Commun. Math. Phys. 85, 275–299 (1982)

Communications in Mathematical Physics

Boundary Regularity for the Navier-Stokes Equations in a Half-Space*

Vladimir Scheffer

Department of Mathematics, Rutgers University, New Brunswick, NJ 08903, USA

Abstract. Weak solutions to the nonstationary Navier-Stokes equations in a half-space are locally bounded at the boundary except for a closed set with finite one-dimensional Hausdorff measure.

1. Introduction

The purpose of this paper is to show that weak solutions u to the nonstationary Navier-Stokes equations in a half-space satisfy a regularity condition at the boundary. This regularity condition says that, except for a closed singular set whose one-dimensional Hausdorff measure is finite, u is locally bounded at the boundary of the half-space. The precise statement of this result is contained in Theorem 1.1 below.

In [1] it was proved that, at least in the case of a bounded domain, the interior singularities of the vorticity of u are concentrated in a locally closed set whose one-dimensional Hausdorff measure is finite. The vorticity of u can be replaced by u in the preceding statement. Theorem 1.1 extends that research to the boundary of the domain. It is interesting to note that the dimension does not jump up when we reach the boundary.

Our half-space will be $U=\{(x_1,x_2,x_3)\in R^3 : x_3>0\}$, its boundary will be denoted $B(U)$, and the set of positive times will be $R^+=\{t:t>0\}$. The weak solution u is a function which is defined on $U\times R^+$. It is convenient to extend u by zero, so that it becomes a function on $R^3\times R^+$. The spatial gradient of u (where we do not include the partial derivative with respect to time) will be written Du.

Theorem 1.1. *If $\bar{w}:R^3\to R^3$ is an L^2 function, $\bar{w}(x)=0$ when $x\notin U$, and* $\mathrm{div}(\bar{w})=0$ *then there exist $u:R^3\times R^+\to R^3$ and $S\subset B(U)\times[0,\infty)$ such that the following conditions hold:*

(1) *$u(x,t)=0$ when $x\notin U$; $Du\in L^2$.*

(2) *u is a weak solution to the nonstationary Navier-Stokes equations of incompressible fluid flow in U with viscosity $=1$ and initial condition $\bar{w}$.*

* This research was supported in part by the National Science Foundation Grant MCS-81-02737

0010-3616/82/0085/0275/$05.00

(3) *S is a closed set.*

(4) *The one-dimensional Hausdorff measure of $S \cap (R^3 \times \{t\})$ is a bounded function of $t \in R^+$.*

(5) *If $(a,b) \in B(U) \times R^+$ and $(a,b) \notin S$ then there exists $\varepsilon > 0$ such that u is bounded on the set $\{(x,t) : |x-a|^2 + |t-b|^2 < \varepsilon^2\}$.*

Condition (2) says that u satisfies properties (2), (3) of Theorem 1.2 of [1] (with w^0 replaced by $\bar{w}$). Condition (1) implies that u is zero on $B(U)$ (in a weak sense). This is the adherence condition at the boundary. The proof actually shows that the one-dimensional Hausdorff measure of $S \cap (R^3 \times \{t\})$ is at most $C\|\bar{w}\|_2^2$, where C is a constant.

Definition 1.2. Most of our notation is taken from [1]. In particular, we will use the notation $I(f,A)$, $M(f,A)$, $B(x,r)$, $K(x,t,r,s)$, $D(t)$ introduced in Definition 2.1 of [1]. If $B \subset R$, f is a function defined on $R^3 \times B$, and g is a function defined on R^3, then we set

$$(f*g)(x,t) = \int_{R^3} f(y,t)\, g(x-y)\, dy.$$

2. Solutions to Linearized Equations

Definition 2.1. Let X be the Hilbert space of all L^2 functions $f: R^3 \to R^3$ with the usual inner product $(f,g) = \int f_i(x)\, g_i(x)\, dx$. Let W be the closed linear subspace of X consisting of all $w \in X$ such that $w(x) = 0$ for almost every $x \notin U$ and $\operatorname{div}(w) = 0$ [so that $(w, \operatorname{grad}(g)) = 0$ for every $g \in C_0^\infty(R^3, R)$]. The orthogonal projection of X onto W will be called P. If $f: R^3 \to R^3$ is any function we define the reflection $f': R^3 \to R^3$ by means of the conditions

$$f_i'(x_1,x_2,x_3) = f_i(x_1,x_2,-x_3) \quad \text{if} \quad i \in \{1,2\}; \; f_3'(x_1,x_2,x_3) = -f_3(x_1,x_2,-x_3).$$

The function $\Gamma: R^3 \times R \to R$ is defined by

$$\Gamma(x,t) = 0 \quad \text{if} \quad t \leqq 0, \qquad \Gamma(x,t) = (4\pi t)^{-3/2} \exp(-|x|^2/(4t)) \quad \text{if} \quad t > 0.$$

We will also write $\Gamma_t(x) = \Gamma(x,t)$. The function $J: R^3 \sim \{0\} \to R$ is given by $J(x) = -(4\pi|x|)^{-1}$.

Definition 2.2. We fix, once and for all, a smoothing function $\theta \in C_0^\infty(R^3, R)$ such that $\theta(x) \geqq 0$, $\theta(x) = 0$ if $|x| \geqq 1$, $\theta(x) = \theta(-x)$, and $\|\theta\|_1 = 1$. If $\varepsilon > 0$ then $\theta_\varepsilon: R^3 \to R$ is defined by $\theta_\varepsilon(x) = \varepsilon^{-3}\theta(\varepsilon^{-1}x)$.

Lemma 2.3. *We have* $\operatorname{curl}(f') = -(\operatorname{curl}(f))'$ *and hence* $\operatorname{curl}(\operatorname{curl}(f')) = (\operatorname{curl}(\operatorname{curl}(f)))'$. *If* $\operatorname{div}(f) = 0$ *then* $\operatorname{div}(f') = 0$.

Proof. This is a straightforward computation.

Lemma 2.4. *Suppose $\varepsilon > 0$, $\delta > 0$, $r > 0$, $\beta \in \{1,2,3\}$, the function $f: R^3 \to R^3$ is defined by*

$$f_\beta(x_1,x_2,x_3) = \theta_\delta(x_1,x_2,x_3 - r), \; f_i(x) = 0 \quad \text{if} \quad i \neq \beta,$$

$g = -\operatorname{curl}(\operatorname{curl}(f*J))$, *and $F: R^3 \times [0,\infty) \to R^3$ is given by*

$$F(x,0) = g(x) + g'(x), \; F(x,t) = ((g+g')*\Gamma_t)(x) \quad \text{if} \quad t > 0.$$

Then $F(x,0)=P(f+f')(x)$ *for almost every* $x\in U$, $F_3(x_1,x_2,0,t)=0$, $(D_tF-\Delta F)(x,t)=0$ *if* $t>0$, $\operatorname{div}(F)=0$, *and the following inequalities are satisfied:*

$$|F(x_1,x_2,x_3,t)|\leqq C(|x_1|+|x_2|+|x_3-r|+t^{1/2})^{-3} \quad \text{if} \quad x_3\geqq 0,$$

$$|(F*\theta_\varepsilon)(x_1,x_2,x_3,t)|\leqq C(|x_1|+|x_2|+|x_3-r|+t^{1/2})^{-3} \quad \text{if} \quad x_3\geqq 0,$$

$$|DF(x_1,x_2,x_3,t)|\leqq C(|x_1|+|x_2|+|x_3-r|+t^{1/2})^{-4} \quad \text{if} \quad x_3\geqq 0.$$

Proof. Define $h:R^3\to R^3$ by $h(x)=(g+g')(x)$ when $x\in U$, $h(x)=0$ when $x\notin U$. Since g is a curl we conclude $\operatorname{div}(g)=0$. Now Lemma 2.3 yields $\operatorname{div}(g+g')=0$. Combining this with $(g_3+g_3')(x_1,x_2,0)=0$ we find

$$(h,\operatorname{grad}(p))=\int_U (g+g')_i(x)\,D_ip(x)\,dx=0$$

for any $p\in C_0^\infty(R^3,R)$. This implies $h\in W$. If $w\in W$ then Lemma 2.3 gives us

$$\begin{aligned}(h,w)&=\int_U (g_i+g_i')(x)\,w_i(x)\,dx=(g+g',w)\\&=(-(\operatorname{curl}(\operatorname{curl}((f+f')*J))),w)\\&=(-(\operatorname{curl}(\operatorname{curl}((f+f')*J)))+\operatorname{grad}(\operatorname{div}(f+f')*J),w)\\&=(\Delta((f+f')*J),w)=(f+f',w).\end{aligned}$$

Hence $f+f'-h$ is orthogonal to W. All this implies $h=P(f+f')$ and hence $F(x,0)=P(f+f')(x)$ for almost every $x\in U$. Since $F(x,t)=(g*\Gamma_t)(x)+(g*\Gamma_t)'(x)$ we obtain $F_3(x_1,x_2,0,t)=0$. The next two assertions follow from the fact $(D_t\Gamma-\Delta\Gamma)(x,t)=0$ when $t>0$, $\operatorname{div}(g)=0$, and Lemma 2.3. The function g satisfies

$$|(g*\Gamma_t)(x_1,x_2,x_3)|\leqq C(|(x_1,x_2,x_3-r)|+t^{1/2})^{-3},$$

$$|(g*\Gamma_t*\theta_\varepsilon)(x_1,x_2,x_3)|\leqq C(|(x_1,x_2,x_3-r)|+t^{1/2})^{-3},$$

$$|D(g*\Gamma_t)(x_1,x_2,x_3)|\leqq C(|(x_1,x_2,x_3-r)|+t^{1/2})^{-4}.$$

Hence g' satisfies

$$|(g'*\Gamma_t)(x_1,x_2,x_3)|\leqq C(|(x_1,x_2,-x_3-r)|+t^{1/2})^{-3},$$

$$|(g'*\Gamma_t*\theta_\varepsilon)(x_1,x_2,x_3)|\leqq C(|(x_1,x_2,-x_3-r)|+t^{1/2})^{-3},$$

$$|D(g'*\Gamma_t)(x_1,x_2,x_3)|\leqq C(|(x_1,x_2,-x_3-r)|+t^{1/2})^{-4}.$$

The three inequalities in the conclusion of the lemma follow from the above, the fact that $|(x_1,x_2,-x_3-r)|\geqq|(x_1,x_2,x_3-r)|$ when $x_3\geqq 0$, and $|x_1|+|x_2|+|x_3-r|\leqq C|(x_1,x_2,x_3-r)|$.

Lemma 2.5. *If* $0<d_1\leqq d_2$ *and* $(a_1,a_2)\in R^2$ *then*

$$\begin{aligned}&\int_{R^2} (|x_1-a_1|+|x_2-a_2|+d_1)^{-3}(|x_1|+|x_2|+d_2)^{-3}\,dx_1\,dx_2\\&\quad=\int_{R^2} (|x_1|+|x_2|+d_1)^{-3}(|x_1-a_1|+|x_2-a_2|+d_2)^{-3}\,dx_1\,dx_2\\&\quad\leqq C(|a_1|+|a_2|+d_2)^{-3}\,d_1^{-1}.\end{aligned}$$

Proof. This is straightforward.

Lemma 2.6. *Suppose* $\varepsilon>0$, $\delta>0$, $r>0$, $\beta\in\{1,2,3\}$ *and the function* f *is defined as in Lemma 2.4. Then there is a solution*

$$V:\mathrm{closure}(U)\times[0,\infty)\to R^3$$

to the linearized Navier-Stokes equations

$$\mathrm{div}(V)=0, D_tV-\Delta V \text{ is a spatial gradient, } V(x_1,x_2,0,t)=0 \quad \text{if} \quad t>0, \tag{2.1}$$

$$V(x,0)=P(f+f')(x) \text{ for almost every } x\in U$$

satisfying the inequalities

$$\begin{aligned} |V(x_1,x_2,x_3,t)|&\leqq C(|x_1|+|x_2|+|x_3-r|+t^{1/2})^{-3} \quad \text{if} \quad x_3>0, t>0,\\ |(V*\theta_\varepsilon)(x_1,x_2,x_3,t)|&\leqq C(|x_1|+|x_2|+|x_3-r|+t^{1/2})^{-3} \quad \text{if} \quad x_3\geqq 2\varepsilon, t>0 \end{aligned} \tag{2.2}$$

and

$$\int_0^s\int_U |DV(x,t)|\,dx\,dt\leqq Cs^{1/2} \quad \text{if} \quad s>0. \tag{2.3}$$

Proof. We adopt the terminology of Lemma 2.4. Solonnikoff [2, pp. 243, 248] proved that the system

$$\mathrm{div}(v)=0, D_tv-\Delta v \text{ is a spatial gradient},$$

$$v(x_1,x_2,0,t)=F(x_1,x_2,0,t) \quad \text{when} \quad t>0, v(x,0)=0$$

for a function v: $\mathrm{closure}(U)\times[0,\infty)\to R^3$ is solved by

$$v_i(a,b)=\sum_{j=1}^{2}\int_0^b\int_{R^2} F_j(x_1,x_2,0,t)\,G_{ij}(a_1-x_1,\, a_2-x_2,\, a_3,\, b-t)\,dx_1\,dx_2\,dt, \tag{2.4}$$

where $G_{ij}: U\times R^+\to R$ satisfies the inequalities

$$|G_{ij}(x,t)|\leqq C_\lambda t^{-1/2}(x_3t^{-1/2})^\lambda(|x|+t^{1/2})^{-3}, \quad \text{when} \quad 0\leqq\lambda\leqq 1, \tag{2.5}$$

$$|DG_{ij}(x,t)|\leqq Ct^{-1/2}(|x|+t^{1/2})^{-3}(x_3+t^{1/2})^{-1}, \tag{2.6}$$

when $i\in\{1,2,3\}$ and $j\in\{1,2\}$. Observe that j does not have to take the value 3 because Lemma 2.4 yields $F_3(x_1,x_2,0,t)=0$. The properties of v and Lemma 2.4 imply that the function V defined by

$$V(x,t)=F(x,t)-v(x,t) \quad \text{for} \quad x_3\geqq 0, t\geqq 0 \tag{2.7}$$

satisfies (2.1). Now we fix $a\in U$ and $b>0$. If t satisfies

$$0<t<b \quad \text{and} \quad r+t^{1/2}\leqq a_3+(b-t)^{1/2}, \tag{2.8}$$

then Lemma 2.4, (2.5) with $\lambda=0$, and Lemma 2.5 yield

$$\begin{aligned} &\left|\int_{R^2} F_j(x_1,x_2,0,t)\,G_{ij}(a_1-x_1,a_2-x_2,a_3,b-t)\,dx_1\,dx_2\right|\\ &\leqq \int_{R^2} C(|x_1|+|x_2|+r+t^{1/2})^{-3}(b-t)^{-1/2}\\ &\qquad (|a_1-x_1|+|a_2-x_2|+a_3+(b-t)^{1/2})^{-3}\,dx_1\,dx_2\\ &\leqq C(b-t)^{-1/2}(|a_1|+|a_2|+a_3+(b-t)^{1/2})^{-3}(r+t^{1/2})^{-1}. \end{aligned}$$

The assumption (2.8) implies

$$2(|a_1|+|a_2|+a_3+(b-t)^{1/2})\geqq|a_1|+|a_2|+a_3+(b-t)^{1/2}+r+t^{1/2}$$
$$\geqq|a|+(b-t)^{1/2}+t^{1/2}+r\geqq|a|+b^{1/2}+r. \tag{2.9}$$

Hence condition (2.8) gives us

$$\left|\int_{R^2} F_j(x_1,x_2,0,t)\,G_{ij}(a_1-x_1,a_2-x_2,a_3,b-t)\,dx_1\,dx_2\right|$$
$$\leqq C(b-t)^{-1/2}t^{-1/2}(|a|+b^{1/2}+r)^{-3}. \tag{2.10}$$

If, on the other hand, t satisfies

$$0<t<b \quad\text{and}\quad r+t^{1/2}>a_3+(b-t)^{1/2}, \tag{2.11}$$

then the same arguments with $\lambda=1/2$ yield

$$2(|a_1|+|a_2|+r+t^{1/2})>|a_1|+|a_2|+r+t^{1/2}+a_3+(b-t)^{1/2}$$
$$\geqq|a|+t^{1/2}+(b-t)^{1/2}+r\geqq|a|+b^{1/2}+r, \tag{2.12}$$

and hence

$$\left|\int_{R^2} F_j(x_1,x_2,0,t)\,G_{ij}(a_1-x_1,a_2-x_2,a_3,b-t)\,dx_1\,dx_2\right|$$
$$\leqq \int_{R^2} C(|x_1|+|x_2|+r+t^{1/2})^{-3}(b-t)^{-3/4}a_3^{1/2}$$
$$\cdot(|a_1-x_1|+|a_2-x_2|+a_3+(b-t)^{1/2})^{-3}\,dx_1\,dx_2$$
$$\leqq C(b-t)^{-3/4}a_3^{1/2}(|a_1|+|a_2|+r+t^{1/2})^{-3}(a_3+(b-t)^{1/2})^{-1}$$
$$\leqq C(b-t)^{-3/4}a_3^{1/2}(a_3+(b-t)^{1/2})^{-1}(|a|+b^{1/2}+r)^{-3}. \tag{2.13}$$

Breaking up the integral of (2.4) into the cases (2.8), (2.11), and using (2.10), (2.13) we obtain

$$|v(a,b)|\leqq C(|a|+b^{1/2}+r)^{-3}\int_0^b (b-t)^{-1/2}t^{-1/2}\,dt$$
$$+C(|a|+b^{1/2}+r)^{-3}\int_0^b (b-t)^{-3/4}a_3^{1/2}(a_3+(b-t)^{1/2})^{-1}\,dt$$
$$\leqq C(|a|+b^{1/2}+r)^{-3}.$$

When $a_3\geqq 2\varepsilon$, the above implies $|(v*\theta_\varepsilon)(a,b)|\leqq C(|a|+b^{1/2}+r)^{-3}$. Combining this with the estimate

$$(|a|+b^{1/2}+r)^{-3}\leqq C(|a_1|+|a_2|+b^{1/2}+(a_3+r))^{-3}$$
$$\leqq C(|a_1|+|a_2|+b^{1/2}+|a_3-r|)^{-3}$$

(which is true because $a_3>0$) and using Lemma 2.4 and (2.7) we obtain the two estimates in (2.2).

Again we fix $a\in U$ and $b>0$. If condition (2.8) holds, then Lemma 2.4, (2.6), Lemma 2.5, (2.9) [which is a consequence of (2.8)] and

$$2(a_3+(b-t)^{1/2})\geqq a_3+(b-t)^{1/2}+r+t^{1/2}\geqq a_3+b^{1/2}+r$$

yield

$$\left|\int_{R^2} F_j(x_1, x_2, 0, t)\, DG_{ij}(a_1 - x_1, a_2 - x_2, a_3, b-t)\, dx_1\, dx_2\right|$$
$$\leqq C(|a_1| + |a_2| + a_3 + (b-t)^{1/2})^{-3}(r + t^{1/2})^{-1}(b-t)^{-1/2}(a_3 + (b-t)^{1/2})^{-1}$$
$$\leqq C(|a| + b^{1/2} + r)^{-3}(r + t^{1/2})^{-1}(b-t)^{-1/2}(a_3 + b^{1/2} + r)^{-1}.$$

If condition (2.11) holds, then Lemma 2.4, (2.6), Lemma 2.5, and (2.12) [which is a consequence of (2.11)] yield

$$\left|\int_{R^2} F_j(x_1, x_2, 0, t)\, DG_{ij}(a_1 - x_1, a_2 - x_2, a_3, b-t)\, dx_1\, dx_2\right|$$
$$\leqq C(|a_1| + |a_2| + r + t^{1/2})^{-3}(a_3 + (b-t)^{1/2})^{-1}(b-t)^{-1/2}(a_3 + (b-t)^{1/2})^{-1}$$
$$\leqq C(|a| + b^{1/2} + r)^{-3}(a_3 + (b-t)^{1/2})^{-2}(b-t)^{-1/2}.$$

Once again, consideration of the two cases (2.8), (2.11) and use of (2.4) gives us

$$\begin{aligned} |Dv(a,b)| &\leqq C(|a| + b^{1/2} + r)^{-3}(a_3 + b^{1/2} + r)^{-1} \int_0^b t^{-1/2}(b-t)^{-1/2}\, dt \\ &\quad + C(|a| + b^{1/2} + r)^{-3} \int_0^b (a_3 + (b-t)^{1/2})^{-2}(b-t)^{-1/2}\, dt \\ &\leqq C a_3^{-1}(|a| + b^{1/2} + r)^{-3}. \end{aligned} \tag{2.14}$$

Using (2.2) we obtain $\int_U |V(x,b)|^2\, dx \leqq Cb^{-3/2}$ for $b>0$. Therefore the fundamental energy estimate for the Navier-Stokes equations yields

$$\int_b^\infty \int_U |DV(x,t)|^2\, dx\, dt \leqq Cb^{-3/2} \quad \text{for} \quad b>0. \tag{2.15}$$

Now we fix $b>0$ and define

$$S = \{(x_1, x_2, x_3) \in R^3 : 0 < x_3 < (|x_1| + |x_2| + b^{1/2})^{-3} b^2\},$$
$$T = \{(x_1, x_2, x_3) \in R^3 : (|x_1| + |x_2| + b^{1/2})^{-3} b^2 \leqq x_3 \leqq b^{1/2}\}.$$

Then (2.14) yields

$$\begin{aligned} \int_T |Dv(x,b)|\, dx &\leqq C \int_T x_3^{-1}(|x_1| + |x_2| + b^{1/2})^{-3}\, dx_1\, dx_2\, dx_3 \\ &= C \int_{R^2} \log_e[(|x_1| + |x_2| + b^{1/2})^3 b^{-3/2}](|x_1| + |x_2| + b^{1/2})^{-3}\, dx_1\, dx_2 \\ &\leqq C \int_{R^2} \log_e[(|(x_1,x_2)|\sqrt{2} + b^{1/2})^3 b^{-3/2}](|(x_1,x_2)|\sqrt{2} + b^{1/2})^{-3} \\ &\quad dx_1\, dx_2 \\ &= C \int_0^\infty \log_e[(s + b^{1/2})^3 b^{-3/2}](s + b^{1/2})^{-3} s\, ds \leqq Cb^{-1/2}. \end{aligned}$$

In addition, (2.14) yields

$$\int_{U\sim(T\cup S)} |Dv(x,b)|\,dx$$

$$\leqq \int_{b^{1/2}}^{\infty}\int_{R^2} Cx_3^{-1}(|x_1|+|x_2|+x_3+b^{1/2})^{-3}\,dx_1\,dx_2\,dx_3$$

$$\leqq C\int_{b^{1/2}}^{\infty} x_3^{-1}(x_3+b^{1/2})^{-1}\,dx_3 \leqq Cb^{-1/2}.$$

Using the above, we find $\int_{U\sim S} |Dv(x,b)|\,dx \leqq Cb^{-1/2}$. Since $\int_U |DF(x,b)|\,dx \leqq Cb^{-1/2}$ is a consequence of Lemma 2.4, we can use (2.7) to conclude

$$\int_{U\sim S} |DV(x,b)|\,dx \leqq Cb^{-1/2}. \tag{2.16}$$

We also have

$$\int_S |DV(x,b)|\,dx \leqq [\text{volume}(S)]^{1/2}\left(\int_S |DV(x,b)|^2\,dx\right)^{1/2}$$

$$\leqq Cb^{3/4}\left(\int_U |DV(x,b)|^2\,dx\right)^{1/2}.$$

Combining this with (2.15), (2.16) we find

$$\int_0^t\int_U |DV(x,b)|\,dx\,db \leqq \int_0^t Cb^{-1/2}\,db + \int_0^t Cb^{3/4}\left(\int_U |DV(x,b)|^2\,dx\right)^{1/2} db$$

$$\leqq Ct^{1/2} + \sum_{i=0}^{\infty} C(t2^{-i})^{3/4} \int_{t2^{-i-1}}^{t2^{-i}} \left(\int_U |DV(x,\,b)|^2\,dx\right)^{1/2} db$$

$$\leqq Ct^{1/2} + \sum_{i=0}^{\infty} C(t2^{-i})^{3/4}$$

$$\cdot\left(\int_{t2^{-i-1}}^{t2^{-i}}\int_U |DV(x,b)|^2\,dx\,db\right)^{1/2}(t2^{-i})^{1/2} \leqq Ct^{1/2}.$$

This concludes the proof of the lemma.

3. An Approximate Solution

In this section we fix a positive number ε.

Definition 3.1. If w is a function from an open subset of R into X and $(w(s+h)-w(s))/h$ converges in L^2 as h approaches zero, then the limit will be denoted $D_t w(s)$. If f is a function with domain R^3 then $\tilde{f}$ will be given by $\tilde{f}(x_1,x_2,x_3)=f(x_1,x_2,x_3-4\varepsilon)$. When the domain of f is a subset of $R^3\times R$ then we will also write $\tilde{f}(x_1,x_2,x_3,t)=f(x_1,x_2,x_3-4\varepsilon,t)$.

For every $\alpha>0$ we can use a slight modification of the construction in [1, pp. 20, 21] (with Ψ replaced by θ_ε and Ω replaced by θ_α) to find a continuous function $w^\alpha:[0,\infty)\to W$ (where W has the norm topology) such that, setting $u^\alpha(x,t)=(w^\alpha(t))(x)$, we obtain $u^\alpha(x,0)=\bar{w}(x)$,

$$\int_{R^3} |u^\alpha(x,t)|^2\,dx \leqq \|\bar{w}\|_2^2, \tag{3.1}$$

$$\int_0^\infty \int_{R^3} |D(u^\alpha*\theta_\alpha)(x,t)|^2\,dx\,dt \leqq (1/2)\,\|\bar{w}\|_2^2, \tag{3.2}$$

$D_t w^\alpha$ is a norm continuous function on R^+, and

$$D_t w^\alpha(s)=P(-((\tilde{w}_j^\alpha(s)*\theta_\varepsilon)\,D_j(w^\alpha(s)*\theta_\varepsilon))*\theta_\varepsilon+\Delta(w^\alpha(s)*\theta_\alpha*\theta_\alpha)). \tag{3.3}$$

Let $\alpha_1,\alpha_2,\alpha_3,\dots$ be a sequence of positive numbers converging to zero. If $\alpha=\alpha_k$ then u^α will be denoted v^k for the sake of typographical simplicity. Using (3.1) and the Cantor diagonal process we can pass to a subsequence and assume that there exists a measurable function $u:R^3\times R^+\to R^3$ such that v^k converges weakly in L^2 to u when the domain is restricted to a set of the form $R^3\times(0,b)$, $0<b<\infty$. In view of (3.2), we can also assume that $D(v^k*\theta_{\alpha_k})$ converges weakly in L^2. Taking the inner product with a test function, we find that this weak limit coincides with the distribution Du. This proves the next lemma:

Lemma 3.2. *The distribution Du is an L^2 function and $D(v^k*\theta_{\alpha_k})$ converges weakly to it.*

Lemma 3.3. *By passing to a subsequence, we may assume $\lim_{k\to\infty}(v^k*\theta_\varepsilon)(x,t)=(u*\theta_\varepsilon)(x,t)$ for almost every (x,t). In addition, one can modify $u*\theta_\varepsilon$ on a set of measure zero so that it becomes a continuous function on $R^3\times[0,\infty)$.*

Proof. For any $\alpha>0$, (3.1) yields

$$\begin{aligned}&\|((\tilde{w}_j^\alpha(s)*\theta_\varepsilon)\,D_j(w^\alpha(s)*\theta_\varepsilon))*\theta_\varepsilon\|_2\\ &\leqq \|(\tilde{w}_j^\alpha(s)*\theta_\varepsilon)(w^\alpha(s)*D_j\theta_\varepsilon)\|_2\,\|\theta_\varepsilon\|_1 \leqq \|\tilde{w}^\alpha(s)*\theta_\varepsilon\|_2\,\|w^\alpha(s)*D\theta_\varepsilon\|_\infty\\ &\leqq \|\tilde{w}^\alpha(s)\|_2\,\|\theta_\varepsilon\|_1\,\|w^\alpha(s)\|_2\,\|D\theta_\varepsilon\|_2 \leqq \bar{w}\|_2^2\|D\theta_\varepsilon\|_2.\end{aligned} \tag{3.4}$$

Let $a\in R^3$, $\beta\in\{1,2,3\}$ and define $f:R^3\to R^3$ by $f_\beta(x)=\theta_\varepsilon(x-a)$, $f_i(x)=0$ if $i\neq\beta$. Let $g=P(f)$. For any $\delta>0$ we let $g^\delta(x)=(g*\theta_\delta)(x-(0,0,\delta))$. If $0<s_1<s_2$ and $\alpha=\alpha_k$ for some k, then the fact $w^\alpha(s)\in W$, (3.1), (3.3), the fact $(P(h),g^\delta)=(h,g^\delta)$ (which follows from $g^\delta\in W$), (3.4), the estimate $\|w^\alpha(s)*\theta_\alpha*\theta_\alpha\|_2\leqq\|w^\alpha(s)\|_2\,\|\theta_\alpha\|_1\,\|\theta_\alpha\|_1=\|w^\alpha(s)\|_2$, and (3.1) yield

$$\begin{aligned}&|(v_\beta^k*\theta_\varepsilon)(a,s_2)-(v_\beta^k*\theta_\varepsilon)(a,s_1)|\\ &=\left|\int_{R^3} w_i^\alpha(s_2)(x)\,f_i(x)\,dx-\int_{R^3} w_i^\alpha(s_1)(x)\,f_i(x)\,dx\right|\\ &=\left|\int_{R^3} w_i^\alpha(s_2)(x)\,P(f)_i(x)\,dx-\int_{R^3} w_i^\alpha(s_1)(x)\,P(f)_i(x)\,dx\right|\end{aligned}$$

$$\leqq \left| \int_{R^3} w_i^\alpha(s_2)(x)\, g_i^\delta(x)\, dx - \int_{R^3} w_i^\alpha(s_1)(x)\, g_i^\delta(x)\, dx \right|$$
$$+ \|w^\alpha(s_2)\|_2 \, \|Pf - g^\delta\|_2 + \|w^\alpha(s_1)\|_2 \, \|Pf - g^\delta\|_2$$
$$\leqq \left| \int_{s_1}^{s_2} \int_{R^3} D_t w_i^\alpha(s)(x)\, g_i^\delta(x)\, dx\, ds \right| + 2\|\bar{w}\|_2 \, \|Pf - g^\delta\|_2$$
$$\leqq \int_{s_1}^{s_2} \int_{R^3} |(((\bar{w}_j^\alpha(s) * \theta_\varepsilon)\, D_j(w_i^\alpha(s) * \theta_\varepsilon)) * \theta_\varepsilon)(x)\, g_i^\delta(x)|\, dx\, ds$$
$$+ \int_{s_1}^{s_2} \left| \int_{R^3} \Delta(w_i^\alpha(s) * \theta_\alpha * \theta_\alpha)(x)\, g_i^\delta(x)\, dx \right| ds + 2\|\bar{w}\|_2 \, \|Pf - g^\delta\|_2$$
$$\leqq \int_{s_1}^{s_2} \|\bar{w}\|_2^2 \, \|D\theta_\varepsilon\|_2 \, \|g^\delta\|_2 \, ds + \int_{s_1}^{s_2} \|w^\alpha(s) * \theta_\alpha * \theta_\alpha\|_2 \, \|\Delta g^\delta\|_2 \, ds + 2\|\bar{w}\|_2 \, \|Pf - g^\delta\|_2$$
$$\leqq (s_2 - s_1)\, \|\bar{w}\|_2^2 \, \|D\theta_\varepsilon\|_2 \, \|g^\delta\|_2 + (s_2 - s_1)\, \|\bar{w}\|_2 \, \|\Delta g^\delta\|_2 + 2\|\bar{w}\|_2 \, \|Pf - g^\delta\|_2 . \quad (3.5)$$

If $\gamma > 0$ then $2\|\bar{w}\|_2 \, \|Pf - g^\delta\|_2$ can be made smaller than γ by choosing δ sufficiently small. Then the other two terms at the end of (3.5) can be made less than γ by choosing $s_2 - s_1$ to be small enough. This shows that $(v^k * \theta_\varepsilon)(a, s)$ gives us a uniformly equicontinuous family of functions of the variable s for every fixed point a. If $\{a_1, a_2\} \subset R^3$ then (3.1) yields

$$\begin{aligned} &|(v^k * \theta_\varepsilon)(a_1, s) - (v^k * \theta_\varepsilon)(a_2, s)| \\ &= \left| \int_{R^3} v^k(x, s)\, (\theta_\varepsilon(a_1 - x) - \theta_\varepsilon(a_2 - x))\, dx \right| \\ &\leqq \|\bar{w}\|_2 \left(\int_{R^3} |\theta_\varepsilon(a_1 - x) - \theta_\varepsilon(a_2 - x)|^2\, dx \right)^{1/2} \\ &= \|\bar{w}\|_2 \left(\int_{R^3} |\theta_\varepsilon(a_1 - a_2 + x) - \theta_\varepsilon(x)|^2\, dx \right)^{1/2} . \end{aligned} \quad (3.6)$$

Since the last line of (3.6) approaches zero as $a_1 - a_2$ goes to zero, we use the previous result to conclude that $v^k * \theta_\varepsilon$ is an equicontinuous family of functions when restricted to $K \times [0, \infty)$ for every compact set $K \subset R^3$. Using (3.1) we get $|(v^k * \theta_\varepsilon)(x, s)| \leqq \|\bar{w}\|_2 \, \|\theta_\varepsilon\|_2$. Now Ascoli's theorem implies that, passing to a subsequence, we may assume that $v^k * \theta_\varepsilon$ converges uniformly on compact sets to a continuous function. Since $v^k * \theta_\varepsilon$ converges weakly to $u * \theta_\varepsilon$ on every subdomain $R^3 \times (0, b)$, the pointwise limit of $v^k * \theta_\varepsilon$ must coincide with $u * \theta_\varepsilon$ almost everywhere. The proof of the lemma is now complete.

In view of this lemma, we may assume that $u * \theta_\varepsilon$ is a continuous function. We also have [from $w^\alpha(s) \in W$]

$$\operatorname{div}(u) = 0, u(x, t) = 0 \quad \text{if} \quad x \notin U . \quad (3.7)$$

For the next argument we will need to convolve with respect to the time variable. If the domain of f is a subset of $R^3 \times R$ and the domain of g is R then $(f * g)(x, t) = \int_R f(x, s)\, g(t - s)\, ds$ when the integral makes sense. We use the same definition when f is defined only on $R^3 \times R^+$ but $g(t - s) = 0$ whenever $s \leqq 0$. Then the values of f outside $R^3 \times R^+$ are irrelevant.

Lemma 3.4. *Suppose* $a \in U$, $0<b'<b$, $0<\delta<a_3$, $\beta \in \{1,2,3\}$, $\Upsilon \in C_0^\infty(R,R)$, *support* $(\Upsilon) \subset [-b',b']$, $\Upsilon(t)=\Upsilon(-t)$, *and* $\eta>0$. *Then*

$$\begin{aligned}
&|(u_\beta * \Upsilon * \theta_\delta)(a,b)| \\
&\leqq \int_{R^3} C|(u*\Upsilon)(x,b')|\,(|x-a|+(b-b')^{1/2})^{-3}\,dx \\
&\quad + (C\eta)\max\{|(((\tilde{u}_j*\theta_\varepsilon)(u*\theta_\varepsilon))*\Upsilon)(x,s)| : (x,s)\in K(a,b,\eta,\eta^2),\ j=1,2,3\} \\
&\quad + \int_{b'}^{b}\int_{R^3} C|(((\tilde{u}_j*\theta_\varepsilon)(D_j u*\theta_\varepsilon))*\Upsilon)(x,s)|\,(|x-a|+(b-s)^{1/2}+\eta)^{-3}\,dx\,ds.
\end{aligned}$$

Proof. Set $r=a_3$. We let V, f be the functions of Lemma 2.6 corresponding to our choices of δ, r, β. The function $\phi: R^3\times R \to R^3$ is defined by

$$\phi(x,t)=V(x_1-a_1, x_2-a_2, x_3, b-t) \quad \text{if} \quad x\in U \quad \text{and} \quad b'\leqq t \leqq b,$$

and $\phi(x,t)=0$ otherwise. The restriction of $u*\Upsilon$ to $R^3\times[b',b]$ has these two properties: It is zero outside $U\times[b',b]$ and its spatial gradient is in L^2 [see (3.7) and Lemma 3.2]. Hence we can say $(u*\Upsilon)(x_1,x_2,0,s)=0$ for a.e. x_1, x_2, s with $b'\leqq s\leqq b$. Lemma 2.6 implies that the restriction of $D_t\phi+\Delta\phi$ to $U\times(b',b)$ is a spatial gradient, and (3.7) yields $\operatorname{div}(u*\Upsilon)=0$. All this implies

$$\begin{aligned}
&\int_{b'}^{b}\int_U D_t(u_i*\Upsilon)(x,s)\,\phi_i(x,s)\,dx\,ds \\
&\qquad + \int_{b'}^{b}\int_U D_j(u_i*\Upsilon)(x,s)\,D_j\phi_i(x,s)\,dx\,ds \\
&\quad = \int_U (u_i*\Upsilon)(x,b)\,\phi_i(x,b)\,dx - \int_U (u_i*\Upsilon)(x,b')\,\phi_i(x,b')\,dx. \qquad (3.8)
\end{aligned}$$

Let k be a positive integer and set $\alpha=\alpha_k$. For each s, the function $g(x)=(\phi*\Upsilon)(x,s)$ is an element of W (see Lemma 2.6). Using this fact, (3.3), and $\operatorname{div}(w^\alpha(s))=0$ we find

$$\begin{aligned}
&\int_{b'}^{b}\int_{R^3} D_t(v_i^k*\Upsilon)(x,s)\,\phi_i(x,s)\,dx\,ds \\
&\quad = \int_0^\infty\int_{R^3} D_t v_i^k(x,s)\,(\phi_i*\Upsilon)(x,s)\,dx\,ds \\
&\quad = \int_0^\infty\int_{R^3} D_t w_i^\alpha(s)(x)\,(\phi_i*\Upsilon)(x,s)\,dx\,ds \\
&\quad = -\int_0^\infty\int_{R^3} (((\tilde{w}_j^\alpha(s)*\theta_\varepsilon)\,D_j(w_i^\alpha(s)*\theta_\varepsilon))*\theta_\varepsilon)(x)\,(\phi_i*\Upsilon)(x,s)\,dx\,ds \\
&\qquad + \int_0^\infty\int_{R^3} \Delta(w_i^\alpha(s)*\theta_\alpha*\theta_\alpha)(x)\,(\phi_i*\Upsilon)(x,s)\,dx\,ds \\
&\quad = \int_0^\infty\int_{R^3} (\tilde{w}_j^\alpha(s)*\theta_\varepsilon)(x)\,(w_i^\alpha(s)*\theta_\varepsilon)(x)\,(D_j\phi_i*\Upsilon*\theta_\varepsilon)(x,s)\,dx\,ds \\
&\qquad - \int_0^\infty\int_{R^3} D_j(w_i^\alpha(s)*\theta_\alpha)(x)\,D_j(\phi_i*\Upsilon*\theta_\alpha)(x,s)\,dx\,ds
\end{aligned}$$

$$= \int_0^\infty \int_{R^3} (\tilde{v}_j^k * \theta_\varepsilon)(x,s)(v_i^k * \theta_\varepsilon)(x,s)(D_j\phi_i * \Upsilon * \theta_\varepsilon)(x,s)\,dx\,ds$$
$$- \int_0^\infty \int_{R^3} D_j(v_i^k * \theta_{\alpha_k})(x,s) D_j(\phi_i * \Upsilon * \theta_{\alpha_k})(x,s)\,dx\,ds. \tag{3.9}$$

Parts (2.3), (2.1) of Lemma 2.6 imply that $D\phi * \Upsilon * \theta_\varepsilon$ is an L^1 function. Hence Lemma 3.3, the estimate $|(v^k * \theta_\varepsilon)(x,s)| \leq \|\bar{w}\|_2 \|\theta_\varepsilon\|_2$ [see (3.1)], and the Lebesgue dominated convergence theorem yield

$$\lim_{k\to\infty} \int_0^\infty \int_{R^3} (\tilde{v}_j^k * \theta_\varepsilon)(x,s)(v_i^k * \theta_\varepsilon)(x,s)(D_j\phi_i * \Upsilon * \theta_\varepsilon)(x,s)\,dx\,ds$$
$$= \int_0^\infty \int_{R^3} (\tilde{u}_j * \theta_\varepsilon)(x,s)(u_i * \theta_\varepsilon)(x,s)(D_j\phi_i * \Upsilon * \theta_\varepsilon)(x,s)\,dx\,ds. \tag{3.10}$$

The weak convergence of v^k, the fact $D_t(v^k * \Upsilon) = v^k * D_t\Upsilon$, (3.10), Lemma 3.2, and $D\phi \in L^2$ [see (2.1)] imply that (3.9) yields

$$\int_{b'}^{b} \int_{R^3} D_t(u_i * \Upsilon)(x,s)\,\phi_i(x,s)\,dx\,ds$$
$$= \int_0^\infty \int_{R^3} (\tilde{u}_j * \theta_\varepsilon)(x,s)(u_i * \theta_\varepsilon)(x,s)(D_j\phi_i * \Upsilon * \theta_\varepsilon)(x,s)\,dx\,ds$$
$$- \int_0^\infty \int_{R^3} D_j u_i(x,s) D_j(\phi_i * \Upsilon)(x,s)\,dx\,ds.$$

Hence (3.8), the fact $\phi(x,t) = 0$ when $x \notin U$, and (3.7) yield

$$\int_{R^3} (u_i * \Upsilon)(x,b)\,\phi_i(x,b)\,dx - \int_{R^3} (u_i * \Upsilon)(x,b')\,\phi_i(x,b')\,dx$$
$$= -\int_0^\infty \int_{R^3} (((\tilde{u}_j * \theta_\varepsilon) D_j(u_i * \theta_\varepsilon)) * \Upsilon)(x,s)(\phi_i * \theta_\varepsilon)(x,s)\,dx\,ds. \tag{3.11}$$

Lemma 2.6 yields

$$\phi(x,b) = V(x_1 - a_1, x_2 - a_2, x_3, 0) = P(f + f')(x_1 - a_1, x_2 - a_2, x_3)$$

if $x \in U$. In addition, $\delta < a_3 = r$ and (3.7) imply (see Lemma 2.6)

$$\int_{R^3} (u_i * \Upsilon)(x,b)\,f_i'(x_1 - a_1, x_2 - a_2, x_3)\,dx = \int_{R^3} 0\,dx = 0.$$

Hence (3.7) and the symmetry of θ yield

$$\int_{R^3} (u_i * \Upsilon)(x,b)\,\phi_i(x,b)\,dx$$
$$= \int_{R^3} (u_i * \Upsilon)(x,b)(P(f + f'))_i(x_1 - a_1, x_2 - a_2, x_3)\,dx$$
$$= \int_{R^3} (u_i * \Upsilon)(x,b)(f + f')_i(x_1 - a_1, x_2 - a_2, x_3)\,dx$$
$$= \int_{R^3} (u_i * \Upsilon)(x,b)\,f_i(x_1 - a_1, x_2 - a_2, x_3)\,dx$$
$$= \int_{R^3} (u_\beta * \Upsilon)(x,b)\,\theta_\delta(x_1 - a_1, x_2 - a_2, x_3 - a_3)\,dx$$
$$= (u_\beta * \Upsilon * \theta_\delta)(a,b). \tag{3.12}$$

Also, (2.2) yields

$$\begin{aligned}\left|\int_{R^3} (u_i*\Upsilon)(x,b')\,\phi_i(x,b')\,dx\right| &\leqq \int_U |(u*\Upsilon)(x,b')|\,|V(x_1-a_1,x_2-a_2,x_3,b-b')|\,dx\\ &\leqq \int_U C|(u*\Upsilon)(x,b')|\,(|x_1-a_1|+|x_2-a_2|+|x_3-a_3|+(b-b')^{1/2})^{-3}\,dx. \qquad (3.13)\end{aligned}$$

Now (3.11)–(3.13) and the definition of ϕ imply

$$\begin{aligned}|(u_\beta*\Upsilon*\theta_\delta)(a,b)| \leqq &\int_{R^3} C|(u*\Upsilon)(x,b')|\,(|x-a|+(b-b')^{1/2})^{-3}\,dx\\ &+\left|\int_{b'}^{b}\int_{R^3} (((\tilde{u}_j*\theta_\varepsilon)\,D_j(u_i*\theta_\varepsilon))*\Upsilon)\,(x,s)\,(\phi_i*\theta_\varepsilon)\,(x,s)\,dx\,ds\right|. \qquad (3.14)\end{aligned}$$

Let $g: R^3\times R\to[0,1]$ be a function such that $g(x,s)=1$ when $|x-a|\leqq\eta/2$ and $b-\eta^2\leqq s\leqq b$, $g(x,s)=0$ when $|x-a|\geqq\eta$ and $b-\eta^2\leqq s\leqq b$, $g(x,s)=0$ when $s\notin[b-\eta^2,b]$, and $\|Dg\|_\infty\leqq 4\eta^{-1}$.

Let $K=K(a,b,\eta,\eta^2)$. Using the *first* inequality in (2.2) we find $\int_K |(\phi*\theta_\varepsilon)(x,s)|\,dx\,ds\leqq C\eta^2$. Property (2.3) yields

$$\begin{aligned}\int_K |(D\phi*\theta_\varepsilon)(x,s)|\,dx\,ds &\leqq \int_{b-\eta^2}^{b}\int_{R^3} |(D\phi*\theta_\varepsilon)(x,s)|\,dx\,ds\\ &\leqq C(\eta^2)^{1/2}\,\|\theta_\varepsilon\|_1 = C\eta.\end{aligned}$$

Using the above and $\operatorname{div}(u*\theta_\varepsilon)=0$ [see (3.7)] we find

$$\begin{aligned}&\left|\int_{b'}^{b}\int_{R^3} (((\tilde{u}_j*\theta_\varepsilon)\,D_j(u_i*\theta_\varepsilon))*\Upsilon)\,(x,s)\,(\phi_i*\theta_\varepsilon)\,(x,s)\,dx\,ds\right|\\ &=\left|\int_{b'}^{b}\int_{R^3} (((\tilde{u}_j*\theta_\varepsilon)\,D_j(u_i*\theta_\varepsilon))*\Upsilon)\,(x,s)\;g(x,s)\;(\phi_i*\theta_\varepsilon)\,(x,s)\,dx\,ds\right.\\ &\quad\left.+\int_{b'}^{b}\int_{R^3} (((\tilde{u}_j*\theta_\varepsilon)\,D_j(u_i*\theta_\varepsilon))*\Upsilon)\,(x,s)\;(1-g)\,(x,s)\;(\phi_i*\theta_\varepsilon)\,(x,s)\,dx\,ds\right|\\ &=\left|-\int_{b'}^{b}\int_{R^3} (((\tilde{u}_j*\theta_\varepsilon)\,(u_i*\theta_\varepsilon))*\Upsilon)\,(x,s)\;D_jg(x,s)\;(\phi_i*\theta_\varepsilon)\,(x,s)\,dx\,ds\right.\\ &\quad-\int_{b'}^{b}\int_{R^3} (((\tilde{u}_j*\theta_\varepsilon)\,(u_i*\theta_\varepsilon))*\Upsilon)\,(x,s)\;g(x,s)\;(D_j\phi_i*\theta_\varepsilon)\,(x,s)\,dx\,ds\\ &\quad\left.+\int_{b'}^{b}\int_{R^3} (((\tilde{u}_j*\theta_\varepsilon)\,D_j(u_i*\theta_\varepsilon))*\Upsilon)\,(x,s)\;(1-g)\,(x,s)\;(\phi_i*\theta_\varepsilon)\,(x,s)\,dx\,ds\right|\\ &\leqq (C\eta)\max\{|(((\tilde{u}_j*\theta_\varepsilon)\,(u*\theta_\varepsilon))*\Upsilon)\,(x,s)| : (x,s)\in K,\ j\in\{1,2,3\}\}\\ &\quad+\int_{b'}^{b}\int_{R^3} |(((\tilde{u}_j*\theta_\varepsilon)\,D_j(u_i*\theta_\varepsilon))*\Upsilon)\,(x,s)\;(1-g)\,(x,s)\;(\phi_i*\theta_\varepsilon)\,(x,s)|\,dx\,ds.\end{aligned}$$

If $x=(x_1, x_2, x_3)$ and $x_3 \geqq 2\varepsilon$ then the second inequality of (2.2) yields $|(\phi*\theta_\varepsilon)(x,s)| \leqq C(|x-a|+(b-s)^{1/2})^{-3}$. If, in addition, $(1-g)(x,s) \neq 0$ then $|x-a|+(b-s)^{1/2} \geqq \eta/2$ and we conclude $|(\phi*\theta_\varepsilon)(x,s)| \leqq C(|x-a|+(b-s)^{1/2}+\eta)^{-3}$. If $x_3 < 2\varepsilon$ then $(\tilde{u}*\theta_\varepsilon)(x,t)=0$ for each t [see (3.7)] and hence

$$(((\tilde{u}_j*\theta_\varepsilon)\, D_j(u*\theta_\varepsilon))*\Upsilon)(x,s)=0.$$

All this implies

$$\begin{aligned}
&\int_{b'}^{b} \int_{R^3} |(((\tilde{u}_j*\theta_\varepsilon)\, D_j(u_i*\theta_\varepsilon))*\Upsilon)(x,s)\,(1-g)(x,s)\;(\phi_i*\theta_\varepsilon)(x,s)|\, dx\, ds \\
&\quad \leqq \int_{b'}^{b} \int_{R^3} C|(((\tilde{u}_j*\theta_\varepsilon)(D_j u*\theta_\varepsilon))*\Upsilon)(x,s)|\;(|x-a|+(b-s)^{1/2}+\eta)^{-3}\, dx\, ds.
\end{aligned}$$

This inequality and (3.14), (3.15) yield the conclusion of the lemma.

Lemma 3.5. *Suppose* $c \in R^3$, $d>0$, $\tau>h>0$, $h^2<d$, $2\varepsilon<\tau$, *and* $0<\eta \leqq h$. *Then*

$$\begin{aligned}
|(u*\theta_\varepsilon)(c,d)| \leqq & \int_{d-h^2}^{d} \int_{R^3} C|u(x,s)|\,(|x-c|+\tau)^{-3} h^{-2}\, dx\, ds \\
& + \int_{d-h^2}^{d} C\Big(\int_{B(c,\tau)} |u(x,s)|^3\, dx\Big)^{1/3} h^{-3}\, ds \\
& + (C\eta)\max\{|(\tilde{u}*\theta_\varepsilon)(x,s)|\,|(u*\theta_\varepsilon)(x,s)| : (x,s) \in K(c,d,\eta,\eta^2)\} \\
& + \int_{d-h^2}^{d} \int_{R^3} C|(\tilde{u}*\theta_\varepsilon)(x,s)|\;|D(u*\theta_\varepsilon)(x,s)|\;(|x-c|+(d-s)^{1/2}+\eta)^{-3}\, dx\, ds.
\end{aligned}$$

Proof. Suppose first that we have the case $\varepsilon < c_3$ [where $c=(c_1, c_2, c_3)$]. If we set $\delta=\varepsilon$, $a=c$, $b=d$ then use of Lemmas 3.3 and 3.4, with a sequence of functions Υ converging to the Dirac delta function, gives us

$$\begin{aligned}
|(u*\theta_\varepsilon)(a,b)| \leqq & \int_{R^3} C|u(x,b')|\,(|x-a|+(b-b')^{1/2})^{-3}\, dx \\
& + (C\eta)\max\{|(\tilde{u}*\theta_\varepsilon)(x,s)|\;|(u*\theta_\varepsilon)(x,s)| : (x,s) \in K(a,b,\eta,\eta^2)\} \\
& + \int_{b'}^{b} \int_{R^3} C|(\tilde{u}*\theta_\varepsilon)(x,s)|\;|D(u*\theta_\varepsilon)(x,s)|\;(|x-a|+(b-s)^{1/2}+\eta)^{-3}\, dx\, ds
\end{aligned}$$

for almost every b' such that $b-h^2<b'<b-(1/2)h^2$. Averaging the above over all such b' and setting $b'=s$ we find

$$\begin{aligned}
|(u*\theta_\varepsilon)(a,b)| \leqq & \int_{b-h^2}^{b-h^2/2} \int_{R^3} C|u(x,s)|\,(|x-a|+h/\sqrt{2})^{-3} 2h^{-2}\, dx\, ds \\
& + (C\eta)\max\{|(\tilde{u}*\theta_\varepsilon)(x,s)|\,|(u*\theta_\varepsilon)(x,s)| : (x,s) \in K(a,b,\eta,\eta^2)\} \\
& + \int_{b-h^2}^{b} \int_{R^3} C|(\tilde{u}*\theta_\varepsilon)(x,s)|\,|D(u*\theta_\varepsilon)(x,s)|\;(|x-a|+(b-s)^{1/2}+\eta)^{-3}\, dx\, ds.
\end{aligned}$$

Combining this with

$$\int_{R^3} |u(x,s)|\,(|x-a|+h)^{-3}\,dx$$
$$\leq \int_{R^3\sim B(a,\tau/2)} |u(x,s)|\,(|x-a|+h)^{-3}\,dx + C\left(\int_{B(a,\tau/2)} |u(x,s)|^3\,dx\right)^{1/3} h^{-1}$$
$$\leq \int_{R^3} C|u(x,s)|\,(|x-a|+\tau)^{-3}\,dx + C\left(\int_{B(a,\tau/2)} |u(x,s)|^3\,dx\right)^{1/3} h^{-1} \tag{3.16}$$

we obtain the conclusion of the lemma in case $\varepsilon < c_3$.

Now we observe what happens in the case $\varepsilon < \eta/2$. Using Lemma 3.4 with a sequence of numbers δ converging to zero, a sequence of functions Υ converging to the Dirac delta, and $\eta/2$ in place of η we obtain

$$|u(a,b)| \leq \int_{R^3} C|u(x,b')|\,(|x-a|+(b-b')^{1/2})^{-3}\,dx$$
$$+(C\eta/2)\max\{|(\tilde{u}*\theta_\varepsilon)(x,s)|\,|(u*\theta_\varepsilon)\,(x,s)| : (x,s)\in K(a,b,\eta/2,\eta^2/4)\}$$
$$+\int_{b'}^{b}\int_{R^3} C|(\tilde{u}*\theta_\varepsilon)(x,s)|\;|D(u*\theta_\varepsilon)(x,s)|\;(|x-a|+(b-s)^{1/2}+\eta/2)^{-3}\,dx\,ds \tag{3.17}$$

for almost all a, b, b' such that $a\in U$, $0<b'<b$. Property (3.7) implies that (3.17) is still true when $a\notin U$ [because then $u(a,b)=0$]. If we integrate the above over $a\in B(c,\varepsilon)$ and use the corresponding inequality

$$3(|x-a|+(b-s)^{1/2}+\eta/2) \geqq |x-a|+\eta/2+(b-s)^{1/2}+\eta$$
$$\geqq |x-a|+\varepsilon+(b-s)^{1/2}+\eta \geqq |x-a|+|a-c|+(b-s)^{1/2}+\eta$$
$$\geqq |x-c|+(b-s)^{1/2}+\eta\,,$$

we find

$$\int_{B(c,\varepsilon)} |u(a,b)|\,da$$
$$\leq \int_{B(c,\varepsilon)}\int_{R^3} C|u(x,b')|\,(|x-a|+(b-b')^{1/2})^{-3}\,dx\,da$$
$$+C\eta\varepsilon^3\max\{|(\tilde{u}*\theta_\varepsilon)(x,s)|\,|(u*\theta_\varepsilon)(x,s)| : (x,s)\in K(c,b,\eta/2+\varepsilon,\eta^2/4)\}$$
$$+\int_{b'}^{b}\int_{R^3} C\varepsilon^3|(\tilde{u}*\theta_\varepsilon)(x,s)|\;|D(u*\theta_\varepsilon)(x,s)|\;(|x-c|+(b-s)^{1/2}+\eta)^{-3}\,dx\,ds \tag{3.18}$$

for almost all b, b' such that $0<b'<b$. If $a\in B(c,\varepsilon)$ then $2\varepsilon<\tau$ yields

$$2(|x-a|+\tau) \geqq |x-a|+\varepsilon+\tau \geqq |x-a|+|a-c|+\tau \geqq |x-c|+\tau\,.$$

Property $2\varepsilon<\tau$ also implies $B(a,\tau/2)\subset B(c,\tau/2+\varepsilon)\subset B(c,\tau)$. Hence the argument of (3.16) yields

$$\int_{R^3} |u(x,b')|\,(|x-a|+(b-b')^{1/2})^{-3}\,dx$$
$$\leq \int_{R^3} C|u(x,b')|\,(|x-a|+\tau)^{-3}\,dx + C\left(\int_{B(a,\tau/2)} |u(x,b')|^3\,dx\right)^{1/3}(b-b')^{-1/2}$$
$$\leq \int_{R^3} C|u(x,b')|\,(|x-c|+\tau)^{-3}\,dx + C\left(\int_{B(c,\tau)} |u(x,b')|^3\,dx\right)^{1/3}(b-b')^{-1/2}. \tag{3.19}$$

If we fix b such that $b>h^2$, average (3.18) over $b'\in[b-h^2, b-h^2/2]$, use (3.19) and $\varepsilon<\eta/2$, and substitute later $b'=s$, then we find

$$\begin{aligned}|(u*\theta_\varepsilon)(c,b)| &\leqq \int_{B(c,\varepsilon)} C\varepsilon^{-3}|u(a,b)|\,da\\ &\leqq \int_{b-h^2}^{b}\int_{R^3} C|u(x,s)|\,(|x-c|+\tau)^{-3}h^{-2}\,dx\,ds\\ &\quad+\int_{b-h^2}^{b} C\Big(\int_{B(c,\tau)}|u(x,s)|^3\,dx\Big)^{1/3}h^{-3}\,ds\\ &\quad+(C\eta)\max\{|(\tilde u*\theta_\varepsilon)(x,s)|\,|(u*\theta_\varepsilon)(x,s)|:(x,s)\in K(c,b,\eta,\eta^2)\}\\ &\quad+\int_{b-h^2}^{b}\int_{R^3} C|(\tilde u*\theta_\varepsilon)(x,s)|\,|D(u*\theta_\varepsilon)(x,s)|\,(|x-c|+(b-s)^{1/2}+\eta)^{-3}\,dx\,ds\end{aligned}$$

for almost every $b>h^2$. Now the conclusion of the lemma follows in this case from the contunuity of $u*\theta_\varepsilon$ (Lemma 3.3) and the substitution $b=d$.

It remains to examine the case $c_3\leqq\varepsilon$, $\varepsilon\geqq\eta/2$. If $a\in U\cap B(c,\varepsilon)$, $\beta\in\{1,2,3\}$, $0<b'<b$, and δ, Υ are as in Lemma 3.4 then we conclude (3.14) just as before. The function ϕ appearing in (3.14) was defined at the start of the proof of Lemma 3.4. Since $(\tilde u*\theta_\varepsilon)(x,t)=0$ whenever $x_3\leqq 2\varepsilon$ [see (3.7) and Definition 3.1], property (2.2) yields

$$\begin{aligned}&\left|\int_{b'}^{b}\int_{R^3}(((\tilde u_j*\theta_\varepsilon)\,D_j(u_i*\theta_\varepsilon))*\Upsilon)(x,s)\,(\phi_i*\theta_\varepsilon)(x,s)\,dx\,ds\right|\\ &\quad\leqq\int_{b'}^{b}\int_{R^3} C|(((\tilde u_j*\theta_\varepsilon)\,D_j(u*\theta_\varepsilon))*\Upsilon)(x,s)|\,(|x-a|+(b-s)^{1/2})^{-3}\,dx\,ds.\end{aligned}\tag{3.20}$$

Since $a\in B(c,\varepsilon)$ and $c_3\leqq\varepsilon$ we find $a_3\leqq 2\varepsilon$. If $|x-a|\leqq\varepsilon$ then $x_3\leqq 3\varepsilon$, and hence (3.7) and Definition 3.1 yield $(\tilde u*\theta_\varepsilon)(x,t)=0$. Hence $|x-a|>\varepsilon$ must hold when x is such that the integrand on the right hand side of (3.20) is not zero. Since we have

$$\begin{aligned}3(|x-a|+(b-s)^{1/2}) &> |x-a|+2\varepsilon+(b-s)^{1/2}\\ &\geqq |x-a|+\varepsilon+\eta/2+(b-s)^{1/2}\\ &\geqq |x-a|+|a-c|+\eta/2+(b-s)^{1/2}\\ &\geqq |x-c|+\eta/2+(b-s)^{1/2}\end{aligned}$$

in such cases, (3.14) and (3.20) imply

$$\begin{aligned}|(u_\beta*\Upsilon*\theta_\delta)(a,b)| &\leqq \int_{R^3} C|(u*\Upsilon)(x,b')|\,(|x-a|+(b-b')^{1/2})^{-3}\,dx\\ &\quad+\int_{b'}^{b}\int_{R^3} C|(((\tilde u_j*\theta_\varepsilon)\;D_j(u*\theta_\varepsilon))*\Upsilon)(x,s)|\,(|x-c|+(b-s)^{1/2}+\eta/2)^{-3}\,dx\,ds.\end{aligned}$$

Using a sequence of functions Υ converging to the Dirac delta and a sequence of numbers δ converging to zero, we find

$$\begin{aligned}|u(a,b)| &\leqq \int_{R^3} C|u(x,b')|\,(|x-a|+(b-b')^{1/2})^{-3}\,dx\\ &\quad+\int_{b'}^{b}\int_{R^3} C|(\tilde u*\theta_\varepsilon)(x,s)|\,|D(u*\theta_\varepsilon)(x,s)|\,(|x-c|+(b-s)^{1/2}+\eta/2)^{-3}\,dx\,ds\end{aligned}$$

for almost all a, b, b' such that $a \in U \cap B(c, \varepsilon)$, $0 < b' < b$. Property (3.7) implies that the restriction $a \in U$ is unnecessary. Integrating the above over $a \in B(c, \varepsilon)$, we obtain (3.18) without the "max" term. Now the argument that follows after (3.18) gives us the conclusion of the lemma because the absence of a "max" term makes the earlier assumption $\varepsilon < \eta/2$ unnecessary.

4. An Estimate for Approximate Solutions

We continue working with the same number ε fixed in the previous section. Recall that the functions u, $\tilde{u}$ are defined in terms of ε. For the remainder of this section, we fix $(a, b) \in R^3 \times R$ and $\tau > 0$. If $r > 0$, $i \in \{1, 2, 3, \ldots\}$ and $s \in R$ we set

$$L(r) = \{(x, t) : \tau - r \leqq |x - a| \leqq \tau + r, b - \tau^2 \leqq t \leqq b\}, \tag{4.1}$$

$$G(i) = K(a, b, \tau(1 - 2^{-i}), \tau^2(1 - 2^{-2i})), \tag{4.2}$$

$$D(s) = \{(x, t) : t \leqq s\}. \tag{4.3}$$

The following assumptions [(4.4) through (4.8)] will be in effect throughout this section:

$$a_3 = -4\varepsilon \,[\text{where } a = (a_1, a_2, a_3)],\ \tau > 2\varepsilon,\ b > \tau^2\ ; \tag{4.4}$$

M_1 is a positive number such that

$$\int_{L(r)} |D(u * \theta_\varepsilon)(x, t)|^2 + |D(\tilde{u} * \theta_\varepsilon)(x, t)|^2 \, dx \, dt \leqq M_1 r \text{ if } 0 < r \leqq \tau\ ; \tag{4.5}$$

if $0 < s \leqq \tau^2$, $t_1 = b - \tau^2$, $t_2 = b - \tau^2 + s$ then

$$\int_{t_1}^{t_2} \left(\int_{B(a, 2\tau)} |u(x, t)|^3 \, dx \right)^{1/3} dt \leqq M_1 s, \tag{4.6}$$

$$\int_{t_1}^{t_2} \int_{R^3} |(\tilde{u} * \theta_\varepsilon)(x, t)| \, |D(u * \theta_\varepsilon)(x, t)| \, (|x - a| + \tau)^{-3} \, dx \, dt \leqq M_1 \tau^{-3} s, \tag{4.7}$$

$$\int_{t_1}^{t_2} \int_{R^3} |u(x, t)| \, (|x - a| + \tau)^{-3} \, dx \, dt \leqq M_1 \tau^{-1} s. \tag{4.8}$$

Lemma 4.1. *Suppose* $(c, d) \in R^3 \times R$, $M_2 > 0$, *and*

$$|(u * \theta_\varepsilon)(x, t)| \leqq M_2 (\tau 2^{-i})^{-1} \quad \text{if} \quad (x, t) \in G(i) \cap D(d), \quad i \in \{1, 2, 3, \ldots\}. \tag{4.9}$$

Suppose also that $|a - c| < \tau$, $b - \tau^2 < d < b$, *and* n, p *are defined by*

$$2^{-(n+1)} < \tau^{-1}(\tau - |a - c|) \leqq 2^{-n}, \qquad n \text{ is an integer}, \tag{4.10}$$

$$2^{-2(p+1)} < \tau^{-2}(d - (b - \tau^2)) \leqq 2^{-2p}, \quad p \text{ is an integer}. \tag{4.11}$$

Let $q = \max\{n, p\}$. *Then* $|(u * \theta_\varepsilon)(c, d)| \leqq C_1 M_1 (\tau 2^{-q})^{-1} + C_1 M_2^2 (\tau 2^{-q})^{-1}$ *for some absolute constant* C_1.

Proof. Using (4.10), (4.11), $\tau>0$, and $d<b$ we find

$$2^{-(n+1)}<\tau^{-1}(\tau-|a-c|)\leqq\tau^{-1}\tau=1,$$
$$2^{-2(p+1)}<\tau^{-2}(d-(b-\tau^2))\leqq\tau^{-2}(b-(b-\tau^2))=\tau^{-2}\tau^2=1.$$

Since n, p are integers and $q=\max\{n,p\}$, we find

$$n\geqq 0,\ p\geqq 0,\ q\geqq 0. \tag{4.12}$$

Using (4.4), the resulting inclusion

$$B(a-(0,0,4\varepsilon),\tau(1-2^{-i}))\subset B(a,\tau(1-2^{-i}))\cup\{(x_1,x_2,x_3):x_3\leqq -6\varepsilon\},$$

property (4.9), the fact that $(u*\theta_\varepsilon)(x,t)=0$ when $x_3\leqq-\varepsilon$, (4.2), and (4.3) we find

$$|(\tilde{u}*\theta_\varepsilon)(x,t)|\leqq M_2(\tau 2^{-i})^{-1} \quad \text{if} \quad (x,t)\in G(i)\cap D(d),\ i\in\{1,2,3,\dots\}. \tag{4.13}$$

Using (4.11) and $q\geqq p$ we deduce

$$d-(b-\tau^2)>\tau^2 2^{-2(p+1)}\geqq\tau^2 2^{-2(q+1)}>\tau^2 2^{-2(q+2)}+\tau^2 2^{-2(q+2)}.$$

The above and $d<b$ yield

$$b>d>d-\tau^2 2^{-2(q+2)}>b-\tau^2(1-2^{-2(q+2)}).$$

If $|x-c|\leqq\tau 2^{-(q+2)}$ then (4.10) and $q\geqq n$ yield

$$|x-a|\leqq|x-c|+|c-a|\leqq\tau 2^{-(q+2)}+\tau-\tau 2^{-(n+1)}$$
$$\leqq\tau 2^{-(q+2)}+\tau-\tau 2^{-(q+1)}=\tau-\tau 2^{-(q+2)}.$$

The above, (4.12), (4.2), and (4.3) yield

$$K(c,d,\tau 2^{-(q+2)},\tau^2 2^{-2(q+2)})\subset G(q+2)\cap D(d).$$

Combining this with (4.9), (4.13) we find

$$\max\{|(\tilde{u}*\theta_\varepsilon)(x,t)|\ |(u*\theta_\varepsilon)(x,t)|:(x,t)\in K(c,d,\tau 2^{-(q+2)},\tau^2 2^{-2(q+2)})\}$$
$$\leqq M_2^2(\tau 2^{-(q+2)})^{-2}. \tag{4.14}$$

We define

$$\eta=\tau 2^{-(q+2)},\ h=\tau 2^{-(p+2)}. \tag{4.15}$$

Our strategy is to use Lemma 3.5. Properties (4.11), (4.12) yield

$$b-\tau^2<d-h^2,\ h\leqq\tau/4. \tag{4.16}$$

We have $2(|x-c|+\tau)\geqq|x-c|+\tau+\tau>|x-c|+|c-a|+\tau\geqq|x-a|+\tau$. Hence (4.16), (4.11), (4.12), (4.8) yield

$$\int_{d-h^2}^{d}\int_{R^3}|u(x,t)|(|x-c|+\tau)^{-3}h^{-2}\,dx\,dt$$
$$\leqq\int_{b-\tau^2}^{d}\int_{R^3}C|u(x,t)|(|x-a|+\tau)^{-3}h^{-2}\,dx\,dt$$
$$\leqq CM_1\tau^{-1}(d-(b-\tau^2))h^{-2}\leqq CM_1\tau^{-1}\tau^2 2^{-2p}h^{-2}\leqq CM_1h^{-1}=CM_1\tau^{-1}2^{p+2}$$
$$\leqq CM_1\tau^{-1}2^{q+2}. \tag{4.17}$$

Also, $|c-a|<\tau$, (4.16), (4.6), and the argument in (4.17) yield

$$\begin{aligned}
&\int_{d-h^2}^{d}\left(\int_{B(c,\tau)}|u(x,t)|^3\,dx\right)^{1/3}h^{-3}\,dt\\
&\leqq \int_{b-\tau^2}^{d}\left(\int_{B(a,2\tau)}|u(x,t)|^3\,dx\right)^{1/3}h^{-3}\,dt\\
&\leqq M_1(d-(b-\tau^2))h^{-3}\leqq M_1\tau^2 2^{-2p}h^{-3}\\
&\leqq CM_1\tau^{-1}2^{p+2}\leqq CM_1\tau^{-1}2^{q+2}\,.
\end{aligned}\tag{4.18}$$

Now $|c-a|<\tau$ and (4.16) give us $|x-a|+\tau\leqq|x-c|+|c-a|+\tau<|x-c|+2\tau$ $\leqq(2\tau h^{-1})\,(|x-c|+h)$. Hence (4.16), (4.7) and the argument in (4.17) yield

$$\begin{aligned}
&\int_{d-h^2}^{d}\int_{R^3}|(\tilde{u}*\theta_\varepsilon)(x,t)|\,|D(u*\theta_\varepsilon)(x,t)|\,(|x-c|+(d-t)^{1/2}+h)^{-3}\,dx\,dt\\
&\leqq(2\tau h^{-1})^3\int_{b-\tau^2}^{d}\int_{R^3}|(\tilde{u}*\theta_\varepsilon)(x,t)|\,|D(u*\theta_\varepsilon)(x,t)|\,(|x-a|+\tau)^{-3}\,dx\,dt\\
&\leqq(2\tau h^{-1})^3M_1\tau^{-3}(d-(b-\tau^2))\leqq CM_1h^{-3}\tau^2 2^{-2p}\\
&\leqq CM_1h^{-1}\leqq CM_1\tau^{-1}2^{q+2}\,.
\end{aligned}\tag{4.19}$$

If $q=p$ then $h=\eta$ [see (4.15)], (4.14)–(4.19), and Lemma 3.5 yield the conclusion of the lemma. Therefore, we may assume $q>p$. This implies [see (4.12)]

$$q=n>p,\ n\geqq 1\,.\tag{4.20}$$

We fix an integer k such that $p+2\leqq k\leqq q+1$. From (4.10) and (4.20) we obtain $\tau-|a-c|\leqq\tau 2^{-n}\leqq\tau/2$, and hence $|a-c|\geqq\tau/2$. Then (4.12) yields $|a-c|\geqq\tau/2$ $>\tau 2^{-(p+2)}\geqq\tau 2^{-k}$. This implies that we can define e_k to be the point on the line segment joining a and c such that

$$|e_k-c|=(3/4)\tau 2^{-k},\ |e_k-a|=|a-c|-|e_k-c|=|a-c|-(3/4)\tau 2^{-k}.$$

The above and $|a-c|<\tau$ yield

$$B(e_k,(1/4)\tau 2^{-k})\subset B(a,\tau-\tau 2^{-(k+1)})\,.\tag{4.21}$$

Using (4.11) and $p+2\leqq k$ we find

$$d-(b-\tau^2)>\tau^2 2^{-2(p+1)}\geqq\tau^2 2^{-2(k-1)}>\tau^2 2^{-2k}+\tau^2 2^{-2(k+1)}\,.$$

Hence we conclude $d-\tau^2 2^{-2k}>b-\tau^2(1-2^{-2(k+1)})$. Combining this with $d<b$, (4.21), (4.2), (4.3) we conclude

$$\begin{aligned}
&K(e_k,d,(1/4)\tau 2^{-k},\tau^2 2^{-2k})\\
&\quad\subset K(a,b,\tau(1-2^{-(k+1)}),\tau^2(1-2^{-2(k+1)}))\cap D(d)=G(k+1)\cap D(d)\,.
\end{aligned}\tag{4.22}$$

The definition of e_k yields $B(e_k,(1/4)\tau 2^{-k})\subset B(c,\tau 2^{-k})$. This inclusion and the proof of Lemma 2.2 of [1] give us

$$\begin{aligned}
&I(|\tilde{u}*\theta_\varepsilon|^2,K(c,d,\tau 2^{-k},\tau^2 2^{-2k}))\\
&\leqq C(\tau 2^{-k})^5\,(\max\{|(\tilde{u}*\theta_\varepsilon)(x,t)|^2:(x,t)\in K(e_k,d,(1/4)\tau 2^{-k},\tau^2 2^{-2k})\})\\
&\quad+C(\tau 2^{-k})^2\,I(|D(\tilde{u}*\theta_\varepsilon)|^2,K(c,d,\tau 2^{-k},\tau^2 2^{-2k}))\,.
\end{aligned}\tag{4.23}$$

From (4.10) and $k \leqq q+1=n+1$ [see (4.20)] we conclude

$$B(c,\tau 2^{-k}) \subset \{x : \tau-\tau 2^{-k+2} \leqq |x-a| \leqq \tau+\tau 2^{-k+2}\}. \tag{4.24}$$

From $d<b$, $p+2 \leqq k$, (4.12), and (4.11) we conclude

$$b>d>d-\tau^2 2^{-2k} \geqq d-\tau^2 2^{-2(p+2)} > d-\tau^2 2^{-2(p+1)} > b-\tau^2. \tag{4.25}$$

Now (4.24), (4.25), (4.1) yield

$$K(c,d,\tau 2^{-k},\tau^2 2^{-2k}) \subset L(\tau 2^{-k+2}). \tag{4.26}$$

Using (4.26), (4.5), $p+2 \leqq k$, (4.12) we find

$$\begin{aligned} &I(|D(u*\theta_\varepsilon)|^2, K(c,d,\tau 2^{-k},\tau^2 2^{-2k})) \\ &\quad \leqq I(|D(u*\theta_\varepsilon)|^2, L(\tau 2^{-k+2})) \leqq M_1(\tau 2^{-k+2}). \end{aligned} \tag{4.27}$$

The same argument also yields

$$I(|D(\tilde{u}*\theta_\varepsilon)|^2, K(c,d,\tau 2^{-k},\tau^2 2^{-2k})) \leqq M_1(\tau 2^{-k+2}). \tag{4.28}$$

Using (4.23), (4.22), (4.13), (4.28) we find

$$\begin{aligned} &I(|\tilde{u}*\theta_\varepsilon|^2, K(c,d,\tau 2^{-k},\tau^2 2^{-2k})) \\ &\quad \leqq C(\tau 2^{-k})^5 M_2^2(\tau 2^{-(k+1)})^{-2} + C(\tau 2^{-k})^2 M_1(\tau 2^{-k+2}) \\ &\quad \leqq CM_1(\tau 2^{-k})^3 + CM_2^2(\tau 2^{-k})^3. \end{aligned} \tag{4.29}$$

The inequality

$$|\tilde{u}*\theta_\varepsilon|\,|D(u*\theta_\varepsilon)| \leqq (1/2)(\tau 2^{-k})^{-1}|\tilde{u}*\theta_\varepsilon|^2 + (1/2)(\tau 2^{-k})\,|D(u*\theta_\varepsilon)|^2$$

and (4.27), (4.29) give us

$$I(|\tilde{u}*\theta_\varepsilon|\,|D(u*\theta_\varepsilon)|, K(c,d,\tau 2^{-k},\tau^2 2^{-2k})) \leqq CM_1(\tau 2^{-k})^2 + CM_2^2(\tau 2^{-k})^2, \tag{4.30}$$

when $p+2 \leqq k \leqq q+1$. Now (4.15), (4.20), the estimate

$$\begin{aligned} &\int_{d-h^2}^{d} \int_{R^3} |(\tilde{u}*\theta_\varepsilon)(x,s)|\,|D(u*\theta_\varepsilon)(x,s)|\,(|x-c|+(d-s)^{1/2}+\eta)^{-3}\,dx\,ds \\ &\quad \leqq \int_{d-h^2}^{d} \int_{R^3} |(\tilde{u}*\theta_\varepsilon)(x,s)|\,|D(u*\theta_\varepsilon)(x,s)|\,(|x-c|+(d-s)^{1/2}+h)^{-3}\,dx\,ds \\ &\quad\quad + \sum_{k=p+2}^{q+1} C(\tau 2^{-k})^{-3}\, I(|\tilde{u}*\theta_\varepsilon|\,|D(u*\theta_\varepsilon)|,\ K(c,d,\tau 2^{-k},\tau^2 2^{-2k})), \end{aligned}$$

properties (4.14)–(4.19), (4.30) and Lemma 3.5 yield the conclusion of the lemma.

Lemma 4.2. *There exist absolute constants $C_2>0$, C_3 such that the following is true: If $M_1 \leqq C_2$ then $|(u*\theta_\varepsilon)(x,t)| \leqq 2C_3\tau^{-1}$ for every $(x,t) \in K(a,b,\tau/2, 3\tau^2/4)$.*

Proof. We choose $C_3>0$ so that $C_1C_3^2 \leqq (1/4)C_3$. Then we choose $C_2>0$ so that $C_1C_2 \leqq (1/4)C_3$. Let f : interior$(K(a,b,\tau,\tau^2)) \to R^+$ be a continuous function such that

$$C_3 2^i \tau^{-1} \geqq f(x,t) \geqq C_3 2^{i-1}\tau^{-1} \quad \text{if} \quad (x,t) \in G(i) \sim G(i-1) \tag{4.31}$$

for $i\in\{1,2,3,\ldots\}$. Here $G(0)$ is the empty set. In particular, we get

$$f(x,t)\leqq C_3 2^i\tau^{-1} \quad \text{if} \quad (x,t)\in G(i). \tag{4.32}$$

We will prove

$$|(u*\theta_\varepsilon)(x,t)|\leqq f(x,t) \quad \text{when} \quad (x,t)\in \text{interior}(K(a,b,\tau,\tau^2)). \tag{4.33}$$

Assume that (4.33) is false. Then the nature of f and the continuity of f and $u*\theta_\varepsilon$ (see Lemma 3.3) imply the existence of $(c,d)\in\text{interior}(K(a,b,\tau,\tau^2))$ such that

$$|(u*\theta_\varepsilon)(c,d)|=f(c,d), \tag{4.34}$$

$$|(u*\theta_\varepsilon)(x,t)|\leqq f(x,t) \quad \text{if} \quad (x,t)\in D(d)\cap\text{interior}(K(a,b,\tau,\tau^2)). \tag{4.35}$$

Then c, d satisfy $|a-c|<\tau$, $b-\tau^2<d<b$. We define n, p, q as in Lemma 4.1 and set $M_2=C_3$. Then (4.32), (4.35) imply that (4.9) is satisfied. All this implies implies that the hypotheses of Lemma 4.1 are satisfied and hence we get (using $M_1\leqq C_2$)

$$|(u*\theta_\varepsilon)(c,d)|\leqq C_1C_2 2^q\tau^{-1}+C_1C_3^2 2^q\tau^{-1}. \tag{4.36}$$

The definition of p, n, q yields $(c,d)\notin\text{interior}(G(q))$. Hence (4.34), (4.31) yield $|(u*\theta_\varepsilon)(c,d)|\geqq C_3 2^q\tau^{-1}$. Combining this with (4.36) we get $C_3\leqq C_1C_2+C_1C_3^2$. Now the definition of C_2, C_3 yields $C_3\leqq(1/4)C_3+(1/4)C_3$, which is a contradiction. Hence (4.33) is true. Setting $i=1$ in (4.32) and using (4.33), (4.2) we obtain the conclusion of the lemma.

5. Isolating the Singular Set

Once again, we fix $\varepsilon>0$ and consider the corresponding functions u, $\tilde{u}$. Lemma 3.2 and (3.2) yield

$$\int_0^\infty\int_{R^3}|Du(x,t)|^2\,dx\,dt\leqq(1/2)\,\|\bar{w}\|_2^2. \tag{5.1}$$

A consequence of (5.1) and $\|\theta_\varepsilon\|_1=1$ is

$$\begin{aligned}\int_0^\infty\int_{R^3}|D(u*\theta_\varepsilon)(x,t)|^2\,dx\,dt&\leqq\|\theta_\varepsilon\|_1^2\left(\int_0^\infty\int_{R^3}|Du(x,t)|^2\,dx\,dt\right)\\&\leqq(1/2)\,\|\bar{w}\|_2^2.\end{aligned} \tag{5.2}$$

Using (3.1) we find

$$\int_s^{s'}\int_{R^3}|u(x,t)|^2\,dx\,dt\leqq(s'-s)\,\|\bar{w}\|_2^2 \quad \text{when} \quad 0\leqq s<s'<\infty. \tag{5.3}$$

This also yields

$$\begin{aligned}\int_s^{s'}\int_{R^3}|(u*\theta_\varepsilon)(x,t)|^2\,dx\,dt&\leqq\|\theta_\varepsilon\|_1^2\left(\int_s^{s'}\int_{R^3}|u(x,t)|^2\,dx\,dt\right)\\&\leqq(s'-s)\,\|\bar{w}\|_2^2.\end{aligned} \tag{5.4}$$

Lemma 5.1. *There exists an absolute constant $C_4>0$ with the following property: Suppose $(a,b)\in R^3\times R$, $\sigma>0$, $b>\sigma^2$,*

$$\int_{b-\sigma^2}^{b}\int_{B(a,2\sigma)}|D(u*\theta_\varepsilon)(x,t)|^2+|D(\tilde{u}*\theta_\varepsilon)(x,t)|^2\,dx\,dt\leqq C_4\sigma, \tag{5.5}$$

$$\int_{b-\sigma^2}^{b}\left(\int_{B(a,2\sigma)}|u(x,t)|^3\,dx\right)^{1/3}dt\leqq C_4\sigma^2, \tag{5.6}$$

$$\int_{b-\sigma^2}^{b}\int_{R^3}|(\tilde{u}*\theta_\varepsilon)(x,t)|\,|D(u*\theta_\varepsilon)(x,t)|\,(|x-a|+\sigma)^{-3}\,dx\,dt\leqq C_4\sigma^{-1}, \tag{5.7}$$

and

$$\int_{b-\sigma^2}^{b}\int_{R^3}|u(x,t)|\,(|x-a|+\sigma)^{-3}\,dx\,dt\leqq C_4\sigma. \tag{5.8}$$

Then there exists τ such that $\sigma/2<\tau<\sigma$ and properties (4.5)–(4.8) *are satisfied when $M_1=C_2$.*

Proof. This is a consequence of the Hardy-Littlewood weak-type inequality for L^1 and the fact that $\sigma/2<\tau<\sigma$ implies $(|x-a|+\tau)^{-3}\leqq 8(|x-a|+\sigma)^{-3}$.

Lemma 5.2. *Suppose $(a,b)\in R^3\times R$, $a_3=-4\varepsilon$ (where $a=(a_1,a_2,a_3)$), $\sigma>4\varepsilon$, $b>\sigma^2$, and properties* (5.5)–(5.8) *are satisfied. Then*

$$|(u*\theta_\varepsilon)(x,t)|\leqq 4C_3\sigma^{-1}\quad \text{if}\quad (x,t)\in K(a,b,\sigma/4,3\sigma^2/16).$$

Proof. This follows from Lemmas 5.1 and 4.2.

Lemma 5.3. *If $(a,b)\in R^3\times R$, $a_3<0$, and $b>\sigma^2$ then*

$$\int_{b-\sigma^2}^{b}\left(\int_{B(a,2\sigma)}|u(x,t)|^3\,dx\right)^{1/3}dt \leqq (C\sigma^{3/2})\left(\int_{b-\sigma^2}^{b}\int_{B(a,4\sigma)}|Du(x,t)|^2\,dx\,dt\right)^{1/2}$$

and

$$\int_{b-\sigma^2}^{b}\int_{R^3}|u(x,t)|\,(|x-a|+\sigma)^{-3}\,dx\,dt \leqq (C\sigma^{1/2})\left(\int_{b-\sigma^2}^{b}\int_{R^3}|u(x,t)|^2\,(|x-a|+\sigma)^{-5/2}\,dx\,dt\right)+(1/2)C_4\sigma.$$

Proof. From $a_3<0$ and (3.7) we conclude

$$\int_{B(a,4\sigma)}|u(x,t)|^2\,dx\leqq C\sigma^2\left(\int_{B(a,4\sigma)}|Du(x,t)|^2\,dx\right)$$

for almost all $t>0$. Hence Lemma 2.6 of [1] yields

$$\begin{aligned}
&\int_{b-\sigma^2}^{b}\left(\int_{B(a,2\sigma)}|u(x,t)|^3\,dx\right)^{1/3}dt\\
&\leqq \int_{b-\sigma^2}^{b} C(2\sigma)^{-1/2}\sigma\left(\int_{B(a,4\sigma)}|Du(x,t)|^2\,dx\right)^{1/2}dt\\
&\quad+\int_{b-\sigma^2}^{b} C(2\sigma)^{1/2}\left(\int_{B(a,4\sigma)}|Du(x,t)|^2\,dx\right)^{1/2}dt\\
&\leqq C\sigma^{3/2}\left(\int_{b-\sigma^2}^{b}\int_{B(a,4\sigma)}|Du(x,t)|^2\,dx\,dt\right)^{1/2}.
\end{aligned}$$

In addition,

$$\begin{aligned}
&\int_{R^3}|u(x,t)|\,(|x-a|+\sigma)^{-3}\,dx\\
&\leqq \int_{R^3} C\gamma|u(x,t)|^2(|x-a|+\sigma)^{-5/2}\sigma^{1/2}\,dx+\int_{R^3} C\gamma^{-1}(|x-a|+\sigma)^{-7/2}\sigma^{-1/2}\,dx\\
&\leqq \int_{R^3} C\gamma|u(x,t)|^2(|x-a|+\sigma)^{-5/2}\sigma^{1/2}\,dx+C_5\gamma^{-1}\sigma^{-1}
\end{aligned}$$

for almost every $t>0$ and every $\gamma>0$. The second inequality of the lemma follows by substituting $\gamma=(C_5^{-1}(1/2)C_4)^{-1}$ and integrating over t.

Lemma 5.4. *There exists an absolute constant C_6 with the following property: Suppose $(a,b)\in R^3\times R$, $a_3=-4\varepsilon$, $\sigma>4\varepsilon$, $b>\sigma^2$,*

$$\int_{b-\sigma^2}^{b}\int_{B(a,4\sigma)}|Du(x,t)|^2+|D(\tilde{u}*\theta_\varepsilon)(x,t)|^2+|D(u*\theta_\varepsilon)(x,t)|^2\,dx\,dt\leqq C_6\sigma, \tag{5.9}$$

$$\int_{b-\sigma^2}^{b}\int_{R^3}|(\tilde{u}*\theta_\varepsilon)(x,t)|\,|D(u*\theta_\varepsilon)(x,t)|\,(|x-a|+\sigma)^{-3}\,dx\,dt\leqq C_6\sigma^{-1}, \tag{5.10}$$

and

$$\int_{b-\sigma^2}^{b}\int_{R^3}|u(x,t)|^2(|x-a|+\sigma)^{-5/2}\,dx\,dt\leqq C_6\sigma^{1/2}. \tag{5.11}$$

Then $|(u*\theta_\varepsilon)(x,t)|\leqq 4C_3\sigma^{-1}$ whenever $(x,t)\in K(a,b,\sigma/4,3\sigma^2/16)$.

Proof. This follows from Lemmas 5.2 and 5.3.

Definition 5.5. For each $\sigma>0$ we choose a countable set $Z(\sigma)\subset R^2$ such that

$$R^2\times\{0\}\subset\bigcup_{(c_1,c_2)\in Z(\sigma)}\text{interior of } B((c_1,c_2,0),\sigma/4), \tag{5.12}$$

$$|c-c'|\geqq\sigma/4 \quad\text{if}\quad \{c,c'\}\subset Z(\sigma)\quad\text{and}\quad c\neq c'. \tag{5.13}$$

Lemma 5.6. *There is an absolute constant C_7 with the following property: Let $\sigma>0$, $b>\sigma^2$ and define*

$$S_1=\{(c_1,c_2,-4\varepsilon):(c_1,c_2)\in Z(\sigma)\quad\textit{and (5.9) is false when}\quad a=(c_1,c_2,-4\varepsilon)\},$$

$S_2 = \{(c_1, c_2, -4\varepsilon) : (c_1, c_2) \in Z(\sigma)$ *and* (5.10) *is false when* $a = (c_1, c_2, -4\varepsilon)\}$,

$S_3 = \{(c_1, c_2, -4\varepsilon) : (c_1, c_2) \in Z(\sigma)$ *and* (5.11) *is false when* $a = (c_1, c_2, -4\varepsilon)\}$.

Then $S_1 \cup S_2 \cup S_3$ *is a set with at most* $C_7 \|\bar{w}\|_2^2 \sigma^{-1}$ *elements.*

Proof. Using (5.13), (5.1), (5.2) and Definition 3.1 we find

$$(C_6\sigma)(\text{cardinality of } S_1)$$
$$\leqq \sum_{a \in S_1} \int_{b-\sigma^2}^{b} \int_{B(a,4\sigma)} |Du(x,t)|^2 + |D(u*\theta_\varepsilon)(x,t)|^2 + |D(\tilde{u}*\theta_\varepsilon)(x,t)|^2 \, dx \, dt$$
$$\leqq C\left(\int_{b-\sigma^2}^{b} \int_{R^3} |Du(x,t)|^2 + |D(u*\theta_\varepsilon)(x,t)|^2 + |D(\tilde{u}*\theta_\varepsilon)(x,t)|^2 \, dx \, dt\right)$$
$$\leqq C\|\bar{w}\|_2^2.$$

Similarly, (5.13), (5.4), (5.2) yield

$$(C_6\sigma^{-1})(\text{cardinality of } S_2)$$
$$\leqq \sum_{a \in S_2} \int_{b-\sigma^2}^{b} \int_{R^3} |(\tilde{u}*\theta_\varepsilon)(x,t)| \, |D(u*\theta_\varepsilon)(x,t)| \, (|x-a|+\sigma)^{-3} \, dx \, dt$$
$$\leqq \int_{b-\sigma^2}^{b} \int_{R^3} C|(\tilde{u}*\theta_\varepsilon)(x,t)| \, |D(u*\theta_\varepsilon)(x,t)| \sigma^{-3} \, dx \, dt$$
$$\leqq \int_{b-\sigma^2}^{b} \int_{R^3} C|(\tilde{u}*\theta_\varepsilon)(x,t)|^2 \sigma^{-4} \, dx \, dt + \int_{b-\sigma^2}^{b} \int_{R^3} C|D(u*\theta_\varepsilon)(x,t)|^2 \sigma^{-2} \, dx \, dt$$
$$\leqq C\sigma^{-2}\|\bar{w}\|_2^2 + C\sigma^{-2}\|\bar{w}\|_2^2.$$

Using (5.13), (5.3) we find

$$(C_6\sigma^{1/2})(\text{cardinality of } S_3)$$
$$\leqq \sum_{a \in S_3} \int_{b-\sigma^2}^{b} \int_{R^3} |u(x,t)|^2 (|x-a|+\sigma)^{-5/2} \, dx \, dt$$
$$\leqq \int_{b-\sigma^2}^{b} \int_{R^3} C|u(x,t)|^2 \sigma^{-5/2} \, dx \, dt \leqq C\sigma^{-1/2}\|\bar{w}\|_2^2.$$

Lemma 5.7. *Suppose* $\sigma > 4\varepsilon$ *and* $b > \sigma^2$. *Then there exists a set of points* $\{(c_{i1}, c_{i2}) : i = 1, 2, \ldots, N\} \subset Z(\sigma)$ *such that* $N \leqq C_7 \|\bar{w}\|_2^2 \sigma^{-1}$ *and the following property holds: If* $(c_1, c_2) \in Z(\sigma)$ *and* (c_1, c_2) *is not one of the* (c_{i1}, c_{i2}) *then*

$$|(u*\theta_\varepsilon)(x,t)| \leqq 4C_3\sigma^{-1} \quad \text{whenever} \quad (x,t) \in K((c_1, c_2, -4\varepsilon), b, \sigma/4, 3\sigma^2/16).$$

Proof. Let $\{(c_{i1}, c_{i2})\}$ be an enumeration of all (c_1, c_2) such that $(c_1, c_2, -4\varepsilon) \in S_1 \cup S_2 \cup S_3$ (see Lemma 5.6). The conclusion follows from the definition of the S_i and Lemma 5.4.

6. The Limit as Epsilon Approaches Zero

We choose a sequence $\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots$ of positive numbers converging to zero, and we let $u^1, u^2, u^3, \ldots$ be the corresponding functions u constructed in Sect. 3 (with $\varepsilon = \varepsilon_n$).

Using (5.3) (which is valid for every $u=u^n$) and passing to a subsequence, we find $u: R^3 \times [0, \infty) \to R^3$ such that $u^n \to u$ weakly in L^2 when the domain is restricted to a set of the form $R^3 \times [0, T]$, $T < \infty$. In addition, (5.1) implies

$$\|Du^n\|_2^2 \leqq (1/2)\|\bar{w}\|_2^2, \ \|Du\|_2^2 \leqq (1/2)\|\bar{w}\|_2^2. \tag{6.11}$$

For any fixed $\delta > 0$, a slight variation of the argument in the proof of Lemma 3.3 shows that the functions $u^n * \theta_\delta$ are equicontinuous and uniformly bounded on compact subsets of $R^3 \times [0, \infty)$. From the inequality $\|(f * \theta_\delta) - f\|_2 \leqq C\delta \|Df\|_2$ (used for $f = u^n$ and $f = u$) and (6.1) we conclude (using Ascoli's theorem and passing to a subsequence) that u^n converges to u in L^2 norm when the functions are restricted to a compact subset of $R^3 \times [0, \infty)$. This implies

$$\lim_{n\to\infty} \int_0^\infty \int_{R^3} (\tilde{u}_j^n * \theta_{\varepsilon_n})(x,t)(u_i^n * \theta_{\varepsilon_n})(x,t)(D_j g_i * \theta_{\varepsilon_n})(x,t)\,dx\,dt$$
$$= \int_0^\infty \int_{R^3} u_j(x,t)\,u_i(x,t)\,D_j g_i(x,t)\,dx\,dt$$

if $g \in C_0^\infty(R^3 \times R, R^3)$ and $\varepsilon = \varepsilon_n$ is used in the definition of $\tilde{u}^n$. Now the construction at the start of Sect. 3, (3.7), and (6.1) imply that u satisfies properties (1), (2) of Theorem 1.1.

Lemma 6.1. *Suppose $\sigma > 0$ and $b > \sigma^2$. Then there exists a set $Y(\sigma, b) \subset Z(\sigma)$ (see Definition 5.5) such that the cardinality of $Y(\sigma, b)$ is at most $C_7 \|\bar{w}\|_2^2 \sigma^{-1}$ and the following property holds: If $(c_1, c_2) \in Z(\sigma)$ and $(c_1, c_2) \notin Y(\sigma, b)$ then*

$$|u(x,t)| \leqq 4C_3\sigma^{-1} \quad \textit{whenever} \quad (x,t) \in K((c_1, c_2, 0), b, \sigma/4, 3\sigma^2/16).$$

Proof. This follows from Lemma 5.7 (which applies to the functions u^n) and a subsequence argument.

Now we construct the (possibly empty) singular set S. For $i \in \{1, 2, 3, \ldots\}$ and for every integer j we define $b(i,j) = i^{-2}j/8$ and

$$S(i) = \{(x_1, x_2, 0, t) \in R^3 \times R : 0 \leqq t \leqq b(i, 8)\}$$
$$\cup \bigcup_{j=9}^{\infty} \bigcup_{(c_1,c_2) \in Y(1/i, b(i,j))} K((c_1, c_2, 0), b(i,j), i^{-1}/4, 3i^{-2}/16).$$

Let $S = \bigcap_{i=1}^{\infty} S(i)$. The set S is a closed subset of $B(U) \times [0, \infty)$ (see Sect. 1). We will show that, for any fixed $t > 0$, the one-dimensional Hausdorff measure of $S \cap (R^3 \times \{t\})$ is at most $C_7 \|\bar{w}\|_2^2$.

Let $\delta > 0$ and choose an integer i large enough so that $i^{-1}/2 < \delta$ and $i^{-2} < t$. Then

$$S \cap (R^3 \times \{t\}) \subset S(i) \cap (R^3 \times \{t\})$$
$$\subset \bigcup_{j=9}^{\infty} \bigcup_{(c_1,c_2) \in Y(1/i, b(i,j))} K((c_1, c_2, 0), b(i,j), i^{-1}/4, 3i^{-2}/16) \cap (R^3 \times \{t\}).$$

There exists an integer J such that $i^{-2}(J-1)/8 < t \leqq i^{-2}J/8$. The above implies that we can write

$$S \cap (R^3 \times \{t\}) \subset \bigcup_{j=J}^{J+1} \bigcup_{(c_1, c_2) \in Y(1/i, b(i,j))} K((c_1, c_2, 0), b(i,j), i^{-1}/4, 3i^{-2}/16) \cap (R^3 \times \{t\}).$$

Since $Y(1/i, b(i,j))$ has at most $C_7 \|\bar{w}\|_2^2 i$ elements (see Lemma 6.1) and the diameter of $K(a, b, i^{-1}/4, 3i^{-2}/16) \cap (R^3 \times \{t\})$ is at most $i^{-1}/2 < \delta$, we conclude that $S \cap (R^3 \times \{t\})$ can be covered by sets $A_1, A_2, \ldots, A_N$ where $N \leqq 2C_7 \|\bar{w}\|_2^2 i$ and diameter $(A_k) \leqq i^{-1}/2 < \delta$. Since

$$\sum_{k=1}^{N} \text{diameter}(A_k) \leqq (2C_7 \|\bar{w}\|_2^2 i)(i^{-1}/2) = C_7 \|\bar{w}\|_2^2$$

is valid in such cases, we conclude that the one-dimensional Hausdorff measure of $S \cap (R^3 \times \{t\})$ is at most $C_7 \|\bar{w}\|_2^2$.

Now we prove the last property in the conclusion of Theorem 1.1. Let $(a, b) \in B(U) \times R^+$ such that $(a, b) \notin S$. Then there exists i such that $(a, b) \notin S(i)$. This implies (since the third component of a is zero) $b > b(i, 8)$. There exists an integer j such that $b(i,j) - 3i^{-2}/16 < b < b(i,j)$, and we must have $j \geqq 9$. From (5.12) and $a \in B(U)$ we conclude

$$a \in \text{interior of } B((c_1, c_2, 0), i^{-1}/4)$$

for some $(c_1, c_2) \in Z(i^{-1})$. Hence we get

$$(a, b) \in \text{interior of } K((c_1, c_2, 0), b(i,j), i^{-1}/4, 3i^{-2}/16).$$

The facts $j \geqq 9$ and $(a, b) \notin S(i)$ imply that (c_1, c_2) is not an element of $Y(1/i, b(i,j))$. Now Lemma 6.1 tells us that u is bounded on the open set

$$\text{interior of } K((c_1, c_2, 0), b(i,j), i^{-1}/4, 3i^{-2}/16),$$

which contains (a, b).

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Communicated by J. Glimm

Received January 15, 1982