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On Equilibria in Repeated Games With Absorbing States[1]

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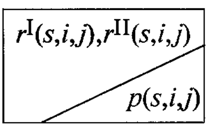
Abstract: We prove the existence of ϵ-(Nash) equilibria in two-person non-zerosum limiting average repeated games with absorbing states. These are stochastic games in which all states but one are absorbing. A state is absorbing if the probability of ever leaving that state is zero for all available pairs of actions.

1 The Stochastic Game Model: Definitions and Notations

We consider two-person stochastic games with finite state and action spaces and with $\mathbf{N} := \{1,2,...\}$ as set of stages.

A stochastic game situation can be seen as a finite set of matrices $\{A_1, A_2, ..., A_z\}$, corresponding to the set of states $S := \{1,2,...,z\}$, in which entry (i,j) of A_s, $s \in S$, has the following shape:



For all s,i and j: $r^{\mathrm{I}}(s,i,j), r^{\mathrm{II}}(s,i,j) \in \mathbf{R}$ and $p(s,i,j) = (p(1|s,i,j), p(2|s,i,j), ..., p(z|s,i,j)$ is a probability vector in $\mathbf{R}^z$.

The stochastic game corresponding to this situation is played in the following way. At each stage $n \in \mathbf{N}$ the system is in one of the states. Say it is in state s at stage n. Simultaneously and independently player I chooses a row, i for instance, and player II chooses a column, j for instance. These choices may be seen as outcomes of probability experiments done by the players. Then the players are informed of each other's choices, player I receives the immediate payoff $r^{\mathrm{I}}(s,i,j)$, player II receives the immediate payoff $r^{\mathrm{II}}(s,i,j)$, and next the system moves to a subsequent state determined by a probability experiment according to $p(s,i,j)$, i.e. a move to state t occurs with probability $p(t|s,i,j)$, for each $t \in S$. In this new state choices have to be made by the players at stage $n+1$, etc..

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A player's strategy is a specification of a probability distribution at each stage, and for each state, over his available actions, conditional on the history of the game up to that stage. Notice that we assume the players to have complete information about everything that has happened in the past. A strategy for player I will be denoted by σ and for player II by τ. A strategy is a Markov strategy if the specified probability distribution at each stage only depends on the current state and on the stage number. A strategy is called a stationary strategy if it is a Markov strategy and the specified probability distributions are the same for all stages. Stationary strategies are denoted by x for player I and by y for player II.

A pair of strategies (σ,τ) together with an initial state $s \in S$ determine a stochastic process on the sets of immediate payoffs. For all $n \in \mathbf{N}$ let $R^k(n)$ denote the stochastic variable representing the immediate payoff to player $k \in \{\mathrm{I,II}\}$ at stage n. Since we will assume that both players want to maximize their expected overall reward, the players have to use some criterion to evaluate streams of payoffs. The two main evaluation criteria are β-discounted, where players discount future payoffs, and limiting average, where players, roughly speaking, evaluate by looking at the expected average reward. To be more exact: for a pair of strategies (σ,τ) and an initial state s

$$\gamma_\beta^k(s,\sigma,\tau) = E_{s\sigma\tau}\left((1-\beta)\sum_{n=1}^{\infty} \beta^{n-1} R^k(n)\right)$$

is the (expected) β-discounted reward to player k, where $\beta \in (0,1)$;

$$\gamma^k(s,\sigma,\tau) = E_{s\sigma\tau}\left(\liminf_{T\to\infty} \frac{1}{T}\sum_{n=1}^{T} R^k(n)\right)$$

is the (expected) limiting average reward to player k.

Depending on which criterion is used (both players use the same) the game is called a β-discounted stochastic game or a limiting average stochastic game.

The game is called a zerosum stochastic game if $r^{\mathrm{II}}(s,i,j) = -r^{\mathrm{I}}(s,i,j)$ for all s,i,j. In case the game is zerosum there is no need to denote the $r^{\mathrm{II}}(s,i,j)$'s, they will be skipped.

A zerosum game has value $\mathrm{val} \in \mathbf{R}^z$, if for all $\epsilon > 0$ there exist strategies $\underline{\sigma}$ for player I and $\underline{\tau}$ for player II, such that for all σ and τ and for all $s \in S$:

$$e(s,\underline{\sigma},\tau) \geq \mathrm{val}(s) - \epsilon \text{ and } e(s,\sigma,\underline{\tau}) \leq \mathrm{val}(s) + \epsilon ,$$

where $e = \gamma_\beta^{\mathrm{I}}$ $(\beta \in (0,1))$ or $e = \gamma^{\mathrm{I}}$ corresponding to whether we observe the game as a β-discounted game or as a limiting average game. The β-discounted value will be denoted by v_β; the limiting average value will be denoted by v. In case the value exists, strategies $\underline{\sigma}$ and $\underline{\tau}$, with the above property, are called ϵ-optimal strategies for player I and player II respectively. If we can take $\epsilon = 0$, then $\underline{\sigma}$ is called an optimal strategy for player I; a similar definition goes for player II.

In non-zerosum games one is interested in ϵ-equilibria. An ϵ-equilibrium ($\epsilon > 0$) is a pair of strategies $(\underline{\sigma},\underline{\tau})$ such that for all σ and τ and all starting states $s \in S$:

$$e^{\mathrm{I}}(s,\sigma,\underline{\tau}) \leq e^{\mathrm{I}}(s,\underline{\sigma},\underline{\tau}) + \epsilon \text{ and}$$
$$e^{\mathrm{II}}(s,\underline{\sigma},\tau) \leq e^{\mathrm{II}}(s,\underline{\sigma},\underline{\tau}) + \epsilon,$$

where again $e = \gamma_\beta$ ($\beta \in (0,1)$) or $e = \gamma$; thus we speak of a β-discounted ϵ-equilibrium and a limiting average ϵ-equilibrium respectively. A 0-equilibrium will be called an equilibrium. The interpretation of an ϵ-equilibrium is that, once the players have "agreed" to play some specific ϵ-equilibrium, neither player can gain more than ϵ by changing his strategy, if his opponent does not change strategy. Thus, an ϵ-equilibrium is in a way self-enforcing; once it is agreed upon, it is not worthwhile to deviate from it.

A uniform limiting average ϵ-equilibrium is a pair of strategies $(\underline{\sigma},\underline{\tau})$ which is an average ϵ-equilibrium for all N-stage games for N sufficiently large. In remark 3 of section 4 we indicate that the ϵ-equilibria we construct are uniform.

In non-zerosum games one has also the notion of Nash Equilibrium Payoffs, NEPs for short. A NEP is a density point of rewards corresponding to ϵ-equilibria ($\epsilon > 0$) for ϵ tending to 0. To be more precise: $(r^{\mathrm{I}},r^{\mathrm{II}})$ is a NEP for the game with starting state s if for all $\epsilon > 0$ there exist $(\underline{\sigma},\underline{\tau})$ for player I and player II respectively such that:

i) $(\underline{\sigma},\underline{\tau})$ is an ϵ-equilibrium
ii) $|e^k(s,\underline{\sigma},\underline{\tau}) - r^k| \leq \epsilon$ for $k \in \{\mathrm{I},\mathrm{II}\}$,

where again $e = \gamma_\beta$ ($\beta \in (0,1)$) or $e = \gamma$ depending on whether we consider the game to be β-discounted or limiting average respectively. In the sequel we will examine β-discounted as well as limiting average equilibria and NEPs for the same stochastic game situation.

We will close this section with one more definition: A repeated game with absorbing states is a stochastic game in which all states but one are absorbing. A state s is called absorbing if for all i,j and $t \neq s$ it holds that $p(t \mid s,i,j) = 0$ (and $p(s \mid s,i,j) = 1$).

2 Historical Review

Shapley (1953), initiator of the theory of stochastic games, showed that zerosum β-discounted stochastic games have a value and that both players possess optimal stationary strategies. For non-zerosum β-discounted stochastic games Fink (1964) showed the existence of equilibria consisting of stationary strategies.

Limiting average stochastic games appear to be more troublesome. Gillette (1957) gave an example of a limiting average zerosum stochastic game for which it was not clear whether it had a value or not. The game was baptized "the big match" and it was first in 1968 that Blackwell and Ferguson (1968) were able to prove that the big match has a value; however, to obtain ϵ-optimality, one player has to use a history dependent strategy. This big match is an example of a zerosum repeated game with absorbing states. These zerosum repeated games with absorbing states were studied more extensively by Kohlberg (1974), who showed that all such games possess a limiting average value. Later, Mertens and Neyman (1981) were able to prove that all limiting average zerosum stochastic games have a value. Their proof is based on the work of Blackwell and Ferguson (1968), of Kohlberg (1974) and of Bewley and Kohlberg (1976), who used Puiseux series to prove relations between β-discounted and limiting average stochastic games.

Until now the existence of ϵ-equilibria has not been proved for general limiting average non-zerosum stochastic games. Nevertheless, the existence turned out to be true for several subclasses, i.e. stochastic games with special properties for the reward or transition functions. Rogers (1969), for instance, showed the existence of stationary equilibria in irreducible stochastic games, i.e. stochastic games for which, for every pair of stationary strategies, the underlying Markov chain is irreducible. Other classes, for which the existence of stationary equilibria has been shown, are for example: stochastic games in which one player controls the transitions (Parthasarathy and Raghavan (1981)), stochastic games with state independent transitions and separable rewards (Parthasarathy et al (1984)).

In this paper we will prove the existence of ϵ-equilibria in limiting average non-zerosum repeated games with absorbing states. We have some good indications that our approach can be extended to solve the question of existence of ϵ-equilibria in general.

For zerosum stochastic games it holds that $v = \lim_{\beta \uparrow 1} v_\beta$, as is shown by Mertens and Neyman (1981). One may think that a similar statement could be true for the limiting average and β-discounted NEPs in non-zerosum stochastic games. Sorin (1986) showed that this is not the case. He also showed that a limiting average NEP need not be a limit of average NEPs in the corresponding finite stage games. Nevertheless, we will show that sequences of stationary β-discounted equilibria, converging for β tending to 1, are very useful to construct limiting average ϵ-equilibria.

3 The Main Theorem

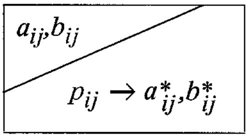
Theorem

For every repeated game with absorbing states there exist limiting average ϵ-equilibria for all $\epsilon > 0$.

Our proof of this theorem is a constructive one and is given in section 3.3. First we introduce some notations in section 3.1 and derive some preliminary results in section 3.2.

3.1 Notations

Without loss of generality we suppose the absorbing states to be of size 1x1 (else take some equilibrium point in the associated bimatrix game). The initial state will always be the non-absorbing one. Then we can describe the repeated game with absorbing states by one mxn-matrix in which entry (i,j) is of the following shape:



where $a_{ij},b_{ij},a^*_{ij},b^*_{ij}\in\mathbf{R}$ and $p_{ij}\in[0,1]$.

Player I is the row player, having available the choices $1,2,...,m$ and player II is the column player who can choose among the actions $1,2,...,n$.

If entry (i,j) is chosen, then player I receives a_{ij}, player II receives b_{ij} and with probability p_{ij} the system moves to an absorbing state, where player I always receives a^*_{ij}, and player II always receives b^*_{ij}; with probability 1-p_{ij} the system remains in the initial state, and actions are chosen again at the next stage.

A stationary strategy x for player I is a probability vector $(x_1,x_2,...,x_m)$ and likewise for player II: $y=(y_1,y_2,...,y_n)$.

Associated with a non-zerosum game we can distinguish two zerosum games: one with the payoffs to player I in which player I is the maximizing player and player II the minimizer, and one with the payoffs to player II in which player II is the maximizing player and player I is the minimizer. Let $\mathrm{val}^{\mathrm{I}}$ and $\mathrm{val}^{\mathrm{II}}$ be the respective values of these games (val being v_β or v). Then $(\mathrm{val}^{\mathrm{I}}, \mathrm{val}^{\mathrm{II}})$ can be considered as threatpoint in the sense that, player II can prevent player I from earning more than $\mathrm{val}^{\mathrm{I}}$ (possibly up to $\delta > 0$) and player I can prevent player II from earning more than $\mathrm{val}^{\mathrm{II}}$ (possibly up to $\delta > 0$).

Definition 1

Let $\delta > 0$. A δ-threat strategy $\sigma^*(\delta)$ for player I is a strategy that keeps player II's limiting average income below $v^{\mathrm{II}} + \delta$. Similar a δ-threat strategy $\tau^*(\delta)$ for player II keeps player I's limiting average income below $v^{\mathrm{I}} + \delta$.

On the other hand, player I can guarantee himself $\mathrm{val}^{\mathrm{I}}$ (possibly up to $\delta > 0$) and player II can guarantee himself $\mathrm{val}^{\mathrm{II}}$ (possibly up to $\delta > 0$). Hence for every ϵ-equilibrium $(\underline{\sigma},\underline{\tau})$ it holds that

$$e^k(\underline{\sigma},\underline{\tau}) \geq \mathrm{val}^k - \epsilon \text{ for } k\in\{\mathrm{I},\mathrm{II}\}, \tag{1}$$

where e is γ or γ_β and val is v or v_β respectively.

From the result of Fink (1964) we know that, for every $\beta \in (0,1)$ there exists a stationary β-discounted equilibrium, i.e. a β-discounted equilibrium consisting of stationary strategies.

By compactness arguments one can find a sequence of β's tending to one and a sequence of stationary β-discounted equilibria $(\underline{x}_\beta, \underline{y}_\beta)$ such that the following definitions make sense:

$$\underline{x} := \lim_{\beta \uparrow 1} \underline{x}_\beta \qquad \underline{y} := \lim_{\beta \uparrow 1} \underline{y}_\beta \tag{2}$$

$$G^{\mathrm{I}} := \lim_{\beta \uparrow 1} \gamma_\beta^{\mathrm{I}}(\underline{x}_\beta, \underline{y}_\beta) \qquad G^{\mathrm{II}} := \lim_{\beta \uparrow 1} \gamma_\beta^{\mathrm{II}}(\underline{x}_\beta, \underline{y}_\beta). \tag{3}$$

Furthermore, by the definition of equilibrium, it holds that for all σ an τ:

$$\gamma_\beta^{\mathrm{I}}(\sigma, \underline{y}_\beta) \le \gamma_\beta^{\mathrm{I}}(\underline{x}_\beta, \underline{y}_\beta) \text{ and } \gamma_\beta^{\mathrm{II}}(\underline{x}_\beta, \tau) \le \gamma_\beta^{\mathrm{II}}(\underline{x}_\beta, \underline{y}_\beta). \tag{4}$$

From (1), (3) and the mentioned result of Mertens and Neyman (1981) it follows that:

$$G^{\mathrm{I}} = \lim_{\beta \uparrow 1} \gamma_\beta^{\mathrm{I}}(\underline{x}_\beta, \underline{y}_\beta) \ge \lim_{\beta \uparrow 1} v_\beta^{\mathrm{I}} = v^{\mathrm{I}} \tag{5}$$

and likewise

$$G^{\mathrm{II}} \ge v^{\mathrm{II}}. \tag{6}$$

We call a pair of stationary strategies (x,y) absorbing, when an absorbing state will be reached with probability 1 in case these strategies are played.

3.2 Preliminary Results

In this section we state some results which are more or less well-known. The formula's of the following lemmas turn out to be very useful and are of fundamental importance to our proof.

Lemma 2

For a pair of stationary strategies (x,y):

$$\gamma_\beta^{\mathrm{I}}(x,y) = \frac{(1-\beta)\sum_i \sum_j x_i a_{ij} y_j + \beta \sum_i \sum_j x_i p_{ij} a^*_{ij} y_j}{1-\beta + \beta \sum_i \sum_j x_i p_{ij} y_j}$$

and

$$\gamma^{\mathrm{I}}(x,y) = \lim_{\beta \uparrow 1} \frac{(1-\beta)\sum_i \sum_j x_i a_{ij} y_j + \beta \sum_i \sum_j x_i p_{ij} a^*_{ij} y_j}{1-\beta + \beta \sum_i \sum_j x_i p_{ij} y_j}$$

Proof: The first formula is obtained by solving the Shapley-equation: $\gamma_\beta^{\mathrm{I}}(x,y) = \sum_i \sum_j x_i \left((1-\beta)\, a_{ij} + \beta p_{ij} a^*_{ij} + \beta(1-p_{ij})\gamma_\beta^{\mathrm{I}}(x,y)\right) y_j$.
The second formula follows from the first, using a special case of $v = \lim_{\beta \uparrow 1} v_\beta$. ■

The next lemma needs no proof.

Lemma 3

a) If (x,y) is absorbing, then

$$\gamma^{\mathrm{I}}(x,y) = \frac{\sum_i \sum_j x_i p_{ij} a^*_{ij} y_j}{\sum_i \sum_j x_i p_{ij} y_j} \text{ and } \gamma^{\mathrm{II}}(x,y) = \frac{\sum_i \sum_j x_i p_{ij} b^*_{ij} y_j}{\sum_i \sum_j x_i p_{ij} y_j}.$$

b) If (x,y) is non-absorbing, then

$$\gamma^{\mathrm{I}}(x,y) = \sum_i \sum_j x_i a_{ij} y_j \text{ and } \gamma^{\mathrm{II}}(x,y) = \sum_i \sum_j x_i b_{ij} y_j.$$

The following lemma can be derived from lemma 2 and lemma 3.

Lemma 4

Let $\{(x_\beta, y_\beta) : \beta \in (0,1)\}$ be a sequence of pairs of stationary strategies, converging for β tending to 1. Let $(x,y) := \lim_{\beta \uparrow 1} (x_\beta, y_\beta)$.

a) If (x,y) is absorbing, then (x_β, y_β) is absorbing for β close to 1, and $\gamma^k(x,y) = \lim_{\beta \uparrow 1} \gamma^k_\beta(x_\beta, y_\beta)$ for $k \in \{\mathrm{I},\mathrm{II}\}$,

b) If (x_β, y_β) is non-absorbing for β close to 1, then (x,y) is non-absorbing and $\gamma^k(x,y) = \lim_{\beta \uparrow 1} \gamma^k_\beta(x_\beta, y_\beta)$ for $k \in \{\mathrm{I},\mathrm{II}\}$.

From now on, fix a sequence of stationary β-discounted equilibria (x_β, y_β) with the properties (2) and (3). From this sequence we will derive a limiting average ϵ-equilibrium, which is even uniform (cf. Remark 4.3).

The next lemma is a consequence of lemma 4, definitions (2) and (3) and property (4).

Lemma 5

If y is such that $(\underline{x}, y)$ is absorbing, then $\gamma^{\mathrm{II}}(\underline{x}, y) \leq G^{\mathrm{II}}$.

Proof: $G^{\mathrm{II}} = \lim_{\beta \uparrow 1} \gamma_\beta^{\mathrm{II}}(x_\beta, y_\beta) \geq \lim_{\beta \uparrow 1} \gamma_\beta^{\mathrm{II}}(x_\beta, y) = \gamma^{\mathrm{II}}(\underline{x}, y)$. ■

From this lemma we will conclude in the sequel, that player II cannot profitably deviate in an absorbing way from his "equilibrium strategy".

Let τ be arbitrary. Let $q^n(\underline{x}, \tau)$ be the probability that the system is still "alive" at stage n, when the players use $\underline{x}$ and τ. Let τ prescribe to play the mixed action y^n at stage n. Then the contribution to $\gamma^{\mathrm{II}}(\underline{x}, \tau)$ via absorption at stage n equals:

$$\gamma_{\text{abs-n}}^{\mathrm{II}}(\underline{x}, \tau) := q^n(\underline{x}, \tau) \sum_i \sum_j \underline{x}_i \, p_{ij} \, b_{ij}^* \, y_j^n \tag{7}$$

In view of lemmas 3 and 5 we get

$$\gamma_{\text{abs-n}}^{\mathrm{II}}(\underline{x}, \tau) \leq q^n(\underline{x}, \tau) \Big(\sum_i \sum_j \underline{x}_i \, p_{ij} \, y_j^n\Big) G^{\mathrm{II}} \tag{8}$$

$$= Pr_{\underline{x}, \tau}\{\text{absorption at stage } n\} G^{\mathrm{II}}.$$

From (7) and (8) we immediately obtain the following lemma.

Lemma 6

If for $\underline{x}$ and τ, $\mu^n(\underline{x}, \tau) \in [0,1]$ is the probability of absorption up to stage n, then the contribution to $\gamma^{\mathrm{II}}(\underline{x}, \tau)$ via absorption up to stage n is at most $\mu^n(\underline{x}, \tau) G^{\mathrm{II}}$, for every $n \in (\mathbb{N} \cup \infty)$.

At this point, define the carrier of a stationary strategy x as $C(x) := \{i \in \{1,2,\ldots, m\}; x(i) > 0\}$ and similar for a stationary strategy y. The next lemma follows directly from the equalizing property of equilibrium strategies.

Lemma 7

If x is such that $C(x) \subset C(x_\beta)$, $\beta \in (0,1)$, then $\gamma_\beta^{\mathrm{I}}(x, y_\beta) = \gamma_\beta^{\mathrm{I}}(x_\beta, y_\beta)$.

Since for β close to 1 it holds that $C(\underline{x}) \subset C(x_\beta)$, lemma 7 has as consequence:

Corollary 8

$\gamma^{\mathrm{I}}_{\beta}(\underline{x},\underline{y}_{\beta}) = \gamma^{\mathrm{I}}_{\beta}(\underline{x}_{\beta},\underline{y}_{\beta})$ for all β close to 1.

Obviously, when interchanging the roles of the players in the lemmas 2-8, analogous statements can be made.

3.3 Main Proof

We now return to the existence of ϵ-equilibria. The proof is divided into three parts:

Case A: $(\underline{x},\underline{y})$ absorbing,
Case B: $(\underline{x},\underline{y})$ non-absorbing and $\gamma^k(\underline{x},\underline{y}) \geq G^k$ for $k = \mathrm{I,II}$
Case C: $(\underline{x},\underline{y})$ non-absorbing and $\gamma^k(\underline{x},\underline{y}) < G^k$ for $k = \mathrm{I}$ or $k = \mathrm{II}$.

Case A: $(\underline{x},\underline{y})$ is absorbing.

Lemma 4 and corollary 8 imply that

$$\gamma^k(\underline{x},\underline{y}) = G^k \text{ for } k = \mathrm{I,II}. \tag{9}$$

Observe that, since $(\underline{x},\underline{y})$ is absorbing, for every $\delta>0$, there exists $N_\delta \in \mathbf{N}$ such that the probability that the system has reached an absorbing state before stage N_δ is at least $1-\delta$.

Let $\epsilon>0$ and take $\delta>0$ such that $(1-\delta)G^k-\delta M \geq G^k-\epsilon/2$ for both $k=\mathrm{I,II}$, where $M = \max_{i,j} \{|a_{ij}|,|b_{ij}|,|a^*_{ij}|,|b^*_{ij}|\}$.

Define $\underline{\sigma}$ and $\underline{\tau}$ as follows:

$\underline{\sigma}$: play $\underline{x}$ stationary up to stage N_δ; if at stage N_δ absorption has not yet occurred, start playing some $\sigma^*(\epsilon/2)$ (cf. definition 1);
$\underline{\tau}$: play $\underline{y}$ stationary up to stage N_δ; if at stage N_δ absorption has not yet occurred, start playing some $\tau^*(\epsilon/2)$ (cf. definition 1).

Lemma 9

In case A, the pair $(\underline{\sigma},\underline{\tau})$ is an ϵ-equilibrium.

Proof: Let player I play $\underline{\sigma}$ and let player II play an arbitrary τ. By lemma 6, the definition of $\sigma^*(\epsilon/2)$ and (6) it follows that

$$\begin{aligned}\gamma^{\mathrm{II}}(\underline{\sigma},\tau) &\leq \mu^{N_\delta}(\underline{x},\tau)G^{\mathrm{II}} + (1-\mu^{N_\delta}(\underline{x},\tau))(v^{\mathrm{II}}+\epsilon/2) \\ &\leq G^{\mathrm{II}} + \epsilon/2.\end{aligned} \tag{10}$$

On the other hand, by (9) and by the definitions of $\underline{\sigma}$ and $\underline{\tau}$:

$$\begin{aligned}\gamma^{\mathrm{II}}(\underline{\sigma},\underline{\tau}) &\geq (1-\delta)\,\gamma^{\mathrm{II}}(\underline{x},\underline{y})-\delta M \\ &= (1-\delta)G^{\mathrm{II}}-\delta M \geq G^{\mathrm{II}}-\epsilon/2.\end{aligned} \tag{11}$$

Combining (10) and (11) yields: $\gamma^{\mathrm{II}}(\underline{\sigma},\tau) \leq \gamma^{\mathrm{II}}(\underline{\sigma},\underline{\tau}) + \epsilon$.

Analogously one can show that $\gamma^{\mathrm{I}}(\sigma,\underline{\tau}) \leq \gamma^{\mathrm{I}}(\underline{\sigma},\underline{\tau}) + \epsilon$ for all σ, which proves the lemma. ■

Case B: $(\underline{x},\underline{y})$ is non-absorbing and $\gamma^k(\underline{x},\underline{y}) \geq G^k$ for $k=\mathrm{I},\mathrm{II}$.

When player I plays $\underline{x}$, then by lemmas 5 and 6, player II cannot gain anything by deviating from $\underline{y}$ in an absorbing way. So, in order to obtain an equilibrium based on $(\underline{x},\underline{y})$, player I should threat with punishment to prevent that player II deviates from $\underline{y}$ in a non-absorbing way, and analogously player II should threat with punishment to prevent that player I deviates from $\underline{x}$ (like in the Folk-theorem, cf Aumann (1981)).

To arrange this, let $Y_N = \frac{1}{N}(Y_{N1},Y_{N2},\ldots,Y_{Nn})$, where Y_{Nj} is the random variable denoting the number of times player II chose column j up to stage N. Observe that if player II uses strategy $\underline{y}$, then $\lim_{N\to\infty} Y_N = \underline{y}$ with probability 1.

Consequently, for every $\delta>0$ and $\alpha\in(0,\epsilon/2M)$ there exists $N_\delta\in\mathbf{N}$, such that $Pr\{|Y_N-\underline{y}| > \alpha$ for some $N\geq N_\delta$, given player II uses $\underline{y}\} < \delta$. Now, choose $\delta>0$ such that $(1-\delta)^2\gamma^k(\underline{x},\underline{y})-(1-(1-\delta)^2)M \geq \gamma^k(\underline{x},\underline{y}) -\epsilon/2$ for $k=\mathrm{I}$ as well as for $k=\mathrm{II}$, and define $\underline{\sigma}$ by:

> keep playing the stationary strategy $\underline{x}$ unless at some stage $N\geq N_\delta$ it would happen that $|Y_N-\underline{y}| > \alpha$; in that case start playing some threat strategy $\sigma^*(\epsilon/2)$.

Define $\underline{\tau}$ analogously.

Thus each player checks the credibility of his opponent's strategy.

Lemma 10

The pair $(\underline{\sigma},\underline{\tau})$ as defined above is an ϵ-equilibrium in case B.

Proof: Let player I play $\underline{\sigma}$ and let player II play an arbitrary strategy τ. Define:

$$\pi(\underline{\sigma},\tau) := Pr_{\underline{\sigma},\tau}\{|Y_N-\underline{y}| > \alpha \text{ for some stage } N\geq N_\delta\} \text{ and}$$

$$\mu(\underline{\sigma},\tau) := Pr_{\underline{\sigma},\tau}\{\text{absorption before player I starts punishment}\}.$$

Then

$1-\pi(\underline{\sigma},\tau)-\mu(\underline{\sigma},\tau) = Pr_{\underline{\sigma},\tau}\{$no absorption and $|Y_N-\underline{y}| \leq \alpha$ for all $N\geq N_\delta\}$. Hence by definition 1, lemma 6, equation (6) and assumption B:

$$\begin{aligned}\gamma^{\mathrm{II}}(\underline{\sigma},\tau) &\leq \pi(\underline{\sigma},\tau)\,(v^{\mathrm{II}}+\epsilon/2)+\mu(\underline{\sigma},\tau)G^{\mathrm{II}}+(1-\pi(\underline{\sigma},\tau)-\mu(\underline{\sigma},\tau))(\gamma^{\mathrm{II}}(\underline{x},\underline{y})+\alpha M) \\ &\leq \gamma^{\mathrm{II}}(\underline{x},\underline{y})+\epsilon/2.\end{aligned} \tag{12}$$

On the other hand

$$\gamma^{\mathrm{II}}(\underline{\sigma},\underline{\tau}) \geq (1-\delta)^2 \gamma^{\mathrm{II}}(\underline{x},\underline{y})-(1-(1-\delta)^2)M \geq \gamma^{\mathrm{II}}(\underline{x},\underline{y})-\epsilon/2. \tag{13}$$

Combining (12) and (13) yields: $\gamma^{\mathrm{II}}(\underline{\sigma},\tau) \leq \gamma^{\mathrm{II}}(\underline{\sigma},\underline{\tau})+\epsilon$ for all τ.
Analogously one can show that $\gamma^{\mathrm{I}}(\sigma,\underline{\tau}) \leq \gamma^{\mathrm{I}}(\underline{\sigma},\underline{\tau})+\epsilon$ for all σ, which proves the lemma. ■

Case C: $(\underline{x},\underline{y})$ is non-absorbing and $\gamma^k(\underline{x},\underline{y}) < G^k$ for $k=\mathrm{I}$ or for $k=\mathrm{II}$.

Without loss of generality, we will assume

$$\gamma^{\mathrm{I}}(\underline{x},\underline{y}) < G^{\mathrm{I}} \tag{14}$$

In case C it holds that $(\underline{x},\underline{y}_\beta)$ is absorbing for β close to 1, whereas $(\underline{x},\underline{y})$ is non-absorbing. This follows because:

Suppose that $(\underline{x},\underline{y}_\beta)$ is non-absorbing for β close to 1. Then by lemma 4, corollary 8 and (3) we have:

$$\gamma^{\mathrm{I}}(\underline{x},\underline{y}) = \lim_{\beta\uparrow 1} \gamma^{\mathrm{I}}_\beta(\underline{x},\underline{y}_\beta) = \lim_{\beta\uparrow 1} \gamma^{\mathrm{I}}_\beta(\underline{x}_\beta,\underline{y}_\beta) = G^{\mathrm{I}}, \tag{15}$$

which contradicts (14).

The main idea in the proof of existence of ϵ-equilibria in case C is to divide each of the strategies $\underline{y}_\beta$ into two separate stationary strategies, one of them is non-absorbing against $\underline{x}$ while the other is absorbing against $\underline{x}$. The limits of these strategies will form the base for the construction of the ϵ-equilibrium strategy of player II.

For β close to 1 define $\tilde{y}_\beta$ and $\tilde{y}^*_\beta \in \mathbf{R}^n$ by:

$$\tilde{y}_{\beta j} := \begin{cases} \underline{y}_{\beta j} & \text{if } (\underline{x},j) \text{ is non-absorbing} \\ 0 & \text{if } (\underline{x},j) \text{ is absorbing} \end{cases} \qquad \text{for } j\in\{1,2,\ldots,n\} \tag{16}$$

$$\tilde{y}^*_{\beta j} := \begin{cases} 0 & \text{if } (\underline{x},j) \text{ is non-absorbing} \\ \underline{y}_{\beta j} & \text{if } (\underline{x},j) \text{ is absorbing} \end{cases} \qquad \text{for } j\in\{1,2,\ldots,n\} \tag{17}$$

Observe that $\underline{y}_\beta=\tilde{y}_\beta+\tilde{y}^*_\beta$, $\tilde{y}^*_\beta\neq 0$ and $\tilde{y}_\beta\neq 0$ for all β close to 1,

$\lim_{\beta\uparrow 1} \tilde{y}^*_\beta=0$ and $\lim_{\beta\uparrow 1} \tilde{y}_\beta=\lim_{\beta\uparrow 1} \underline{y}_\beta = \underline{y}$.

Define y_β and $y^*_\beta \in \mathbf{R}^n$ as the normalizations of $\tilde{y}_\beta$ and $\tilde{y}^*_\beta$:

$$y_{\beta j} := \frac{\tilde{y}_{\beta j}}{\sum_k \tilde{y}_{\beta k}} \text{ for } j\in\{1,2,\ldots,n\} \tag{18}$$

$$y^*_{\beta j} := \frac{\tilde{y}^*_{\beta j}}{\sum_k \tilde{y}^*_{\beta k}} \text{ for } j\in\{1,2,\ldots,n\}. \tag{19}$$

By taking subsequences we may assume that $\lim_{\beta\uparrow 1} y^*_\beta$ exists. Define $y^* := \lim_{\beta\uparrow 1} y^*_\beta$. Obviously $\underline{y} = \lim_{\beta\uparrow 1} \underline{y}_\beta = \lim_{\beta\uparrow 1} y_\beta$.

In the sequel the following definitions will play an important role. First, define ABS$:= \{x;\ C(x) \subset C(\underline{x})$ and (x,y^*) is absorbing$\}$. By definition of y^* it holds that $\underline{x}\in$ABS. Next, for $x\in$ABS define:

$$\lambda_\beta(x) := \frac{1-\beta}{1-\beta+\beta\sum_i \sum_j x_i\, p_{ij}\, \underline{y}_{\beta j}} \text{ and } \lambda(x) := \lim_{\beta\uparrow 1} \lambda_\beta(x),$$

whenever this limit exists. If it does not exist, take some convergent subsequence. It is clear that $\lambda(x)\in[0,1]$.

The following lemma is crucial in our approach.

Lemma 11

For $x\in$ABS: $\lim_{\beta\uparrow 1} \gamma^{\mathrm{I}}_\beta(x,\underline{y}_\beta) = \lambda(x)\gamma^{\mathrm{I}}(x,\underline{y}) + (1-\lambda(x))\,\gamma^{\mathrm{I}}(x,y^*)$.

Proof: From lemma 2, (16) and (17) we obtain:

$$\begin{aligned}\lim_{\beta\uparrow 1} \gamma^{\mathrm{I}}_\beta(x,\underline{y}_\beta) &= \lim_{\beta\uparrow 1} \lambda_\beta(x) \sum_i \sum_j x_i\, a_{ij}(\tilde{y}_{\beta j} + \tilde{y}^*_{\beta j}) \\ &\quad + \lim_{\beta\uparrow 1} \frac{\beta\lambda_\beta(x)}{1-\beta} \sum_i \sum_j x_i\, p_{ij}\, a^*_{ij}\, (\tilde{y}_{\beta j} + \tilde{y}^*_{\beta j}).\end{aligned} \tag{20}$$

From (16) observe that $x_i p_{ij} = 0$ for all i whenever $\tilde{y}_{\beta j} > 0$; hence $\sum_i \sum_j x_i p_{ij} \underline{y}_{\beta j} = \sum_i \sum_j x_i\, p_{ij} \tilde{y}^*_{\beta j}$.

Then from (20), using lemmas 3 and 4, we derive:

$$\lim_{\beta\uparrow 1} \gamma^{\mathrm{I}}_\beta(x,\underline{y}_\beta) = \lambda(x) \sum_i \sum_j x_i\, a_{ij} \underline{y}_j$$

$$+\lim_{\beta\uparrow 1} \frac{\beta\lambda_\beta(x)}{1-\beta} \quad (\sum_i \sum_j x_i p_{ij} \underline{y}_{\beta j}) \frac{\sum_i \sum_j x_i p_{ij} a^*_{ij} \tilde{y}^*_{\beta j}}{\sum_i \sum_j x_i p_{ij} \tilde{y}^*_{\beta j}}$$

$$= \lambda(x)\, \gamma^{\mathrm{I}}\,(x,\underline{y}) \; + \; \lim_{\beta\uparrow 1} \,(1-\lambda_\beta\,(x)) \frac{\sum_i \sum_j x_i p_{ij} a^*_{ij} y^*_{\beta j}}{\sum_i \sum_j x_i p_{ij} y^*_{\beta j}}$$

$$= \lambda(x)\, \gamma^{\mathrm{I}}\,(x,\underline{y}) \; + \; (1-\lambda(x))\, \gamma^{\mathrm{I}}(x,y^*). \quad \blacksquare \tag{21}$$

Using lemma 7 and (3), lemma 11 gives

Corollary 12

$G^{\mathrm{I}} = \lambda(x)\,\gamma^{\mathrm{I}}(x,\underline{y}) + (1-\lambda(x))\,\gamma^{\mathrm{I}}(x,y^*)$ for $x \in \mathrm{ABS}$, and particularly for $\underline{x}$.

Assumption (14) and corollary 12 lead to:

$$\gamma^{\mathrm{I}}(\underline{x},y^*) \geq G^{\mathrm{I}} \text{ and } \lambda(\underline{x}) \neq 1. \tag{22}$$

Also by lemmas 2, 7 and an analogue of lemma 4 we have:

$$\gamma^{\mathrm{II}}\,(\underline{x},y^*) = \lim_{\beta\uparrow 1} \gamma^{\mathrm{II}}_\beta(\underline{x}_\beta,y^*_\beta) = \lim_{\beta\uparrow 1} \gamma^{\mathrm{II}}_\beta(\underline{x}_\beta,\underline{y}_\beta) = G^{\mathrm{II}}. \tag{23}$$

By (22) and (23) it can be seen that, if the players could manage that absorption occurs proportional to $(\underline{x},y^*)$, then both players should be satisfied more or less. Unfortunately, $\underline{x}$ need not be an ϵ-best answer to y^*. Therefore player II has to use a more complex strategy in order to prevent player I from deviating from $\underline{x}$. This strategy will consist of a behavioural strategy on $(\underline{y}, y^*)$, defined like in Kohlberg (1974, p 731), and amplified with a threat strategy as in cases A and B. The strategy will be constructed in such a way that against $\underline{x}$ absorption happens with probability close to 1; hence, the limiting average reward to player $k \in \{\mathrm{I},\mathrm{II}\}$ will be close to $\gamma^k(\underline{x},y^*)$.

For $\epsilon > 0$ let τ_ϵ be the strategy defined by:

If at stage N absorption has not yet occurred, then with probability $\epsilon^2 f(m_N)$ play y^* and with probability $1-\epsilon^2 f(m_N)$ play $\underline{y}$. Here $f(m) := (1-\epsilon)^m$ for $m \in [0,\infty)$ and m_N is given by (cf. Kohlberg (1974, p 731)):

$$m_N := \begin{cases} 0 & \text{for } N=1 \\ \max\ \{0, \sum_{n=1}^{N-1} (\sum_j p_{i_n j}\, a^*_{i_n j}\, y^*_j - \sum_j p_{i_n j} y^*_j \gamma^{\mathrm{I}}(\underline{x},y^*))\} & \text{for } N>1 \end{cases}$$

where i_n is the row chosen by player I at stage $n \in \{1,2,\ldots,N-1\}$.

Lemma 13

For all $\epsilon > 0$ and $k \in \{I, II\}$: under $(\underline{x}, \tau_\epsilon)$ absorption happens with probability 1 and $\gamma^k(\underline{x}, \tau_\epsilon) = \gamma^k(\underline{x}, y^*)$.

Proof: If $(\underline{x}, \tau_\epsilon)$ is being played, then m_N would be 0 for infinitely many N. Hence, player II would play y^* with probability ϵ^2 infinitely often. This results in absorption according to $(\underline{x}, y^*)$ with probability 1. ■

Lemma 14

For all $\epsilon > 0$ and all strategies σ of player I:
$\mathrm{Prob}_{\sigma,\tau_\epsilon}\{\mathbf{A} < \infty\}\ E_{\sigma,\tau_\epsilon}\{\liminf_{N\to\infty} \mathbf{g}_N - \gamma^{I}(\underline{x}, y^*) \mid \mathbf{A} < \infty\} \le \epsilon$, where $\mathbf{A}$ is the random variable denoting the stage at which absorption takes place, and $\mathbf{g}_N$ is the random variable denoting the average of the first N payoffs to player I.

Proof: Analogous to the proof of (2.7) in Kohlberg (1974, p 732-733). ■

We are now ready to define for $\epsilon > 0$ a pair of strategies $(\underline{\sigma}, \underline{\tau})$, which is an ϵ-equilibrium in case C, as will be shown in lemma 15. Let $\epsilon > 0$. Take $\delta > 0$ and $N_\delta \in \mathbf{N}$ such that:

i) $(1-\delta)\gamma^k(\underline{x}, \tau_{\epsilon/2}) - \delta M \ge \gamma^k(\underline{x}, \tau_{\epsilon/2}) - \epsilon/2$ for $k = I, II$
ii) under $(\underline{x}, \tau_{\epsilon/2})$ absorption will happen before stage N_δ with probability $\ge 1-\delta$. (cf. lemma 13).

Define $\underline{\sigma}$ (resp. $\underline{\tau}$) by:
play $\underline{x}$ (resp. $\tau_{\epsilon/2}$) unless
a) no absorption occurs before stage N_δ
b) player II (resp. I) chooses an action outside $C(\underline{y}) \cup C(y^*)$ (resp. $C(\underline{x})$).
In case (a) or (b) immediately start playing an $\epsilon/2$-threat strategy $\sigma^*(\epsilon/2)$ (resp. $\tau^*(\epsilon/2)$), as defined in definition 1.

Lemma 15

In case C the pair $(\underline{\sigma}, \underline{\tau})$, as defined above, is a limiting average ϵ-equilibrium corresponding with the NEP $(\gamma^{I}(\underline{x}, y^*), \gamma^{II}(\underline{x}, y^*))$.

Proof: By the definitions of $\underline{\sigma}$ and $\underline{\tau}$ and by lemma 13:

$$\gamma^{I}(\underline{\sigma}, \underline{\tau}) \ge \gamma^{I}(\underline{x}, y^*) - \epsilon/2 \text{ and } \gamma^{II}(\underline{\sigma}, \underline{\tau}) \ge \gamma^{II}(\underline{x}, y^*) - \epsilon/2 = G^{II} - \epsilon/2 \tag{24}$$

Part 1: Player II cannot gain more than ϵ by deviating against $\underline{\sigma}$. Because: Suppose player I uses $\underline{\sigma}$ and player II uses an arbitrary strategy τ. Let $\mu(\underline{\sigma}, \tau)$ be the probability of absorption before stage N_δ.
Then by lemma 6, the definition of $\underline{\sigma}$ and (6):

$$\gamma^{II}(\underline{\sigma}, \tau) \le \mu(\underline{\sigma}, \tau) G^{II} + (1 - \mu(\underline{\sigma}, \tau))\,(v^{II} + \epsilon/2) \le G^{II} + \epsilon/2. \tag{25}$$

Hence, combining (24) and (25) we have shown:

$$\gamma^{\mathrm{II}}(\underline{\sigma},\tau) \le \gamma^{\mathrm{II}}(\underline{\sigma},\underline{\tau}) + \epsilon \text{ for all strategies } \tau. \tag{26}$$

Part 2: Player I cannot gain more than ϵ by deviating against $\underline{\tau}$. Because: First of all, observe that at each stage $n \le N_\delta$ player II uses y^* with probability less than $(\epsilon/2)^2$. Without loss of generality we can assume $\epsilon \le 1/M$, and hence $(\epsilon/2)^2 \le \epsilon/4M$.

Now, suppose player II uses $\underline{\tau}$ and player I uses an arbitrary strategy σ. Under $(\sigma,\underline{\tau})$ realisations of two types may occur:
type (a): realisations for which player I at some stage before N_δ chooses an action outside $C(\underline{x})$,
type (b): realisations for which player I only chooses actions within $C(\underline{x})$ until stage N_δ.

Let $\tilde{\mu}(\sigma,\underline{\tau})$ be the probability that player I chooses some action outside $C(\underline{x})$ before stage N_δ and let $\mu(\sigma,\underline{\tau})$ be the probability of absorption in such a case.

The contribution to the limiting average reward of player I, by realisations of type (a), is at most (using the player I version of lemma 6, the definition of $\underline{\tau}$ and (6)):

$$\begin{aligned}&\tilde{\mu}(\sigma,\underline{\tau})[\mu(\sigma,\underline{\tau})\,((1-\epsilon/4M)G^{\mathrm{I}} + (\epsilon/4M)M) + (1-\mu(\sigma,\underline{\tau}))\,(v^{\mathrm{I}}+\epsilon/2)]\\ &\le \tilde{\mu}(\sigma,\underline{\tau})\,(G^{\mathrm{I}} + \epsilon/2).\end{aligned} \tag{27}$$

In order to determine the contribution of realisations of type (b), we distinguish realisations of this type with absorption before stage N_δ, and realisations of this type without absorption before stage N_δ. Let $\pi(\sigma,\underline{\tau})$ be the probability that no absorption occurs before stage N_δ. As a consequence of lemma 14, the contribution of realisations of type (b) with absorption before stage N_δ is at most

$$(1-\tilde{\mu}(\sigma,\underline{\tau})-\pi(\sigma,\underline{\tau}))\,(\gamma^{\mathrm{I}}(\underline{x},y^*)+\epsilon/2). \tag{28}$$

The contribution of realisations of type (b) without absorption before stage N_δ is at most:

$$\pi(\sigma,\underline{\tau})\,(v^{\mathrm{I}}+\epsilon/2) \tag{29}$$

Combining (27), (28) and (29) we have:

$$\begin{aligned}\gamma^{\mathrm{I}}(\sigma,\underline{\tau}) &\le \tilde{\mu}(\sigma,\underline{\tau})\,(G^{\mathrm{I}}+\epsilon/2)+(1-\tilde{\mu}(\sigma,\underline{\tau})-\pi(\sigma,\underline{\tau}))\,(\gamma^{\mathrm{I}}(\underline{x},y^*)+\epsilon/2)\\ &\quad + \pi(\sigma,\underline{\tau})\,(v^{\mathrm{I}}+\epsilon/2) \le \gamma^{\mathrm{I}}(\underline{x},y^*)+\epsilon/2.\end{aligned} \tag{30}$$

Hence, combining (30) with (24) we have shown that:

$$\gamma^{\mathrm{I}}(\sigma,\underline{\tau}) \leq \gamma^{\mathrm{I}}(\underline{\sigma},\underline{\tau}) + \epsilon \text{ for all strategies } \sigma. \tag{31}$$

The lemma now follows from (24), (26) and (31). ∎

4 Remarks and Examples

Remark 1

If in case C it holds that $\lambda(\underline{x})>0$, then $\lambda_i := \lambda(i) > 0$ for all $i \in \mathrm{ABS}$. Moreover, from the definition of $\lambda(\ )$ one can derive straightforward that:

If $x \in \mathrm{ABS}$ and $C(x) \subset \mathrm{ABS}$, then

$$\lambda(x) = 1/(\sum_{i \in C(x)} x_i/\lambda_i).$$

Remark 2

If in case C it holds that $\lambda(\underline{x})=0$, then $\lambda_i = 0$ for all $i \in \mathrm{ABS}$. Hence, corollary 12 implies $\gamma^{\mathrm{I}}(i,y^*) = \gamma^{\mathrm{I}}(\underline{x},y^*)$, which means that within $C(\underline{x})$ player I has no profitable deviations against y*.

Therefore $(\underline{x},(1-p)\underline{y}+py^*)$, supplemented which threats as in case A, will constitute an ϵ-equilibrium for $p \in (0,1)$ small enough.

Remark 3

In cases A, B and C the limiting average ϵ-equilibria can be seen as uniform ϵ-equilibria (cf. section 1) by taking uniform $\epsilon/2$-threat strategies. The latter is possible because Kohlberg (1974) shows that there exist uniform limiting average ϵ-optimal strategies in zerosum repeated games with absorbing states.

Remark 4

The limiting average ϵ-equilibria we constructed can also be seen as ϵ-equilibria with respect to the alternative criterion $\liminf_{T\to\infty} \frac{1}{T} \sum_{n=1}^{T} E_{s\sigma\tau}(R^k(n))$. This is due to the fact that Kohlberg (1974) shows that, for zerosum repeated games with absorbing states, there exist strategies which are ϵ-optimal for both criteria. Using such threat strategies in all cases, and such a behavioural "Kohlberg" strategy in case C, gives the result.

Remark 5

If for a general stochastic game we can take a sequence of stationary β-discounted equilibria $(\underline{x}_\beta,\underline{y}_\beta)$ which converges to some pair $(\underline{x},\underline{y})$, and if $\gamma^k(s,\underline{x},\underline{y})\geq G^k(s)$ for $k\in\{\mathrm{I},\mathrm{II}\}$, for all initial states s, and if there are no transient states, then one can easily construct a limiting average ϵ-equilibrium. This can be done along the same lines as in cases A and B.

Examples

We will conclude this paper by giving examples to illustrate that each of the discerned cases A, B, C with $\lambda(\underline{x})>0$ and C with $\lambda(\underline{x})=0$, can actually occur. Let us consider the following game

0,0	−1,0
−1,0	0,0 1→0,0

in which absorption can only occur in entry (2,2).

To illustrate case A notice that $(\underline{x}_\beta,\underline{y}_\beta)=((0,1),\ (0,1))$, $\beta\in(0,1)$, is a sequence of stationary β-discounted equilibria with the desired properties.

We get an example of case B, when in the above game we take $(\underline{x}_\beta,\underline{y}_\beta)\ =\ ((1,0),(1,0))$, $\beta\in(0,1)$.

The above game also contains an example for case C with $\lambda(\underline{x})=0$: take $(\underline{x}_\beta,\underline{y}_\beta)$ $= ((0,1),\ (\dfrac{1-(1-\beta)^{1/2}}{\beta},\dfrac{-1+\beta+(1-\beta)^{1/2}}{\beta}))$, $\beta\in(0,1)$.

So, the only case that remains to be illustrated is case C with $\lambda(\underline{x})>0$. Sorin (1986) examines the following game:

0,1	1,0 1→1,0
1,0	0,2 1→0,2

in which absorption can only occur in the second column.

Here the unique stationary β-discounted equilibria are given by $(\underline{x}_\beta,\underline{y}_\beta)\ =\ ((2/3,1/3),(1/(2-\beta),(1-\beta)/(2-\beta)))$, $\beta\in(0,1)$. It can easily be verified that all the conditions of case C with $\lambda(\underline{x})>0$ are fulfilled.

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