $l_k/\alpha$ and using (8.6) and (8.7), we obtain

$$\frac{1}{2} > \left|l_k - \frac{k}{\alpha}\right| \ge \frac{l_k^{1-r}}{\alpha} \ge \frac{1}{\alpha}\left(\frac{2}{\alpha}\right)^{1-r} k^{1-r}\,.$$

Hence, using (8.5), we obtain

$$\left|\sin\frac{\pi k}{\alpha}\right| = \left|\sin\pi\left(\frac{k}{\alpha} - l_k\right)\right| \ge 2\left|\frac{k}{\alpha} - l_k\right| \ge \left(\frac{2}{\alpha}\right)^{2-r} k^{1-r}\,.$$

Hence, the second of the series in (2.9) converges absolutely and uniformly on any compact subset of $\mathbb{R}$. □

**Proof of Theorem 13.** We shall construct a subset $D$ of $PD^+$ which (i) is dense in $PD^+$, (ii) has cardinality of the continuum, and (iii) is such that, for $(\alpha,\theta) \in D$, both of the series in (2.10) diverges.

Let $\{\sigma_n\}_{n=1}^\infty$ be a sequence of rapidly increasing integers defined by the equations

$$\sigma_1 = 2,\quad \sigma_{n+1} = 2^{3\sigma_n},\quad n = 1, 2, \ldots \tag{8.8}$$

Denote by $\Delta$ the set of all sequences $\{\delta_j\}_{j=1}^\infty$ with terms $\delta_j$ having values 0 or 1 only and satisfying the conditions: (i) $\delta_j$ is allowed to be equal to 1 if $j \in \{\sigma_n\}_{n=1}^\infty$ only; (ii) infinitely many of $\delta_j$'s are equal to 1.

Let $\Omega = \{y : y = \sum_{j=1}^\infty \delta_j 2^{-j}, \{\delta_j\}_{j=1}^\infty \in \Delta\}$. Let $\Lambda$ be the set of numbers in $(0, 2)$ representable by finite binary fractions. Set $E = \{\alpha \in (0,2) : \alpha = x + y, x \in \Lambda, y \in \Omega\}$. Evidently $E$ is dense in $(0, 2)$ and it has cardinality of the continuum. Set $D = \{(\alpha,\theta) \in PD : \alpha \in E, (\alpha + \frac{2\theta}{\pi}) \notin L \bigcup \mathbb{Q}\}$. It is easy to see that $D$ is dense in $PD$, and it has cardinality of the continuum.

It suffices to prove that for any $(\alpha,\theta) \in D$, the first of the series in (2.9) diverges.

If $\alpha \in E$, then there is an integer $m$ such that

$$\alpha = b + \sum_{j=1}^{m} a_j 2^{-j} + \sum_{j=m+1}^{\infty} \delta_j 2^{-j}$$

where $b$, $a_j$ take values 0 or 1, and $\{\delta_j\}_{j=1}^\infty \in \Delta$. Denote by $\{\eta_n\}_{n=1}^\infty$ the subsequence of $\{\sigma_n\}_{n=1}^\infty$ such that $\delta_j = 1$ for $j \in \{\eta_n\}_{n=1}^\infty$ and $\delta_j = 0$ for $j \notin \{\eta_n\}_{n=1}^\infty$. Then for any $\eta_n > m$, we have

$$0 < \alpha - \left(b + \sum_{j=1}^{m} a_j 2^{-j} + \sum_{j=m+1}^{\eta_n} \delta_j 2^{-j}\right) = \sum_{j=\eta_{n+1}}^{\infty} \delta_j 2^{-j} < 2^{-\eta_{n+1}+1}\,.$$

Multiplying this inequality by $2^{\eta_n}$, we see that there is an integer $p_n$ such that

$$0 < \alpha 2^{\eta_n} - p_n < 2^{\eta_n - \eta_{n+1} + 1} < 2^{-\frac{1}{2}\eta_{n+1}} \tag{8.9}$$

for sufficiently large $n$.

Consider the terms of the first of the series in (2.9) with indices $q = q_n = 2^{\eta_n}$. From (8.9) we obtain

$$|\sin \pi q_n \alpha| = |\sin (\pi q_n \alpha - \pi p_n)| < \pi 2^{-\frac{1}{2}\eta_{n+1}} .$$

Since $\alpha + \frac{2\theta}{\pi}$ is an irrational, non-Liouville number, as in the proof of the previous theorem, there is an integer $r \geq 2$ such that

$$\left|\sin \left(\frac{\pi q_n \alpha}{2} + q_n \theta\right)\right| = \left|\sin \frac{\pi 2^{\eta_n}}{2}\left(\alpha + \frac{2\theta}{\pi}\right)\right| \geq 2\left(2^{\eta_n - 1}\right)^{1-r} .$$

Hence, for sufficiently large $n$ we have

$$\left|\frac{(-1)^{q_n+1} \sin \left(\frac{\pi q_n \alpha}{2} + q_n \theta\right) |x|^{q_n \alpha}}{\Gamma\left(q_n \alpha\right) \sin \pi q_n \alpha}\right| \geq \frac{2}{\pi} 2^{(\eta_n - 1)(1-r)} |x|^{q_n \alpha} 2^{-q_n^2 \alpha^2} 2^{\frac{1}{2}\eta_{n+1}} . \tag{8.10}$$

Since $\{\eta_n\}_{n=1}^{\infty}$ is a subsequence of $\{\sigma_n\}_{n=1}^{\infty}$, the following inequality holds:

$$\eta_{n+1} \geq 2^{3\eta_n} = q_n^3 .$$

Hence, from (8.10) the series diverges. □

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### Note

This paper is a shortened version of [7]. Having completed [7] and having published its abstract in *IMS Bulletin,* 24(5), 500, 1995, I was kindly informed by T.J. Kozubowski about his paper (Representation and properties of geometric stable laws, in *Appr. Prob., Related Fields,* Anastassiou, G. and Rachev, S.T., Eds., Plenum Press, New York, 1994, 321–337). His paper contains Theorem 5 (i). Kozubowski's proof of these results is quite different from ours. It is based on the properties of stable densities. Moreover, his paper contains Theorem 3, but without the assertion about mode 0 for $1 < \alpha < 2$. Note that our proof of Theorem 3 is immediate, whereas Kozubowski's proof of his result is based on Yamazato's theorem on the unimodality of distributions of the class $L$.

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