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The Least Square Prenucleolus and the Least Square Nucleolus. Two Values for TU Games Based on the Excess Vector

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Abstract: The nucleolus and the prenucleolus are solution concepts for TU games based on the excess vector that can be associated to any payoff vector. Here we explore some solution concepts resulting from a payoff vector selection based also on the excess vector but by means of an assessment of their relative fairness different from that given by the lexicographical order. We take the departure consisting of choosing the payoff vector which minimizes the variance of the resulting excesses of the coalitions. This procedure yields two interesting solution concepts, both a prenucleolus-like and a nucleolus-like notion, depending on which set is chosen to set up the minimizing problem: the set of efficient payoff vectors or the set of inputations. These solution concepts, which, paralleling the prenucleolus and the nucleolus, we call least square prenucleolus and least square nucleolus, are easy to calculate and exhibit nice properties. Different axiomatic characterizations of the former are established, some of them by means of consistency for a reasonable reduced game concept.

1 Introduction

The nucleolus and the prenucleolus (Schmeidler (1969), Sobolev (1975)) are solution concepts for TU games based on the excess vector which can be associated to any payoff vector. The latter selects the efficient payoff vector which minimizes, according to the lexicographical order, the excess vector, while the nucleolus selects it, according to the same principle, from the set of imputations. In both cases the point is to minimize the maximal complaint, considering the excess of a coalition at a payoff vector as a measure of its dissatisfaction when facing it as a possible final payoff.

Here we explore the solution concepts resulting from a payoff vector selection based also on the excess vector, but by means of an assessment of its relative fairness different from that given by the lexicographical order. That is, starting from the excess vector we take the departure consisting of, instead of minimizing it according to the lexicographical order, choosing the payoff vector which minimizes the variance of the resulting excess of the coalitions, or, more accurate-

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ly, that one whose associated excesses are closest to the average excess under the least square criterion. However hastily egalitarian this procedure may seem at first sight, it yields two interesting solution concepts, both a prenucleolus-like and a nucleolus-like notion, depending on which set is chosen to set up the minimizing problem: the set of efficient payoff vectors or the set of imputations. These solution concepts, which, paralleling the prenucleolus and the nucleolus, we call *least square prenucleolus* and *least square nucleolus*, are easy to calculate and exhibit nice properties.

The paper is organized as follows: in Section 2 both solution concepts, the least square prenucleolus and the least square nucleolus, are introduced and properties of the former are established. In Section 3 reformulations of the prekernel and the kernel coherent with our approach are given and shown equivalent to the least square prenucleolus and nucleolus respectively. In Section 4 the least square prenucleolus is characterized axiomatically, and by means of consistency in Section 5; then, in Section 6 it is shown the close relation of this solution concept with the Banzhaf index. Finally, in Section 7 properties of the least square nucleolus and an algorithm for its calculation are given.

2 The Least Square Prenucleolus and the Least Square Nucleolus

An *n-person game in characteristic function form* or a *transferable utility* (TU) *game* is a pair (N, v) where $N = \{1, 2, \ldots, n\}$ and v is a function $v: 2^N \to \mathbb{R}$, such that $v(\emptyset) = 0$. N represents the set of *players* and 2^N denotes the family of all subsets or *coalitions* of N. For each coalition S, the real number $v(S)$ represents the reward that coalition S can achieve by itself if all its members act together. G_N denotes the $2^n - 1$ dimensional vector space of all n-person games. In this context any $x \in \mathbb{R}^n$ will be called a *payoff vector*, and for any coalition S, s denotes its cardinality and $x(S) := \sum_{i \in S} x_i$. A payoff vector x is said to be *efficient* or a *preimputation* if $x(N) = v(N)$, and an *imputation* if, besides, it is $x_i \geq v(i)$ for all $i \in N$. $I(v)$ and $prI(v)$ denote the sets of imputations and preimputations respectively.

For any payoff vector $x \in \mathbb{R}^n$ and any coalition $S \neq \emptyset$, we denote $e(S, x) := v(S) - x(S)$ and call it the *excess of S on x*. And $\theta(x) := (e(S, x))_{S \subset N}$ will be called the *excess vector* of x. Note $e(S, x)$ can be interpreted as a measure of the dissatisfaction of coalition S if x were suggested as final payoff: the greater $e(S, x)$ the more ill-treated would feel S.

As it has been commented in the introduction, based on this interpretation, the nucleolus and the prenucleolus select the payoff vector which minimizes, according to the lexicographical order, the excess vector (once decreasingly ordered its components) among all efficient payoff vectors, and on the set of imputations respectively. This selection is justified by a principle of fairness and stability: to minimize the maximal complaint that a coalition might raise against a proposed payoff. A twofold philosophy towards the coalitions seems to underly this procedure. It embodies both an egalitarian principle (flattening the excess vector) and an utilitarian principle (doing it by pushing down the excesses). Here, also starting from the excess vector and based on the egalitarian side of this philosophy, we take a different departure to implement it. Instead of pushing down the highest excess to flatten the excess vector, we try selecting the payoff vector which minimizes the variance of the excesses of the coalitions. To be precise, we try selecting the payoff vector for which the resulting excesses are closest to the average excess under the least square criterion. At first sight it seems a sheer if not an exceedingly egalitarian expedient, disregarding any utilitarian consideration. But as it will soon be apparent the latter are of no consequence in this approach. Formally, we consider the following two problems for any game $v \in G_N$:

Problem 1: Minimize $\sum_{S \subset N} (e(S,x) - \bar{e}(v,x))^2$

s.t. $\sum_{i \in N} x_i = v(N)$.

Problem 2: Minimize $\sum_{S \subset N} (e(S,x) - \bar{e}(v,x))^2$

s.t. $\sum_{i \in N} x_i = v(N)$ and $x_i \geq v(i)$ for all $i \in N$.

Where the summations are taken (and so it should be understood in what follows) over all *nonempty* coalitions, and $\bar{e}(v,x)$ is the *average excess at* x, given by

$$\bar{e}(v,x) := \frac{1}{2^n - 1} \sum_{S \subset N} e(S,x).$$

Remark 3: Note the average excess at x is the same for any efficient payoff vector. This is a direct consequence of the following

Lemma 4: For any game, the sum of the excesses of all coalitions is the same for all efficient payoff vectors.

Proof: Let x be any efficient payoff vector. Then

$$\sum_{S \subset N} e(S,x) = \sum_{S \subset N} (v(S) - x(S)) = \sum_{S \subset N} v(S) - \sum_{S \subset N} x(S) = \sum_{S \subset N} v(S) - 2^{n-1} v(N)$$

because $x(S) + x(N \backslash S) = v(N)$ for all $S \subset N$. $\square$

In this sense we say the utilitarian principle is not to be taken into account for a choice among efficient payoff vectors. So in what follows we simply denote $\bar{e}(v) := \bar{e}(v,x)$, for any efficient payoff vector x. Now we can state the following result.

Theorem 5: For any game $v \in G_N$.

(i) There exists a unique solution x^* of Problem 1 and it is given by

$$x_i^* = \frac{v(N)}{n} + \frac{1}{n2^{n-2}}\left[na_i(v) - \sum_{j \in N} a_j(v) \right] \quad (i \in N), \tag{1}$$

where $a_i(v) := \sum_{S:i \in S} v(S)$.
(ii) If the set of imputations of v is nonempty, then there exists a unique solution of Problem 2.

Proof: First, calculating the Hessian matrix it can easily be checked that the objective function in both problems is strictly convex on $\mathbb{R}^n$. On the other hand, it is obviously continuous, so that (ii) follows immediately, for the set of imputations is compact and convex. Now consider Problem 1. It is an equality constrained problem in which the objective function is strictly convex and the feasible set is convex so that, first, there exist at most one optimal solution and, second, Lagrange conditions are necessary and sufficient for a point to be the optimal solution. The Lagrangian of Problem 1 is

$$L(x, \lambda) = \sum_{S \subset N} (v(S) - x(S) - \bar{e}(v))^2 + \lambda\left(\sum_{i \in N} x_i - v(N) \right).$$

Besides the constraint equation, the Lagrange conditions are then

$$L_{x_i}(x, \lambda) = -2 \sum_{S:i \in S} (v(S) - x(S) - \bar{e}(v)) + \lambda = 0 \quad (i \in N).$$

A simple calculation solves this linear system and shows that the unique point x^* satisfying these conditions is that one whose coordinates are given by (1). □

Now, paralleling the prenucleolus and the nucleolus, we can define two solution concepts.

Definition 6: The *least square prenucleolus* (*LS-prenucleolus*) and the *least square nucleolus* (*LS-nucleolus*) of a game are[2], respectively, the solutions of Problem 1 and Problem 2.

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[2] We have been full of doubts as to which names should be proper for both solution concepts. For the latter, other possibilities considered have been: 'least square' value or 'minimal variance of excesses' value (or nucleolus). Or even, for the former, in view of what will be shown in Section 6, 'additive normalization of the Banzhaf index'. In the end, taking into account the way it all started, we chose the simpler 'least square (pre) nucleolus'. After all the prenucleolus and the nucleolus have been the most inspiring solution concepts for this work and its basic term of reference, and these names bring to mind the interesting parallelism between both approaches.

According to Theorem 5 both are well defined, and the latter exists whenever the set of imputations is nonempty. Formula (1) provides an explicit definition of the LS-prenucleolus and allows its easy calculation. As to the calculation of the LS-nucleolus, in Section 7 a simple algorithm will be described. Now we concentrate on the LS-prenucleolus properties and axiomatization.

Remark 7: In view of Remark 3, in both Problem 1 and Problem 2, $\bar{e}(v)$ can be substituted for $\bar{e}(v,x)$ in the objective function. Moreover, the optimal solutions of both problems remain unchanged if we substitute any constant k for $\bar{e}(v,x)$ in the objective function. To see this let $\|\cdot\|$ denote the euclidean norm and let $\mathbf{1}=(1,\dots,1)\in\mathbb{R}^{2^n-1}$. Then for any $k\in\mathbb{R}$

$$\sum_{S\subset N}(e(S,x)-k)^2=\|\theta(x)-k\mathbf{1}\|^2=\sum_{S\subset N}e(S,x)^2+(2^n-1)k^2-2k\sum_{S\subset N}e(S,x),$$

and by Lemma 4 the last summation is the same for all efficient payoff vectors, so that the resulting objective function differs from that of Problems 1 and 2 on a constant on their feasible sets.

In particular, for $k=0$ we conclude that the optimal solution of Problem 1 is that of problem

$$\begin{aligned}\text{minimize}\ &\sum_{S\subset N}(v(S)-x(S))^2\\ \text{s.t.}\ &\sum_{i\in N}x_i=v(N),\end{aligned}$$

and that the optimal solution of Problem 2 is that of problem

$$\begin{aligned}\text{minimize}\ &\sum_{S\subset N}(v(S)-x(S))^2\\ \text{s.t.}\ &\sum_{i\in N}x_i=v(N)\quad\text{and}\quad x_i\geq v(i)\quad\text{for all } i\in N.\end{aligned}$$

These formulations provide another two interesting interpretations of both the LS-prenucleolus and the LS-nucleolus of a game v: first, *it is the efficient payoff vector (resp. imputation) for which the excess vector is closest to vector zero* (or to any vector of equal coordinates); and, second, *it is the efficient payoff vector (resp. imputation) whose associated additive game is closest to game v.* In both cases we mean it under the euclidean distance. These equivalences show that the LS-prenucleolus belongs to the family of 'convex solution nuclei' studied by Charnes and Kortanek (1967). Note also that the equivalent to Problem 2 is very similar to that whose solution defines the 'two center' solution of Spinetto (1971), the only difference being that in Spinetto's formulation the excesses of one-player coalitions are deleted in the objective function.

Henceforth we denote $\lambda(v)$ the LS-prenucleolus of a game v. In order to establish the basic properties of the LS-prenucleolus it will be useful to rewrite formula (1) like this:

$$\lambda_i(v) = \frac{v(N)}{n} + \frac{1}{n2^{n-1}}\left[\sum_{S:i\in S} nv(S) - \sum_{S\subset N} sv(S)\right],$$

or, equivalently,

$$\lambda_1(v) = \frac{v(N)}{n} + \frac{1}{n2^{n-2}}\left[\sum_{S:i\in S} (n-s)v(S) - \sum_{S:i\notin S} sv(S)\right]. \tag{2}$$

Proposition 8: The LS-prenucleolus $\lambda: G_N \to \mathbb{R}^n$, verifies the following properties

(i) Linearity.
(ii) Continuity.
(iii) Inessential Game: for any additive game v and any $i, \lambda_i(v) = v(i)$.
(iv) Strategic Equivalence.
(v) Anonymity.
(iv) Strict Coalitional Monotonicity: for all $v, w \in G_N$ such that $v(S) > w(S)$ for some S and $v(T) = w(T)$ for any $T \neq S$, it is $\lambda_i(v) > \lambda_i(w)$, for all $i \in S$.
(vii) It is Standard for Two-person Games, that is, for any two-person game

$$\lambda_i(v) = v(i) + \tfrac{1}{2}[v(i,j) - v(i) - v(j)].$$

(viii) Duality: for any game v and its dual v^* $(v^*(S) := v(N) - v(N/S))$ it is $\lambda(v) = \lambda(v^*)$.

Proof: They all can easily be derived from formulae (1) and (2). □

Remarks 9: (1) As it is well known, anonymity implies the *Equal Treatment* property, that is, for all $i, j \in N$,

$$[\forall S \subset N\backslash\{i,j\}, \quad v(S\cup\{i\}) = v(S\cup\{j\})] \Rightarrow [\lambda_i(v) = \lambda_j(v)].$$

(2) Strict coalitional monotonicity is a desirable property which is not satisfied by the prenucleolus either the nucleolus. On the other hand, this property implies *weak* coalitional monotonicity, and there is no solution function on the class of balanced games satisfying this condition which always selects a vector on the core (Young (1985)), so that the LS-prenucleolus of a balanced game not always belongs to its core.
(3) The Shapley value satisfies also properties (i)–(viii), while the LS-prenucleolus fails to verify the dummy axiom. But, as a straightforward corollary of Theorem 5, the latter satisfies instead an interesting property which is not verified by the Shapley value.

Corollary 10: For any game v, the sum of the excesses of all coalitions containing a player at the LS-prenucleolus of the game is the same for all players, that is

$$\text{(ix)}\quad \sum_{S:i\in S} e(S,\lambda(v)) = \sum_{S:j\in S} e(S,\lambda(v)) \quad (\forall i,j\in N, \forall v\in G_N).$$

Moreover, the LS-prenucleolus is the unique efficient value satisfying this property.

Proof: Eliminating the Lagrange multiplier from the Lagrange conditions which, together with efficiency, characterize the LS-prenucleolus (see the proof of Theorem 5) yields the equations

$$\sum_{S:i\in S} e(S,x) = \sum_{S:j\in S} e(S,x) \quad (\forall i,j\in N). \quad \square$$

Property (ix) has an interesting interpretation: if each player considers how the coalitions he or she belongs to are treated at a payoff vector and assesses it from their excesses, this property ensures that *on the average* all players are equally treated. In a sense then, this way of implementing the egalitarian principle towards the coalitions turns out to be egalitarian towards the players too, so that its being verified by a solution would contribute to its stability.

3 The Surplus Approach

The kernel (Davis and Maschler (1965)) and the prekernel (Maschler, Peleg and Shapley (1972) and (1979)) are solution concepts based on the idea of surplus of a player against other. In a game (N,v) the *(maximal) surplus* of player i against j at an efficient payoff vector x is defined by

$$s_{ij}(x,v) := \max\{v(S) - x(S)/i\in S, j\notin S\}.$$

It can be interpreted as the most player i can hope to gain without the cooperation of j, departing from x and forming a coalition, supposing his or her partners will be satisfied receiving what they would receive in x.

We now could introduce two parallel solution concepts also based on the idea of surplus, but for a suitable reformulation of the surplus concept that will prove coherent with our approach. We define the *average surplus* of player i against j at an efficient payoff vector x as

$$\sigma_{ij}(x,v) := \frac{1}{2^{n-2}} \sum_{\substack{S:i\in S\\ j\notin S}} v(S) - x(S)).$$

This concept is similar to the usual surplus, but now players measure their relative strength from average instead of maximal expectations. Instead of the optimistic view according to which the maximal surplus is evaluated, assuming the best partner (coalition) will cooperate, now it is evaluated supposing all partners (coalitions) are equally probable. Equivalently, one can think that any player, being offered the same payment he or she would receive in x, will be indifferent between joining or not joining i, so that it is reasonable to assign probability 1/2 to each possibility; this assumption leads then to an expected surplus which is given by the same formula which defines the average surplus. According to the surplus concept so reformulated the prekernel and the kernel could be redefined as follows.

Definition 11: The *average prekernel* and the *average kernel* of a game v are, respectively, the sets

$$av-prK(v):=\{x\in prI(v)/\sigma_{ij}(x,v)=\sigma_{ji}(x,v) \quad (\forall i,j\in N, i\neq j)\},$$

$$av-K(v):=\{x\in I(v)/(\sigma_{ij}(x,v)-\sigma_{ji}(x,v))(x_j-v(j))\leq 0 \quad (\forall i,j\in N, i\neq j)\}.$$

As it is well-known, the prenucleolus always belongs to the prekernel and the nucleolus to the kernel (Schmeidler (1969)). A similar but stronger result can be established about the average-versions of these concepts.

Proposition 12: For any game $v\in G_N$, the LS-prenucleolus is the unique point of the average prekernel.

Proof: It is straighforward that condition $\sigma_{ij}(x,v)=\sigma_{ji}(x,v)$ for all i and j $(i\neq j)$, is equivalent to condition (ix)

$$\sum_{S:i\in S} e(S,x)=\sum_{S:j\in S} e(S,x) \quad (\forall i,j\in N),$$

and by Corollary 10, for all $v\in G_N$, $\lambda(v)$ is the only efficient payoff vector satisfying it. □

A similar result concerning the least square nucleolus and the average kernel will be proved in Section 7. So that Definition 11 turns out to be superfluous, for it is a mere restatement of the LS-prenucleolus and LS-nucleolus concepts. Nevertheless this terminology and notation will be useful later.

4 Axiomatization of the Least Square Prenucleolus

In Section 2 we have seen that property (ix) together with efficiency fully characterizes the LS-prenucleolus (Corollary 10). Therefore this property can be judged too strong in order to establish a comparison with any other solution concept. Now we offer an axiomatization involving axioms which are usual in the characterization of some other solution concepts, plus a new axiom requiring that whenever a player contributes more, on the whole or on the average, to the worth of the coalitions than some other player, he or she should receive more.

Recall the index $a_i(v)=\sum_{S:i\in S} v(S)$ which has been introduced in Section 2. From formula (1) it follows that the LS-prenucleolus verifies the following monotonicity property for a value ψ:

$$\text{(x)}\quad a_i(v)\geq a_j(v)\Rightarrow\psi_i(v)\geq\psi_j(v)\quad(\forall i,j\in N,\forall v\in G_N).$$

That is to say, the bigger the index $a_i(v)$ for a player the greater his value. Equivalent ways of expressing the inequality on the left hand side of this implication yield equivalent formulations of this property, as

$$\text{(x}'\text{)}\quad \sum_{S:i\in S,j\notin S} v(S)\geq \sum_{S:j\in S,i\notin S} v(S)\Rightarrow\psi_i(v)\geq\psi_j(v)\quad(\forall i,j\in N,\forall v\in G_N),$$

or even

$$\text{(x}''\text{)}\quad \sum_{S:i\notin S,j\notin S}(v(S\cup\{i\})-v(S))\geq\sum_{S:j\notin S,i\notin S}(v(S\cup\{j\})-v(S))\Rightarrow\psi_i(v)\geq\psi_j(v)$$
$$(\forall i,j\in N,\forall v\in G_N),$$

which allows a new interpretation: if player i's marginal contribution (aggregated or on the average) to the coalitions not containing i nor j is not less than that of player j, then i should not receive less than j. So we call this property *AverageMarginal Contribution (AMC) Monotonicity.* It seems a reasonable requirement which is not verified by the Shapley value, for it gives a smaller weight to intermediate coalitions, while in our approach all coalitions are considered equally important. Note also that this axiom implies equal treatment.

The following characterization of the LS-prenucleolus will permit us to establish a comparison with some other solution concepts as the Shapley value and the Banzhaf index.

Theorem 13: The LS-prenucleolus is the unique value on G_N which verifies efficiency, linearity, inessential game and average marginal contribution monotonicity.

In order to prove it we first establish the following

Lemma 14: A solution $\psi: G_N\rightarrow\mathbb{R}^n$ verifies efficiency, linearity and average marginal contribution monotonicity, if and only if there exists $\beta\geq 0$ such that

$$\psi_i(v)=\frac{v(N)}{n}+\beta\left[na_i(v)-\sum_{j\in N}a_j(v)\right]\quad(\forall i\in N,\forall v\in G_N). \tag{3}$$

Proof: First, it can easily be checked that for any $\beta \geq 0$, the value defined by (3) verifies these three axioms. To see the converse, let $(w^S)_{S \subset N}$ be the basis of G_N such that, for any nonempty coalition S,

$$w^S(T) := \begin{cases} 1, & \text{if } S = T \\ 0, & \text{if } S \neq T. \end{cases}$$

Then, by efficiency and AMC-monotonicity, $\psi_i(w^N) = 1/n$ ($\forall i \in N$), and for each $S \subsetneq N$, there must exist some $\beta_S \geq 0$ such that

$$\psi_i(w^S) = \begin{cases} \dfrac{\beta_S}{s}, & \text{if } i \in S \\ \dfrac{-\beta_S}{n-s}, & \text{if } i \notin S. \end{cases}$$

On the other hand, by linearity, for any two *disjoint* coalition S and T ($\neq N$) we have

$$\psi_i(w^S + w^T) = \psi_i(w^S) + \psi_i(w^T) = \begin{cases} \dfrac{\beta_S}{s} - \dfrac{\beta_T}{n-t}, & \text{if } i \in S \\ \dfrac{\beta_T}{t} - \dfrac{\beta_S}{n-s}, & \text{if } i \in T. \end{cases}$$

And again by AMC-monotonicity, it should be $\psi_i(w^S + w^T) = \psi_j(w^S + w^T)$, for all $i \in S$ and all $j \in T$. So that $(\beta_S/s) - (\beta_T/(n-t)) = (\beta_T/t) - (\beta_S/(n-s))$, or equivalently $(\beta_S/(s(n-s))) = (\beta_T/(t(n-t)))$. Now let us see this relation holds for *any* two nonempty coalitions different from N. If $S \cap T \neq \varnothing$, and $S \cup T \neq N$, then the relation applies to S and $N \backslash (S \cup T)$, and to T and $N \backslash (S \cup T)$, and therefore to S and T. If $S \cap T \neq \varnothing$, and $S \cup T = N$, then the relation applies to S and $N \backslash S$, and to T and $N \backslash T$, and also to $N \backslash S$ and $N \backslash T$, and therefore to S and T. Thus, denoting $\beta := (\beta_S/(s(n-s)))$, and again by linearity, we conclude that for any game $v \in G_N$,

$$\psi_i(v) = \psi_i\left(\sum_{S \subset N} v(S) w^S\right) = \sum_{S \subset N} v(S)\psi_i(w^S) = v(N)\psi_i(w^N) + \sum_{S: i \in S \neq N} \frac{\beta_S}{s} v(S)$$
$$- \sum_{S: i \notin S} \frac{\beta_S}{n-s} v(S) = \frac{v(N)}{n} + \beta\left[\sum_{S: i \in S} (n-s)v(S) - \sum_{S: i \notin S} s v(S)\right].$$

And this is equivalent to formula (3). □

Proof of Theorem 13: By Proposition 8 and what has been observed at the beginning of this section, the LS-prenucleolus satisfies efficiency, linearity, inessential game and average marginal contribution monotonicity. Now we show it is the only value satisfying these four axioms. Let $\psi: G_N \to \mathbb{R}^n$ be a value satisfying them. By Lemma 14, there is some $\beta \geq 0$ such that ψ is given by formula (3). To end the proof, let us see that the inessential game axiom determines β. For any inessential game v,

$$\begin{aligned} a_i(v) &= \sum_{S: i \in S} v(S) = 2^{n-1}v(i) + \sum_{j \neq i} 2^{n-2}v(j) = 2^{n-1}v(i) + 2^{n-2}[v(N) - v(i)] \\ &= 2^{n-2}[v(N) + v(i)], \end{aligned}$$

and therefore

$$\begin{aligned} \sum_{j \in N} a_j(v) = \sum_{j \in N} 2^{n-2}[v(N) + v(j)] &= 2^{n-2}\left[\sum_{j \in N} v(j) + \sum_{j \in N} v(N)\right] \\ &= 2^{n-2}(n+1)v(N). \end{aligned}$$

Substituting on (3) and using the inessential game axiom

$$\begin{aligned} \psi_i(v) &= \frac{v(N)}{n} + \beta[n2^{n-2}[v(N) + v(i)] - 2^{n-2}(n+1)v(N)] \\ &= \frac{v(N)}{n} + \beta[n2^{n-2}v(i) - 2^{n-2}v(N)] \\ &= \frac{v(N)}{n} + \beta\left[n2^{n-2}v(i) - n2^{n-2}\frac{v(N)}{n}\right] = v(i). \end{aligned}$$

That is

$$\beta n 2^{n-2}\left[v(i) - \frac{v(N)}{n}\right] = v(i) - \frac{v(N)}{n}.$$

Thus, choosing v non-symmetric, $\beta = 1/n2^{n-2}$. Therefore, substituting this value for β on (3), we conclude that the unique value satisfying the four axioms is given by

$$\psi_i(v) = \frac{v(N)}{n} + \frac{1}{n2^{n-2}}\left[na_i(v) - \sum_{j \in N} a_j(v)\right],$$

which is precisely (see (1) in Section 2) the LS-prenucleolus. $\square$

Remark 15: To show the independence of these four axioms, note: (i) the Shapley value satisfies them all except AMC-monotonicity; (ii) the 'Banzhaf value' (see Section 6) only fails to be efficient; (iii) the LS-nucleolus, as it will be shown in

Section 7, satisfies all but linearity; and (iv) any value ψ defined by

$$\psi_i(v) = \frac{v(N)}{n} + \beta\left[na_i(v) - \sum_{j\in N} a_j(v)\right] \quad (\forall i\in N, \forall v\in G_N),$$

where β is any nonnegative real number such that $\beta \neq 1/n2^{n-2}$, verifies efficiency, linearity and AMC-monotonocity (Lemma 14), but not the inessential game axiom.

5 Consistency of the Least Square Prenucleolus

In this section we show the consistency of the LS-prenucleolus for a suitable reduced game concept and characterize it by means of this condition. The consistency principle is a concept associated to a notion of 'reduced game'. Given a game $v\in G_N$, the *reduced game* (RG) on a nonempty coalition S at a payoff vector x, is the game $v_{x,s}\in G_S$ which arises when a group of players S supposes that the remainder, $N\backslash S$, are satisfied with what they are paid in x, and, based on this assumption, they face the possibility of renegotiating among themselves how to divide the rest of the cake, $v(N) - x(N\backslash S)$. This is what it is basically a reduced game, though it is not a sharp definition for it leaves deliberately unspecified what game arises in such circumstances. As a matter of fact, it should be only after close examining each particular application that one or another RG concept could be considered appropriate. Based on such a slack definition of the RG, we can formulate the principle of consistency in a parameterized way, the parameter being the RG concept itself.

Definition 16: A value[3] $\psi: G_N \to \mathbb{R}^n$ is *consistent* with respect to a RG concept, if for all $v\in G_N$ and all nonempty $S \subsetneq N$,

$$\psi_i(v) = \psi_i(v_{\psi(v),S}) \quad (\forall i\in S),$$

where $v_{\psi(v),S}$ denotes the reduced game on S at payoff vector $\psi(v)$.

Consistency is then a requirement of stability or self-consistency for a solution concept. If it fails to be verified by a value, its being accepted by some coalition could be a basis for its complementary to deviate. As it will soon be proved, the LS-prenucleolus is consistent for the following RG concept.

[3] For a general solution concept, not necessarily single valued either nonempty, this definition should be slightly retouched. It should be noted too that this formulation does not cover all consistency concepts which have been proposed.

Definition 17: Given a game (N, v), a nonempty coalition S and a payoff vector x, the *reduced game on S at x*, denoted (S, v_x^S), is the game defined by

$$v_x^S(T) := \begin{cases} 0, & T = \varnothing \\ v(N) - x(N \setminus S), & T = S \\ \dfrac{1}{2^{n-s}} \sum\limits_{Q \subset N \setminus S} [v(T \cup Q) - x(Q)], & T \subsetneq S, T \neq \varnothing. \end{cases}$$

It is similar to the reduced game concept introduced by Davis and Maschler (1965), according to which the consistency of different solution concepts can be established (see e.g. Maschler (1992)). But it differs in the way a nonempty coalition $T \subsetneq S$ assesses its worth. In Davis and Maschler's definition this assessment is based on an optimistic expectation, supposing the best partner will cooperate, that is, for any nonempty coalition $T \subsetneq S$, it is given by

$$v_{x,S}(T) := \max\{v(T \cup Q) - x(Q) / Q \subset N \setminus S\}.$$

Our definition, once again, is based on a more realistic or at least not so optimistic a point of view: all partners are equally probable. Alternatively, one can interpret that any player on $N \setminus S$, being offered the same payment he or she would receive in x, will be indifferent between joining or not joining T, so that it is reasonable to assign the same probability to both events; under this assumption the expected worth for T is precisely $v_x^S(T)$.

Theorem 18: The LS-prenucleolus is the unique value $\psi : G_N \to \mathbb{R}^n$ which is standard for two-person games and it is consistent with respect to the reduced game concept given by Definition 17, that is, such that

$$\psi_i(v) = \psi_i(v_{\psi(v)}^S) \quad (\forall v \in G_N, \forall S \subset N, \forall i \in S).$$

Proof: The LS-prenucleolus is standard for two-person games (Proposition 8). To see that it is consistent, let S be a nonempty coalition $S \subset N$, $v \in G_N$ and x an efficient payoff vector. Then for all $i, j \in S$ $(i \neq j)$, denoting $x^S := (x_k)_{k \in S}$,

$$\begin{aligned} \sigma_{ij}(x^S, v_x^S) &= \frac{1}{2^{s-2}} \sum_{\substack{T: i \in T \subset S \\ j \notin T}} (v_x^S(T) - x^S(T)) \\ &= \frac{1}{2^{s-2}} \sum_{\substack{T: i \in T \subset S \\ j \notin T}} \left[\frac{1}{2^{n-s}} \sum_{Q \subset N \setminus S} (v(T \cup Q) - x(Q)) - x(T) \right] \\ &= \frac{1}{2^{n-2}} \sum_{\substack{T: i \in T \subset S \\ j \notin T}} \sum_{Q \subset N \setminus S} (v(T \cup Q) - x(T \cup Q)) \\ &= \frac{1}{2^{n-2}} \sum_{\substack{P: i \in P \\ j \notin P}} (v(P) - x(P)) = \sigma_{ij}(x, v). \end{aligned}$$

Therefore, if $x \in av - prK(v)$, then $x^S \in av - prK(v_x^S)$. But the LS-prenucleolus is the only point of the average prekernel (Proposition 12), so that its consistency follows.

To see it is the unique value satisfying both conditions, let us first show that any value ψ satisfying them is efficient. This is true for $n = 2$ from being standard for two-person games. For $n > 2$, let S be any two player coalition. Then, by consistency

$$\sum_{j \in N} \psi_j(v) = \sum_{j \in S} \psi_j(v^S_{\psi(v)}) + \psi(N \backslash S).$$

But $v^S_{\psi(v)}(S) = v(N) - \psi(N/S)$ by Definition 17 and ψ is efficient for two-person games, thus ψ is efficient. Now let ψ and ϕ be two values satisfying both properties. Let $v \in G_N$, and let S be any two player coalition $S = \{i, j\} \subset N$. Consider the reduced games $(S, v^S_{\psi(v)})$ and $(S, v^S_{\phi(v)})$, by Definition 17

$$v^S_{\psi(v)}(i) = \frac{1}{2^{n-2}} \sum_{Q \subset N \backslash S} [v(Q \cup \{i\}) - \psi(Q)] \quad \text{and}$$

$$v^S_{\psi(v)}(j) = \frac{1}{2^{n-2}} \sum_{Q \subset N \backslash S} [v(Q \cup \{j\}) - \psi(Q)],$$

therefore, $v^S_{\psi(v)}(i) - v^S_{\psi(v)}(j)$ does not depend on ψ. The same is true for ϕ so that $v^S_{\psi(v)}(i) - v^S_{\psi(v)}(j) = v^S_{\phi(v)}(i) - v^S_{\phi(v)}(j)$. On the other hand, by ψ being standard for two person games

$$\psi_i(v^S_{\psi(v)}) = \tfrac{1}{2}[v^S_{\psi(v)}(S) + v^S_{\psi(v)}(i) - v^S_{\psi(v)}(j)],$$

so that $\psi_i(v^S_{\psi(v)}) >$ (resp. $<$) $\psi_i(v^S_{\phi(v)})$ if and only if $\psi_j(v^S_{\psi(v)}) >$ (resp. $<$) $\psi_j(v^S_{\phi(v)})$ (if and only if $v^S_{\psi(v)}(S)$ $>$(resp. $<$)$v^S_{\phi(v)}(S)$). And by consistency and the standard property

$$\psi_i(v) = \psi_i(v^S_{\psi(v)}) > (\text{resp.} <) \psi_i(v^S_{\phi(v)}) = \phi_i(v^S_{\phi(v)}) = \phi_i(v).$$

So that for all i and j, $\psi_i(v) >$ (resp. $<$)$\phi_i(v)$ if and only $\psi_j(v) >$ (resp. $<$)$\phi_j(v)$. But the efficiency of both values implies then $\psi_i(v) = \phi_i(v)$ for all $i \in N$. □

Note the strong similarity of the characterization (and its proof) of the LS-prenucleolus provided by Theorem 18 and that of the Shapley value given by Hart and Mas-Colell (1989). On the other hand, from it it can easily be derived a second characterization of the LS-prenucleolus, stated in the following theorem, entirely similar to that of the prenucleolus due to Sobolev (1975), reinforcing the parallelism of these concepts. In both cases the only difference rests upon the reduced game concept underlying the notion of consistency.

Theorem 19: The LS-prenucleolus is the unique value $\psi: G_N \to \mathbb{R}^N$ which satisfies anonymity, strategic equivalence and consistency with respect to the reduced game concept given by Definition 17.

Proof: From Proposition 8 and Theorem 18 it follows that the LS-prenucleolus verifies all three properties. Now let ψ be a value satisfying them. It can easily be shown that strategic equivalence together with consistency with respect to the RG concept given by Definition 17 imply efficiency. And efficiency with anonymity and strategic equivalence imply being standard for two-person games. Then, by Theorem 18, ψ must be the LS-prenucleolus. □

It should be noted too that the core is also consistent with respect to this RG concept. Another fact that perhaps deserves to be mentioned is that the reduced game of any constant sum game is constant sum too.

We finish this section with a third characterization of the LS-prenucleolus, now by means of the converse reduced game property. This notion also is always associated to some RG notion, so that it can also be formulated in a parameterized way with respect to the RG concept.

Definition 20: A value $\psi: G_N \to \mathbb{R}^n$ verifies the *converse reduced game property* (*CRGP*) with respect to a RG concept, if for all $v \in G_N$ and all efficient payoff vector x,

$$[x^S = \psi(v_{x,S}) \quad (\forall S \subsetneq N \text{ s.t. } s = 2)] \Rightarrow [x = \psi(v)],$$

where $v_{x,S}$ denotes the reduced game on S at x.

Theorem 21: The LS-prenucleolus is the unique value $\psi: G_N \to \mathbb{R}^n$ which verifies the converse reduced game property with respect to the RG concept given by Definition 17 and it is standard for two-person games.

Proof: To see the LS-prenucleolus verifies the converse reduced game property, let $v \in G_N$ and let x be an efficient payoff vector such that for all two player coalition $S \subsetneq N$ it is $x^S = \lambda(v_x^S)$. Then, by Proposition 12, $av - prK(v_x^S) = \{x^S\}$ and therefore, if $S = \{i, j\}$, $\sigma_{ij}(x^S, v_x^S) = \sigma_{ji}(x^S, v_x^S)$. On the other hand, $\sigma_{ij}(x^S, v_x^S) = v_x^S(i) - x_i = \sigma_{ij}(x, v)$. Thus, $av - prK(v) = \{x\}$ or equivalently, by Proposition 12, $x = \lambda(v)$.

Now let ψ be a value standard for two-person games and verifying the CRGP. Then for any game v, $\psi(v)$ must be the unique efficient payoff vector x such that for any two-person coalition $S = \{i, j\}$ verifies

$$x^S = (x_i, x_j) = (v_x^S(i), v_x^S(j)) + \tfrac{1}{2}[v_x^S(i, j) - v_x^S(i) - v_x^S(j)](1, 1).$$

On the other hand, $v_x^S(i) = x_i + \sigma_{ij}(x, v)$, and substituting on the last equality we have

$$x_i = x_i + \sigma_{ij}(x, v) + \tfrac{1}{2}[v(N) - x(N\backslash\{i, j\}) - x_i - \sigma_{ij}(x, v) - x_j - \sigma_{ji}(x, v)],$$

or, which is equivalent, $\sigma_{ij}(x, v) = \sigma_{ji}(x, v)$ for all $i, j \in N$. And by Proposition 12, the only efficient payoff vector satisfying this condition is the LS-prenucleolus. □

6 The Banzhaf Index and the Least Square Prenucleolus

A generalization to non-simple games of a natural normalization of the Banzhaf–Coleman index, sometimes called 'Banzhaf value' (Banzhaf (1965), Coleman (1971), also Owen (1982)), of a player i in a game (N, v) is given by

$$\beta_i(v) := \frac{1}{2^{n-1}} \sum_{S: i \notin S} [v(S \cup \{i\}) - v(S)].$$

That is to say, it is his or her average marginal contribution to the coalitions not containing him or her. In general vector $\beta(v)$ is not efficient. In order to obtain an 'efficient normalization' of $\beta(v)$ there are two natural procedures. One can get it by means of a *multiplicative* efficient normalization (Dubey and Shapley (1979)). Another possibility is adding the same constant to all its components, so obtaining an *additive* efficient normalization (Hammer and Holzman (1987)) given by

$$\beta^a(v) := \beta(v) + \left[(V(N) - \sum_{i \in N} \beta_i(v))/n \right] \mathbf{1}.$$

Then $\beta^a(v)$ is the orthogonal projection of $\beta(v)$ over the efficient hyperplane or, in other words, the point of this hyperplane closest to $\beta(v)$ according to the euclidean distance. As Hammer and Holzman point out, $\beta^a(v)$ provides the additive game closest to v under the euclidean distance. As a consequence, in view of Remark 7, we have the following result of which we give a proof based on the axiomatic characterization of the LS-prenucleolus.

Theorem 22: The additive normalization of β coincides with the LS-prenucleolus.

Proof: It can easily be checked that β^a, besides efficiency, satisfies linearity, inessential game and average marginal contribution monotonicity. By Theorem 13, the only value satisfying this four properties is the LS-prenucleolus. □

Therefore, denoting $\beta_v(S) := \sum_{i \in S} \beta_i(v)$, the LS-prenucleolus of a game v is also given by

$$\lambda_i(v) = \beta_i(v) + \frac{v(N) - \beta_v(N)}{n} \quad (\forall i \in N).$$

Thus, $\lambda(v)$ is an imputation and therefore coincides with the LS-nucleolus, if and only if

$$\frac{v(N)}{n} - v(i) \geq \frac{\beta_v(N)}{n} - \beta_i(v) \quad (\forall i \in N).$$

7 Properties and Calculation of the Least Square Nucleolus

In general the least square prenucleolus is not an imputation for it possibly fails to verify individual rationality. In order to assure this property, a similar solution concept, the least square nucleolus, has been defined in Section 2 as the optimal solution of Problem 2

$$\begin{aligned} \text{minimize} \quad & \sum_{S \subset N} (e(S,x) - \bar{e}(v,x))^2 \\ \text{s.t.} \quad & \sum_{i \in N} x_i = v(N) \text{ and } x_i \geq v(i) \text{ for all } i \in N, \end{aligned}$$

where $\bar{e}(v,x)$ is the average excess at x and the feasible set has been restricted to the set of imputations. As pointed out in Remark 7, the LS-nucleolus can be redefined equivalently as the optimal solution of problem

$$\begin{aligned} \text{minimize} \quad & \sum_{S \subset N} (v(S) - x(S))^2 \\ \text{s.t.} \quad & \sum_{i \in N} x_i = v(N) \text{ and } x_i \geq v(i) \text{ for all } i \in N. \end{aligned}$$

Remark 23: From the last formulation it is easy to check that the LS-nucleolus verifies strategic equivalence. On the other hand, the LS-nucleolus only makes sense for games with a nonempty imputation set. So in what follows we can restrict to $(0,1)$-normalized games, for any game with a nonempty imputation set is strategically equivalent to a $(0,1)$-normalized game.

In order to prove the coincidence, formerly alluded, of the LS-nucleolus and the average kernel, we need a lemma. It is very similar to a lemma proved in Spinetto (1971) in order to establish the correctness of his algorithm to calculate his two center solution. As it has been commented in Section 2, this solution concept is similar to the LS-nucleolus, but in it the excesses of one-player coalitions are not taken into account. This lemma is also useful in order to prove the correctness of an algorithm for the calculation of the LS-nucleolus later described.

Lemma 24: For any game v and any payoff vector x, let $\mu_i(x) := \sum_{S:i\in S}(v(S) - x(S))$. Then an imputation x is the LS-nucleolus of v if and only if for all $j\in N$,

$$x_j > v(j) \Rightarrow \mu_j(x) = \max\{\mu_i(x)/i = 1,\dots,n\}.$$

Proof: We prove it for a (0, 1)-normalized game. For every player i we define the n-person inessential game e_i as

$$e_i(S) := \begin{cases} 1, & \text{if } i\in S \\ 0, & \text{if } i\notin S. \end{cases}$$

Let C_N be the convex hull of these n games. Given a (0, 1)-normalized game v, every imputation $x\in I(v)$ has an associated additive game, given by $\bar{x}(S) := x(S)$, belonging to C_N. It is well known that this is a one to one correspondence between $I(v)$ and C_N. Then (Remark 7), the LS-nucleolus of v can be interpreted as the imputation whose associated additive game in C_N is closest to the original game v according to the euclidean distance. Let $u\in C_N$, by a well-known theorem on minimal distance over convex sets, we have that for u to be the additive game associated to the LS-nucleolus of v, it is necessary and sufficient that for all $w\in C_N$

$$\sum_{S\subset N} (v(S) - u(S))(w(S) - u(S)) \le 0.$$

Since C_N is the convex hull of $\{e_1,\dots,e_n\}$, the last condition holds for all $w\in C_N$ if and only if it holds for each e_i. So we can rewrite the latter condition as

$$\sum_{S:i\in S} (v(S) - u(S)) \le \sum_{S\subset N} u(S)(v(S) - u(S)) \quad (\forall i\in N).$$

Let x be the imputation associated to u, then the last condition is equivalent to

$$\sum_{S:i\in S} (v(S) - x(S)) \le \sum_{S\subset N} x(S)(v(S) - x(S)) \quad (\forall i\in N),$$

where the right-hand side of the inequality can be rewritten

$$\sum_{S\subset N} x(S)(v(S) - x(S)) = \sum_{j\in N} x_j \sum_{S:j\in S} (v(S) - x(S)).$$

So an imputation x is the LS-nucleolus of game v if and only if

$$\mu_i(x) \le \sum_{j\in N} x_j \mu_j(x) \quad (\forall i\in N),$$

and since $\sum_{j\in N} x_j = 1$, this condition is equivalent to say that if for some j it is $x_j > 0$, then it must be $\mu_j(x) \geq \mu_i(x)$ for all i. The conclusion extends immediately to any game. □

A direct conclusion of this lemma, using the same notation, is the following

Corollary 25: An imputation x is the LS-nucleolus of a game if and only if for all i, j,

$$\mu_j(x) < \mu_i(x) \Rightarrow x_j = v(j).$$

Proposition 26: For any game $v \in G_N$ in which the set of imputations is nonempty, the LS-nucleolus is the unique point of the average kernel.

Proof: The average kernel (Definition 11) can be equivalently defined as

$$av - K(v) := \{x \in I(v) / \sigma_{ij}(x, v) > \sigma_{ji}(x, v) \Rightarrow x_j = v(j), \quad (\forall i, j \in N, i \neq j)\}.$$

And it is straightforward that condition $\sigma_{ij}(x, v) > \sigma_{ji}(x, v) \Rightarrow x_j = v(j)$ is equivalent to $\mu_j(x) < \mu_i(x) \Rightarrow x_j = v(j)$, for all $i, j \in N$, $i \neq j$. Thus, from Corollary 25, the result follows. □

We now describe two algorithms for the calculation of the LS-nucleolus. The first one is an adaptation of the algorithm due to Spinetto (1971) for calculating his two center solution of a game. A simpler and equivalent algorithm is described later. Without loss of generality (Remark 23) we consider games such that $v(i) = 0$, for all $i \in N$.

Algorithm 1: Construct a sequence of pairs (x^i, T^i) $(i = 1, 2, \ldots)$, where x^i is a payoff vector and T^i a subset of N, inductively defined by

(i) $x^1 := \lambda(v)$ and $T^1 := \{j \in N / x^i_j < 0\}$
(ii) x^{i+1} is the solution of problem

$$\text{Min} \sum_{S \subset N} (v(S) - x(S))^2 \quad \text{s.t.} \sum_{i \in N} x_i = v(N) \text{ and } x_j = 0, \forall j \in \bigcup_{k=1}^{k=i} T^k.$$

The sequence stops when $T^i = \emptyset$. Clearly, this process must end after at most $n - 1$ steps and the closing payoff vector is an imputation.

Proposition 27: For any game on G_N in which the imputation set is nonempty, the imputation obtained at the end of the above procedure is its LS-nucleolus.

Proof: The proof, based on Lemma 24, is an adaptation of that of a similar result due to Spinetto (1971). □

In Algorithm 1, at each step a minimization problem has to be solved. Namely, x^i is the optimal solution of problem

$$\operatorname{Min} \sum_{S\subset N} (v(S)-x(S))^2 \quad \text{s.t.} \ \sum_{i\in N} x_i = v(N) \text{ and } x_j = 0, \ \forall j\in M^{i-1},$$

where, for each i, $M^i := \bigcup_{k=1}^{k=i} T^k$ and $M^0 := \emptyset$. This problem is equivalent to problem

$$\operatorname{Min} \sum_{S\subset N} (v(S)-x(S))^2 \quad \text{s.t.} \ \sum_{i\in N} x_i = v(N) \text{ and } x_j = x_j^i, \ \forall j\in M^i,$$

whose solution, for $j\notin M^i$, is given by

$$x_j^i = \frac{v(N)-x^i(M^i)}{n-m_i} + \frac{1}{(n-m_i)2^{n-2}}\left[(n-m_i)a_j(v) - \sum_{k\notin M^i} a_k(v)\right],$$

where m_i denotes the cardinality of M^i. On the other hand, x^{i+1} solves

$$\operatorname{Min} \sum_{S\subset N} (v(S)-x(S))^2 \quad \text{s.t.} \ \sum_{i\in N} x_i = v(N) \text{ and } x_j = 0, \ \forall j\in M^i,$$

and the optimal solution of this problem is given by

$$x_j^{i+1} = \begin{cases} \dfrac{v(N)}{n-m_i} + \dfrac{1}{(n-m_i)2^{n-2}}\left[(n-m_i)a_j(v) - \sum_{k\notin M^i} a_k(v)\right], & \forall j\notin M^i \\ 0, & \forall j\in M^i. \end{cases}$$

Therefore, comparing the expressions which give x_j^i and x_j^{i+1} for $j\notin M^i$, it is immediately concluded that each x^{i+1} can be calculated directly from the preceding x^i, so that the algorithm formerly described is equivalent to the following one.

Algorithm 2: Construct a sequence of pairs (x^i, M^i) $(i=1,2,\ldots)$, where x^i is a payoff vector and M^i a subset of N, inductively defined by

(i) $x^1 := \lambda(v)$ and $M^1 := \{j\in N/\lambda_j(v)<0\}$

(ii) $x_j^{i+1} := \begin{cases} x_j^i + \dfrac{x^i(M^i)}{n-m_i}, & \forall j\notin M^i \\ 0, & \forall j\in M^i. \end{cases}$

and $M^{i+1} := M^i \cup \{j\in N/x_j^{i+1}<0\}$. The sequence stops when $M^i = M^{i-1}$ (with $M^0 := \emptyset$).

That is to say, start with the LS-prenucleolus of the game and, at each step, give 0 to those players with a negative payment in some of the earlier steps and divide the aggregated negative payment in the last step equally among the rest of the players. And stop when no player receives a negative payment.

Example: Let v be the game defined by $v(34)=1$; $v(35)=1$; $v(45)=1$; $v(134)=1$; $v(135)=1$; $v(145)=1$; $v(234)=1.4$; $v(235)=1$; $v(245)=1$; $v(345)=1.75$; $v(1234)=1.75$; $v(1235)=1$; $v(1245)=1$; $v(1345)=2$; $v(2345)=2$; $v(12345)=2$; otherwise $v(S)=0$. Using formula (1), we first obtain its LS-prenucleolus

$$x^1 = \lambda(v) = (-0.0425, 0.0075, 0.72625, 0.72625, 0.5825).$$

Then we give 0 to player 1 and divide -0.0425 equally among players 2, 3, 4 and 5, so that we obtain

$$x^2 = (0, -0.003125, 0.715625, 0.715625, 0.571875).$$

Now we give 0 to players 1 and 2, and divide -0.003125 equally among players 3, 4 and 5. In this way we finally obtain the LS-nucleolus of the game

$$x^3 = (0, 0, 0.7145833.., 0.7145833.., 0.570833..).$$

From the procedures above described to calculate the LS-nucleolus, it can easily be established that it verifies the properties enumerated in the following

Proposition 28: The LS-nucleolus $\Lambda: IG_N \to \mathbb{R}^n$, verifies the following properties on the domain, denoted IG_N, of games with a nonempty set of imputations

(i) Continuity.
(ii) Inessential Game.
(iii) Strategic Equivalence.
(iv) Anonymity.
(v) Equal Treatment.
(vi) Weak Coalitional Monotonicity: for all $v, w \in G_N$ such that $v(S) > w(S)$ for some S and $v(T) = w(T)$ for any $T \neq S$, it is $\Lambda_i(v) \geq \Lambda_i(w)$, for all $i \in S$.
(vii) It is Standard for Two-person Games,
(viii) Average Maginal Contribution Monotonicity.

The loss of linearity is the price of assuring individual rationality. This makes it difficult to obtain a nice axiomatization of the LS-nucleolus.

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