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On Optimum Regularity of Navier-Stokes Solutions at Time $t=0$

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1. Introduction

For the efficient numerical treatment of viscous flow problems it would be highly desirable to work with solutions of the Navier-Stokes initial-boundary value problem, which are as smooth as possible at least at the initial time $t=0$. However in [6, p. 243] Heywood has drawn attention to a compatibility condition, which must be fulfilled by the initial value of any Navier-Stokes solution being strongly H_3 - continuous at time $t=0$. In the joint paper [7, p. 277] with Rannacher the authors pointed out, that because of its non-linear and non-local nature, this condition "is virtually uncheckable for given data". A compatibility condition of this type has already been formulated in Ladyženskaya's book [12, p. 168][1]. Recently Temam [22] has shown this type of a non-local condition to be not only necessary, but also (in the case of initial values from H_3) sufficient for strong H_3-continuity at time $t=0$ for solutions of a class of semilinear evolution equations.

The following note gives an answer to the question, how smooth a Navier-Stokes solution can be at time $t=0$ *without* any compatibility condition mentioned above. For this aim (roughly spoken) we measure the smoothness of a vector function $u(t)$ by means of the exponent α of the highest fractional power A^α of the Stokes operator A, which is applicable on $u(t)$, the function $A^\alpha u(t)$ being strongly continuous at $t=0$.

We will see (Theorems 3.1, 4.1 below), that the exponents being possible without the above compatibility condition are $\alpha<\frac{5}{4}$. This follows with methods from Fujita and Kato's work [4] together with results on interpolation spaces from Lions and Magenes' book [14].

By the way we will find, that Heywood's and Temam's formulations of the compatibility condition differ in so far, as Heywood's condition follows from the strong continuity of the external force $f(t)$ itself at time $t=0$, while Temam's condition is a consequence of the strong continuity of Weyl's orthogonal projection of $f(t)$ at $t=0$, c.p. Sect. 4 below.

[1] c.p. also Solonnikov [20, p. 97]

2. Local Strong Navier-Stokes Solutions. Notations

Let Ω be a bounded open set in the (x^1, x^2, x^3)-space $\mathbb{R}^3$, the boundary $\partial\Omega$ being a compact 2-dimensional C_3-submanifold of $\mathbb{R}^3$. The velocity vector $u(t, x) = (u^1, u^2, u^3)$ and the pressure function $p(t, x)$ of a nonstationary incompressible flow in Ω at times $t \geqq 0$ solve the Navier-Stokes initial-boundary value problem

$$\frac{\partial}{\partial t} u - \Delta u + \nabla p = f - u\nabla u, \qquad \nabla \cdot u = 0 \text{ for } t > 0,$$

$$u_{|\partial\Omega} = 0, \qquad u(0, \cdot) = u_0 \tag{2.1}$$

with the prescribed (density of the) external force $f(t, x) = (f^1, f^2, f^3)$, if we assume the condition of adherence on $\partial\Omega$, and if distance and time are measured in the appropriate units.

A particularly adequate framework for (2.1) give the Hilbert spaces H_m of vector functions (defined almost everywhere) on Ω, which belong to Lebesgue's class $L^2(\Omega)$ together with their spatial derivatives up to the order $m = 0, 1, \ldots$. We write the norm

$$|f|_{H_m} = \Big(\sum_{|n| \leqq m} \int_\Omega |\partial_x^n f(x)|^2 \, dx \Big)^{1/2}$$

on H_m with the usual multi-index $n = (n_1, n_2, n_3)$ containing the integers $n_j \geqq 0$, $|n| = n_1 + n_2 + n_3$, where $|\partial_x^n f(x)|$ stands for the Euclidean norm of the vector $(\partial^{|n|} f(x)/(\partial x^1)^{n_1} (\partial x^2)^{n_2} (\partial x^3)^{n_3}) \in \mathbb{R}^3$. For the $L^2(\Omega)$-norm we will write $\|f\| = |f|_{H_0}$.

By $\mathscr{H}_m$ we denote the closure in H_m of the linear space $D(\Omega)$ of divergence-free C_∞-vector functions having compact support in Ω, P being Weyl's orthogonal projection of H_0 on $\mathscr{H}_0$. Finally let A be Friedrichs' selfadjoint extension of the positive definite, symmetric operator $-P\Delta$ in $\mathscr{H}_0$, with D_A (or D_{A^α}) denoting the domain in $\mathscr{H}_0$ of the "Stokes operator" A (or of the fractional power A^α, respectively, for any real $\alpha \geqq 0$). We recall, that D_{A^α} is a Hilbert space equipped with the usual graph norm $(\|f\|^2 + \|A^\alpha f\|^2)^{1/2} = |f|_{D_{A^\alpha}}$.

With these notations and since P commutes with the strong time derivative ∂_t, the Navier-Stokes initial-boundary value problem (1.1) leads to the evolution equation

$$(\partial_t + A)u = P(f - u\nabla u), \qquad t > 0, \ u(0) = u_0 \tag{2.2}$$

for the $\mathscr{H}_0$-valued function $u(t) = u(t, \cdot)$.

Because we are interested in Navier-Stokes solutions which are as regular as possible at $t = 0$ without any additional compatibility conditions, we assume $u_0 \in \mathscr{H}_1 \cap H_2$ for the initial value, the external force $f\colon [0, \infty) \to H_0$ being uniformly Hölder-continuous with Hölder-exponent $\nu \in (0, 1)$, i.e. $f \in C_\nu([0, \infty), H_0)$.

More generally for any fixed interval $J \subset \mathbb{R}^1$ and any Banach space H with norm $|\cdot|_H$, we will write $C_\nu(J, H)$ for the class of strongly Hölder-continuous functions $f\colon J \to H$ with

$$[f]_\nu = \sup_{\substack{t, s \in J \\ 0 < |t-s| < 1}} \{ |f(t) - f(s)|_H \cdot |t-s|^{-\nu} \} < \infty .$$

By $C_0(J, H)$ we denote the usual Banach space of continuous functions $f: J \to H$.

Then from Fujita and Kato's work [4, p. 293, 303, 312] we know, that on a (possibly small) time interval $[0, T]$, $T>0$, the unique strong solution $u(t) \in \mathscr{H}_1 \cap H_2$ of (2.2) exists, u together with the associated gradient ∇p representing the unique solution of (2.1) on $[0, T]$. For u, the inequalities

$$\text{(a)}\ \|A^\alpha u(t)\| \leqq c_\alpha \quad \text{for } t \in [0, T] \quad \text{and} \quad \text{(b)}\ [A^\alpha u]_\mu \leqq c_{\alpha,\mu} \tag{2.3}$$

hold with any $\alpha \in [0,1)$, $\mu \in (0,1)$ if $\alpha+\mu<1$, the bounds c_α, $c_{\alpha,\mu}>0$ depending only on $|u_0|_{H_2}$, Ω, T, α and μ. In the following, by c, c_0, c_1, ... we will denote positive constants, the value of which may be different in different sections.

3. Strongly Continuous Navier-Stokes Solutions in $D_{A^{1+\varepsilon}}$, $\varepsilon \in [0, \frac{1}{4})$

Let u, ∇p be the local solution of (2.1), u existing on a time interval $[0, T]$, on which (2.3) is valid. We prove

Theorem 3.1. *Assume* $u_0 \in D_{A^{1+\varepsilon}}$, $\varepsilon \in [0, \frac{1}{4})$, $Pf \in C_\mu([0, \infty), D_{A^\varepsilon})$ *with Hölder-exponent* $\mu \in (0,1]$. *Then* $u \in C_0$ $([0, T], D_{A^{1+\varepsilon}})$ *holds. If, in addition* $f \in C_0$ $([0, \infty), H_0)$, *then the unique pressure gradient from* (2.1) *is*

$$\nabla p = (1-P)(f - u \nabla u + \Delta u) \in C_0([0, T], H_0). \tag{3.1}$$

For the *proof*, we use the integral equation

$$\begin{aligned} u(t) &= e^{-tA} u_0 - A^{-1}(1-e^{-tA}) P(u \nabla u - f)(t) \\ &\quad - \int_0^t e^{-(t-s)A} P((u \nabla u - f)(s) - (u \nabla u - f)(t))\, ds \\ &= I_1 + I_2 + I_3, \end{aligned} \tag{3.2}$$

which follows from the integral Eq. (1.11) in Fujita-Kato [4, p. 272] because of the identity

$$\int_0^t e^{-(t-s)A} ds = A^{-1}(1-e^{-tA}),$$

[8, p. 489]. Since the strong Hölder continuity of $P(u \nabla u - f) \in C_\nu([0, T], \mathscr{H}_0)$ guarantees the existence and also a Hölder estimate of $A^\alpha I_3$ even for exponents $\alpha \in [1, 1+\nu)$ [4, p. 281, Lemma 2.13], the belonging of the projection $P(u \nabla u)$ from the second term I_2 in (3.2) to D_{A^ε} will be the decisive point for the following estimates. The statement on $P(u \nabla u) \in D_{A^\varepsilon}$ for any positive $\varepsilon < \frac{1}{4}$ will follow in the framework of Lions and Magenes' interpolation spaces [14]. Results in more general Banach spaces are due to Kielhöfer [10] and v. Wahl [23, 24].

The proof of Theorem 3.1 results from the following Lemmata 3.1–3.4 and Corollaries 3.1–3.3. Firstly we state

Remark 3.1. The equality $\mathscr{H}_1 \cap H_2 = D_A$ holds, since the operator A is defined on $\mathscr{H}_1 \cap H_2$ with values in $\mathscr{H}_0$ and, due to Cattabriga's Theorem [2], any solution v of $Av = g \in \mathscr{H}_0$ belongs to $\mathscr{H}_1 \cap H_2$, c.p. [15, p. 299, 324].

Information on the existence of $A^\varepsilon P(u\nabla u)$ gives us

Lemma 3.1. *Assume $v, w \in D_A$.*

Then $P(v\nabla w) \in D_{A^\varepsilon}$ holds for all $\varepsilon \in [0, \frac{1}{4})$, and the estimate

$$\|A^\varepsilon P(v\nabla w)\| \leqq c\,\|Av\|\,\|Aw\| \tag{3.3}$$

is valid.

For the *proof*, using Hölder's inequality, the multiplicative inequality $|\nabla v|_{L^4} \leqq c_0 \|\nabla\nabla v\|^{3/4} \|\nabla v\|^{1/4}$ [13, p. 62, 63], Sobolev's inequality $|v|_{L^\infty} \leqq c_1 |v|_{H_2}$ and the special form

$$|v|_{H_2} \leqq c_2 \|Av\| \quad \text{for } v \in D_A \tag{3.4}$$

of Cattabriga's inequality [2], we find $v\nabla w \in H_1$ and

$$\begin{aligned} |v\nabla w|_{H_1} &\leqq |v|_{L^\infty} \cdot \|\nabla w\| + |\nabla v|_{L^4} |\nabla w|_{L^4} + |v|_{L^\infty} \|\nabla\nabla w\| \\ &\leqq c_3 \|P\Delta v\| \cdot \|P\Delta w\| \end{aligned}$$

for any $v, w \in D_A$. Therefore [21, p. 18] the projection $P(v\nabla w)$ belongs to the Hilbert space $H_1 \cap \mathscr{H}_0 = PH_1$, and the estimate

$$|P(v\nabla w)|_{H_1} \leqq c_4 \|Av\| \cdot \|Aw\| \tag{3.5}$$

holds.

The imbedding Theorems for fractional order spaces $H_s(\Omega)$ with norm

$$|\varphi|_{H_s} = \left(|\varphi|^2_{H_{[s]}} + \sum_{|n|=[s]} \int_{\Omega\times\Omega} \frac{|\partial^n_x \varphi(x) - \partial^n_y \varphi(y)|^2}{|x-y|^{3+2(s-[s])}}\, dx\, dy \right)^{1/2}$$

for the integer $[s] < s < [s]+1$ show, that the Hilbert space $H_1 \cap \mathscr{H}_0$ is continuously imbedded in $H_s(\Omega) \cap \mathscr{H}_0$ if $s \in [0, 1]$. In the case $s \in [0, \frac{1}{2}]$ the space $H_s(\Omega)$ coincides with the closure $\mathring{H}_s(\Omega)$ in $H_s(\Omega)$ of the space of C_∞-vector functions having compact support in Ω, Lions-Magenes [14, p. 55, Theorem 11.1].[2]

On the other side, $\mathring{H}_s(\Omega)$ is (with equivalent norm) the interpolation space $[\mathring{H}_1(\Omega), H_0]_{1-s} = D_{B^{s/2}}$ for $s \in [0, 1]$, $s \neq \frac{1}{2}$, Lions-Magenes [14, p. 64, Theorem 11.6],[2] $D_{B^{1/2}} = \mathring{H}_1(\Omega)$ being the domain of the square root of the Laplacian $B = (-\Delta)$ in H_0. Finally, Fujita and Morimoto [25] have shown $D_{B^{s/2}} \cap \mathscr{H}_0 = D_{A^{s/2}}$ for $s \in (0, 2)$. Therefore we have $H_s(\Omega) \cap \mathscr{H}_0 = D_{A^{s/2}}$, thus $P(v\nabla w) \in D_{A^\varepsilon}$ for $\varepsilon = \frac{s}{2} \in [0, \frac{1}{4})$ and in addition (3.3) holds because of (3.5) and the continuity of the imbedding $H_1 \subset H_s$. An immediate consequence is

Corollary 3.1. *For any Hölder-continuous solution $u \in C_\nu([0, T], D_A)$ of (2.2) with $\nu > 0$, the estimate*

$$[A^\varepsilon P(u\nabla u)]_\nu \leqq c_5 \sup_{t\in[0,T]} \|Au(t)\|\,[Au]_\nu$$

holds.

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[2] The proof of this Theorem in [14] is valid also under our assumption on $\partial\Omega$

For the *proof*, we only have to use (3.3) with $v=u(t)-u(s)$, $w=u(t)$ or $w=u(s)$, respectively, for t, $s\in[0,T]$.

Remark 3.2. From Fujita and Kato's result

$$|v|_{L^\infty}\leqq c\,\|A^\gamma v\|, \qquad \|P(v\nabla v)\|\leqq c\,\|A^{1/2}v\|\,\|A^\gamma v\|,$$
$$\|P(v\nabla v-w\nabla w)\|\leqq c(\|A^\gamma v\|\,\|A^{1/2}(v-w)\|+\|A^{1/2}w\|\,\|A^\gamma(v-w)\|) \tag{3.6}$$

for any real number $\gamma>\frac{3}{4}$, v, $w\in D_{A^\gamma}$ [4, p. 273, Lemma 1.2], the Hölder estimate

$$[P(u\nabla u)]_\nu\leqq c_6 \tag{3.7}$$

follows for the local strong solution $u\in C_\nu([0,T],D_{A^\gamma})$ of (2.2) for any $\nu\in(0,\frac{1}{4})$, $\gamma\in(\frac{3}{4},1)$ with $\gamma+\nu<1$, $c_6=c(c_\gamma c_{\frac{1}{2},\nu}+c_{\frac{1}{2}}c_{\gamma,\nu})$, c_α, $c_{\alpha,\mu}$ denoting the constants from (2.3).

Lemma 3.2. *Assume* $u_0\in D_{A^{1+\varepsilon}}$, $Pf\in C_\mu([0,\infty),D_{A^\varepsilon})$, $\varepsilon\in(0,\frac{1}{4})$. *Then we have* $[Au]_\nu<\infty$ *for any* $\nu<\min(\varepsilon,\mu,\frac{1}{4})$.

For the *proof*, firstly we remark, that we may apply the operator A on both sides of (3.2). Namely, besides I_2 also the term I_1 in (3.2) belongs to D_A, A commuting with the exponential e^{-tA} of this operator. Finally $I_3\in D_A$ follows by [4, p. 281, Lemma 2.13] from the Hölder continuity of $P(u\nabla u)$ and Pf, which we have stated in Remark 3.2 or assumed for Pf, respectively.

Putting $t=s+h$ for t, $s\in[0,T]$ and $h>0$ we find the estimates

$$\|A(I_1(s+h)-I_1(s))\|=\|e^{-sA}(e^{-hA}-1)Au_0\|\leqq\frac{h^\varepsilon}{\varepsilon}\|A^{1+\varepsilon}u_0\|,$$

using [4, p. 280, Lemma 2.11], and

$$\begin{aligned}\|A(I_2(t)-I_2(s))\|&\leqq\|(1-e^{-tA})P((u\nabla u-f)(t)-(u\nabla u-f)(s))\|\\&\quad+\|e^{-sA}(1-e^{-hA})P(u\nabla u-f)(s)\|\\&\leqq 2([P(u\nabla u)]_\nu h^\nu+[Pf]_\mu h^\mu)+\frac{h^\varepsilon}{\varepsilon}(\|A^\varepsilon P(u\nabla u)(s)\|+\|A^\varepsilon Pf(s)\|)\\&\leqq c_7h^\nu \qquad \text{if } h\in[0,1]\end{aligned}$$

for any $\nu<\min(\varepsilon,\mu,\frac{1}{4})$ by Lemma 3.1, Remark 3.2 and the above Lemma from [4] again. Under the same restriction, a bound for $[AI_3]_\nu$ follows by [4, p. 281 Lemma 2.13] from the Hölder continuity of $P(u\nabla u)$ and Pf.

Lemma 3.3. *Assume* $u_0\in D_{A^{1+\varepsilon}}$, $Pf\in C_\mu([0,\infty),D_{A^\varepsilon})$, $\varepsilon\in[0,\frac{1}{4})$, $\mu>0$. *Then* $\|A^{1+\varepsilon}u(t)\|$ *is uniformly bounded on* $[0,T]$.

For the *proof*, by the same conclusion as above we see from Lemma 3.1 and Corollary 3.1 with Lemma 3.2, that u from (3.2) belongs to $D_{A^{1+\varepsilon}}$, $\varepsilon\in[0,\frac{1}{4})$. For any such ε we have $u_0\in D_A$ because of the continuous imbedding $D_{A^{1+\varepsilon}}\subset D_A$ following from the momentum inequality [3, p. 159].

(a) Therefore a uniform bound on $[0,T]$ for $\|Au(t)\|$ results immediately from (3.2), if for the terms AI_2 and AI_3 we recall (3.6) or (3.7) and [4, p. 281

Lemma 2.13], respectively, and because of

$$\|A^{1+\varepsilon}I_1(t)\| \leqq \|A^{1+\varepsilon}u_0\|.$$

(b) In the case $\varepsilon \in (0, \frac{1}{4})$, from Lemma 3.1 and the uniform bound for $\|Au(t)\|$ on $[0, T]$ just established we get

$$\begin{aligned}\|A^{1+\varepsilon}I_2(t)\| &\leqq \|(1-e^{-tA})A^{\varepsilon}P(u\nabla u - f)(t)\| \\ &\leqq 2(\|A^{\varepsilon}P(u\nabla u)(t)\| + \|A^{\varepsilon}Pf(t)\| \leqq c_8.\end{aligned}$$

The estimate

$$\|A^{1+\varepsilon}I_3(t)\| \leqq c_9 \cdot ([A^{\varepsilon}P(u\nabla u)]_v + [A^{\varepsilon}Pf]_{\mu}) \leqq c_{10}$$

follows by means of Corollary 3.1 with Lemma 3.2 and [4, p. 281, Lemma 2.13].

In addition, a consequence of this Lemma of Fujita and Kato is

Corollary 3.2. *The Hölder estimate* $[A^{1+\varepsilon}I_3]_v \leqq c_{11}$ *holds for all* $v < \min(\varepsilon, \mu, \frac{1}{4})$ *if* $\varepsilon \in (0, \frac{1}{4})$ *and, additionally, for all* $v < \min(\mu, \frac{1}{4})$ *if* $\varepsilon = 0$.

Namely for the *proof* in case $\varepsilon \in (0, \frac{1}{4})$ we recall $A^{\varepsilon}P(u\nabla u - f) \in C_{\lambda}([0, T], \mathscr{H}_0)$ for any $\lambda < \min(\varepsilon, \mu, \frac{1}{4})$ by Lemma 3.2 and Corollary 3.1 and by our assumption on Pf. Additionally recalling Remark 3.2 we find $P(u\nabla u - f) \in C_{\lambda}([0, T], \mathscr{H}_0)$ for $\lambda \leqq \mu$, $\lambda < \frac{1}{4}$.

Lemma 3.4. *Assume* $u_0 \in D_{A^{1+\varepsilon}}$, $Pf \in C_{\mu}([0, \infty), D_{A^{\varepsilon}})$, $\varepsilon \in [0, \frac{1}{4})$, $\mu > 0$. *Then* $A^{1+\varepsilon}u \in C_0([0, T], \mathscr{H}_0)$ *holds.*

Because of the last Corollary, for the *proof* we only have to verify the strong continuity of the terms $A^{1+\varepsilon}I_1$ and $A^{1+\varepsilon}I_2$ in the representation of $A^{1+\varepsilon}u$ from (3.2): Since $A^{1+\varepsilon}$ commutes with e^{-tA}, the exponential being strongly continuous on $[0, T]$, we have

$$\|A^{1+\varepsilon}(I_1(s+h) - I_1(s))\| = \|(e^{-hA} - 1)v\| \to 0$$

if $h \downarrow 0$ with $v = e^{-sA}A^{1+\varepsilon}u_0$.

Finally, recalling Lemma 3.2 and Corollary 3.1 or Remark 3.2 in the cases $\varepsilon \in (0, \frac{1}{4})$ or $\varepsilon = 0$, respectively, we are lead to the inequalities

$$\begin{aligned}&\|A^{1+\varepsilon}(I_2(s+h) - I_2(s))\| \\ &\quad \leqq \|(1-e^{-tA})A^{\varepsilon}P((u\nabla u - f)(s+h) - (u\nabla u - f)(s))\| \\ &\qquad + \|(1-e^{-hA})e^{-sA}A^{\varepsilon}P(u\nabla u - f)(s)\| \\ &\quad \leqq c_{12} \cdot h^v + \|(1-e^{-hA})v\| \to 0\end{aligned}$$

for $h \downarrow 0$ with $v = A^{\varepsilon}P(u\nabla u - f)(s)$, and similarly for $A^{1+\varepsilon}(I_j(t) - I_j(t-h))$, $j = 1, 2$.

Corollary 3.3. (a) *Under the assumptions of Lemma* 3.4 *we have* $\partial_t u \in C_0([0, T], \mathscr{H}_0)$.

(b) *If in addition* $f \in C_0([0, T], H_0)$, *then also* (3.1) $\nabla p = (1-P)(f - u\nabla u + \Delta u) \in C_0([0, T], H_0)$ *holds.*

For the *proof* of (a) we state, that in the evolution Eq. (2.2) the terms $P(u\nabla u)$ and Au are strongly continuous on $[0,T]$ by Remark 3.2 or Lemma 3.4 with $\varepsilon=0$, respectively (for the latter recall $u_0 \in D_A$ because of the continuous imbedding $D_{A^{1+\varepsilon}} \subset D_A$ mentioned in the proof of Lemma 3.3).

(b) By means of Cattabriga's estimate (3.4) we see, that the unique solution $u(t) \in H_2$, $\nabla p(t) \in H_0$ of

$$-\Delta u + \nabla p = g(t), \qquad \nabla \cdot u = 0, \qquad u|_{\partial\Omega} = 0$$

with $g=(f-u\nabla u-\partial_t u) \in C_0([0,T],H_0)$ is strongly continuous. Therefore we have

$$\Delta u \in C_0([0,T],H_0) \quad \text{and} \quad \nabla p \in C_0([0,T],H_0).$$

Finally the Eq. (3.1) follows by subtraction of (2.2) from (2.1) because of $A=-P\Delta$ for $u(t) \in \mathcal{H}_1 \cap H_2$.

Evidently, Lemma 3.4 and the last Corollary verify Theorem 3.1.

4. The Compatibility Conditions for Navier-Stokes Solutions, which are Strongly Continuous in $D_{A^{1+\varepsilon}}$ at $t=0$, $\varepsilon > \frac{1}{4}$

Let u, ∇p be the local solution of (2.1) on a time interval $[0,T]$, on which (2.3) is valid. We prove

Theorem 4.1. *Assume* $u \in C_0([0,T],D_{A^{1+\varepsilon}})$, $\varepsilon > \frac{1}{4}$, $Pf \in C_0([0,T],D_{A^\varepsilon})$. *Then the compatibility condition*

$$(Pf(0) - P(u_0\nabla u_0) + P\Delta u_0)|_{\partial\Omega} = 0 \qquad [22, \text{p. } 20] \tag{4.1}$$

holds in the sense of the fractional order space $H_{2\varepsilon-\frac{1}{2}}(\partial\Omega)$. *If, in addition,* $f \in C_0([0,T],H_{2\varepsilon})$, *then the compatibility condition*

$$(-\Delta u_0 + \nabla p(0))|_{\partial\Omega} = f(0)|_{\partial\Omega} \qquad [7, \text{p. } 14] \tag{4.2}$$

follows with $\nabla p(0)$ *from* (3.1).

For the *proof*, because of the continuous imbedding $D_{A^\alpha} \subset D_{A^\beta}$ for any $\beta < \alpha$, it suffices to consider the cases $\varepsilon \in (\frac{1}{4}, \frac{1}{2}]$, thus

$$D_{A^{s/2}} = [D_{A^{1/2}}, \mathcal{H}_0]_{1-s} = \mathring{H}_s(\Omega) \cap \mathcal{H}_0 \subset H_s$$

with $s = 2\varepsilon \in (\frac{1}{2}, 1]$, [14, p. 64, Theorem 11.6] and [25].[3]

Recalling the continuous imbedding $H_1 \subset H_s$, we conclude the strong continuity in H_s of the term $P(u\nabla u)$ in

$$\partial_t u = Pf - P(u\nabla u) + Au \qquad (2.2) \tag{4.3}$$

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[3] See footnote on p. 144

from the estimate

$$|P(u\nabla u(t)-u\nabla u(s))|_{H_1}\leqq c_5\cdot\|A(u(t)-u(s))\|\sup_{\tau\in[0,T]}\|Au(\tau)\|$$

by Lemma 3.4 with $\varepsilon=0$, the estimate above resulting from (3.5). The other two terms on the right side of (4.3) being strongly continuous in $D_{A^\varepsilon}\subset H_s$ on $[0,T]$ by our assumption, from (4.3) we get $\partial_t u\in C_0([0,T],H_s)$ and therefore $\partial_t u|_{\partial\Omega}\in C_0([0,T],H_{s-\frac{1}{2}}(\partial\Omega))$ on the boundary [14, p. 41–42 Theorem 9.4].[4] Since $\partial_t u|_{\partial\Omega}=0$ for $t>0$ because of the boundary condition $u|_{\partial\Omega}=0$, our result is the compatibility condition

$$0=\lim_{t\downarrow 0}(\partial_t u)|_{\partial\Omega}(t)=(Pf(0)-P(u_0\nabla u_0)+P\Delta u_0)|_{\partial\Omega}. \tag{4.4}$$

If, in addition, $f\in C_0([0,T],H_s)$, we have

$$Pf(0)-P(u_0\nabla u_0)+P\Delta u_0=P(f(0)-u_0\nabla u_0+\Delta u_0) \tag{4.5}$$

by definition of Weyl's orthogonal projection P, which for the function $g=f(0)-u_0\nabla u_0+\Delta u_0\in H_0$ reads

$$Pg=g-\nabla q \tag{4.6}$$

with the unique generalized gradient $\nabla q\in H_0$. From this we conclude

$$\nabla q=(1-P)g=\nabla p(0) \tag{4.7}$$

with $\nabla p(0)$ from (3.1). Since due to [14, p. 41-42, Theorem 9.4][4] the boundary value $f(0)|_{\partial\Omega}$ belongs to $H_{s-\frac{1}{2}}(\partial\Omega)$ and because of $u_0\nabla u_0|_{\partial\Omega}=0$, the result of the last Eqs. (4.4)-(4.7) together is the compatibility condition (4.2).

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[4] See footnote on p. 144

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